Applied Mathematics Letters 22 (2009) 823–826

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

A necessary and sufficient condition for input identifiability for linear time-invariant systems

Tihomir Gyulov, Gheorghe Moroşanu *

Department of Mathematics, Central European University, Nador u. 9, 1051 Budapest, Hungary

ARTICLE INFO

Article history:
Received 22 May 2008
Received in revised form 24 June 2008
Accepted 30 June 2008

Keywords:
Linear time-invariant system
State
Input
Output
Input identification

ABSTRACT

A necessary and sufficient condition for input identifiability for linear autonomous systems is given. The result is based on a finite iterative process and its proof relies on elementary arguments involving matrices, finite dimensional linear spaces, Gronwall's lemma, and linear differential systems. Our condition is equivalent to the classical condition involving the geometrical concept of controlled invariant [V. Basile, G. Marro, Controlled and Conditioned Invariants in Linear System Theory, Prentice Hall, Englewood Cliffs, NJ, 1992, p. 237] and the dimension reduction algorithm that we propose seems to be useful in designing deconvolution methods.

1. Introduction

Let $M_{p\times q}(\mathbb{R})$ denote the set of all $p \times q$ matrices with real entries. Consider in a given finite interval $0 \leq t \leq T$ the following linear time-invariant system:

$$\dot{x} = Ax + Bu, \tag{1}$$

$$y = Hx, \tag{2}$$

where $A \in M_{n\times n}(\mathbb{R})$, $B \in M_{n\times d}(\mathbb{R})$, $H \in M_{m\times n}(\mathbb{R})$, $u = u(t)$ is a control policy (input) taking values from $\mathbb{R}^d$, $x = x(t) \in \mathbb{R}^n$ denotes the state of the system, and $y = y(t) \in \mathbb{R}^m$ is the output trajectory.

By the variation of constants formula we can see that for every initial state $x_0 \in \mathbb{R}^n$ and control (input) $u \in L^1\left(0, T; \mathbb{R}^d\right)$ the corresponding output is given by

$$y(t) = He^{At}x_0 + \int_0^t He^{A(t-s)}Bu(s)\,\mathrm{d}s, \quad \forall t \in [0, T]. \tag{3}$$

Let $\mathcal{AC}([0, T]; \mathbb{R}^m)$ denote the space of absolutely continuous functions on $[0, T]$ with values in $\mathbb{R}^m$. Let $Q : L^1\left(0, T; \mathbb{R}^d\right) \to \mathcal{AC}([0, T]; \mathbb{R}^m)$ be the operator defined by

$$(Qu)(t) = \int_0^t He^{A(t-s)}Bu(s)\,\mathrm{d}s, \quad \forall t \in [0, T]. \tag{4}$$

Obviously, the range of Q, denoted Range Q, does not cover the whole $\mathcal{AC}([0, T]; \mathbb{R}^m)$ (in particular, every function from Range Q vanishes at $t = 0$).

We continue with the following definition related to system (1) and (2) (see, e.g., [1, p. 167]):

* Corresponding author.
E-mail address: Morosanug@ceu.hu (G. Moroşanu).

0893-9659/$ – see front matter © 2008 Elsevier Ltd. All rights reserved.
doi:10.1016/j.aml.2008.06.043

Definition. The system input is said to be *identifiable* (*detectable*) if for every initial state x_0 and output $y = y(t)$ the corresponding input $u = u(t)$ is *unique* (u is supposed to exist as long as an output y is produced).

Remark 1. If u is unique for some $x_0,\ y$, then the same property holds for all inputs u. In fact, *input identifiability* (or *detectability*) for system (1) and (2) means that the kernel of Q is the null space: $\ker\ Q = \{0\}$. If the system input is identifiable, then the system is said to be *left invertible* or *ideally observable* in the Russian literature (see [2]).

Remark 2. If system (1) and (2) is left invertible (i.e., equivalently, $\ker\ Q = \{0\}$), then the following rank condition holds (see [1, p. 168]):

$$\mathrm{Rank}\{B^{\mathrm{T}}H^{\mathrm{T}},\ B^{\mathrm{T}}A^{\mathrm{T}}H^{\mathrm{T}}, \dots, B^{\mathrm{T}}(A^{\mathrm{T}})^{n-1}H^{\mathrm{T}}\} = d,$$

where the superscript T denotes the matrix transpose. The converse implication is not true (i.e., the above rank condition is not sufficient for left invertibility), as the following simple counterexample shows: A = the matrix with rows $(-1, 0, 0),\ (0, 0, 0),\ (0, 0, 1)$, $B =$ the matrix with rows $(1, 0),\ (0, 1),\ (0, 0)$, $H = (1, 1, 0)$. Obviously, the rank condition is satisfied, but $\ker\ Q$ contains the nonzero function $u(t) = col\,(t, -1 + \mathrm{e}^{-t})$. Therefore, Theorem 5.5.2 in [1, p. 167] is false.

2. The main result

In the previous section we have seen that the problem of left invertibility for system (1) and (2) reduces to the condition $\ker\ Q = \{0\}$. In this section we formulate a necessary and sufficient condition on matrices $A,\ B,\ H$ such that $\ker\ Q = \{0\}$. Our result relies on an iterative process. Namely, we construct iteratively a non-increasing sequence of integers $\{d_i\} \subset \mathbb{N}$ as well as sequences of matrices $\{A_i\} \subset M_{n\times n}(\mathbb{R})$, $\{B_i\} \subset M_{n\times d_i}(\mathbb{R})$ and $\{H_i\} \subset M_{m\times n}(\mathbb{R})$ for $i = 0, 1, 2, \dots$ as follows.

Let $A_0 = A,\ B_0 = B,\ H_0 = H$ and $d_0 = d$. Given $A_i,\ B_i,\ H_i$ and d_i, let $\dim\ker(H_iB_i) =: d_{i+1}$. If $d_{i+1} = 0$, the iterations terminate. Let $d_{i+1} > 0$. Then, $\dim \mathrm{Range}\,(H_iB_i) = d_i - d_{i+1}$. Moreover, if $\mathbb{R}^{d_i} = U_i \oplus \ker H_iB_i$ and $\mathbb{R}^m = V_i \oplus \mathrm{Range}\,(H_iB_i)$ (e.g., $U_i := (\ker H_iB_i)^{\perp},\ V_i := (\mathrm{Range}\,(H_iB_i))^{\perp}$), then $U_i \cong \mathrm{Range}\,(H_iB_i) \cong \mathbb{R}^{d_i-d_{i+1}}$.

First, assume that $d_i > d_{i+1}$. Then, there are matrices $M_i \in M_{d_i\times(d_i-d_{i+1})}(\mathbb{R})$, $T_i \in M_{d_i\times d_{i+1}}(\mathbb{R})$, $C_i \in M_{(d_i-d_{i+1})\times m}(\mathbb{R})$, such that

$$\mathrm{Range}\,(M_i) = U_i, \qquad \mathrm{Range}\,(T_i) = \ker H_iB_i,$$
$$C_iH_iB_iM_i = id|_{\mathbb{R}^{d_i-d_{i+1}}}$$

and for any $x \in \mathbb{R}^{d_i}$ there exist vectors $y \in \mathbb{R}^{d_i-d_{i+1}}$, $z \in \mathbb{R}^{d_{i+1}}$, uniquely determined, such that $x = M_iy + T_iz$. Let P_i be the matrix of the projection on V_i with respect to $\mathrm{Range}\,(H_iB_i)$. The matrix C_i may be chosen as

$$C_i = \left(M_i^{\mathrm{T}}B_i^{\mathrm{T}}H_i^{\mathrm{T}}H_iB_iM_i\right)^{-1} M_i^{\mathrm{T}}B_i^{\mathrm{T}}H_i^{\mathrm{T}}.$$

Indeed, the matrix $M_i^{\mathrm{T}}B_i^{\mathrm{T}}H_i^{\mathrm{T}}H_iB_iM_i$ is the Gramm matrix of the vectors defined by the columns of $H_iB_iM_i$ and due to the fact that they are linearly independent C_i is well-defined. Note that the matrices $M_i,\ T_i,\ C_i$ and P_i are not uniquely determined.

We define the matrices A_{i+1}, B_{i+1} and H_{i+1} by

$$A_{i+1} := (I - B_iM_iC_iH_i)\,A_i, \qquad B_{i+1} := B_iT_i, \qquad H_{i+1} := P_iH_iA_i.$$

Now, assume that $d_{i+1} = d_i$. In this case we define

$$A_{i+1} := A_i, \qquad B_{i+1} := B_i, \qquad H_{i+1} := H_iA_i.$$

Let us now state our main result:

Theorem 1. *Let $A \in M_{n\times n}(\mathbb{R})$, $B \in M_{n\times d}(\mathbb{R})$, $H \in M_{m\times n}(\mathbb{R})$, and let Q be the operator defined before by (4). If $\{d_i\} \subset \mathbb{N}$, $\{A_i\} \subset M_{n\times n}(\mathbb{R})$, $\{B_i\} \subset M_{n\times d_i}(\mathbb{R})$ and $\{H_i\} \subset M_{m\times n}(\mathbb{R})$ are the sequences defined above, then* $\ker\ Q = \{0\}$ *if and only if there exists $i \in \mathbb{N}$, $i \le dn$ such that $d_i = 0$. Moreover, if $d_i = d_{i+1} = \cdots = d_{i+n-1} > 0$ for some i, then $d_k = d_i$ for all $k \ge i$.*

Proof. Let $w \in \ker\ Q$, i.e., $w \in L^1\left(0, T; \mathbb{R}^d\right)$ and

$$\int_0^t H\mathrm{e}^{A(t-s)}Bw(s)\,\mathrm{d}s = 0 \quad \forall t \in [0, T]. \tag{5}$$

By differentiation with respect to t we see that for a.a. $t \in [0, T]$

$$HBw(t) + \int_0^t HA\mathrm{e}^{A(t-s)}Bw(s)\,\mathrm{d}s = 0. \tag{6}$$

First, assume that $d_1 = \dim\ker(HB) < d = d_0$. If $d_1 = 0$, then there exists a matrix C such that $CHB = id|_{\mathbb{R}^d}$ which allows us to write

$$w(t) + C\int_0^t HA\mathrm{e}^{A(t-s)}Bw(s)\,\mathrm{d}s = 0. \tag{7}$$

Now, Gronwall's lemma implies that w is the null function. Let $d_1 > 0$ and $w(t) = M_0 v(t) + T_0 w_1(t)$. Then Eq. (6) is equivalent to the system

$$v(t) + C_0 \int_0^t H_0 A_0 e^{A_0(t-s)} B_0 (M_0 v(s) + T_0 w_1(s))\, ds = 0, \tag{8}$$

$$P_0 \int_0^t H_0 A_0 e^{A_0(t-s)} B_0 (M_0 v(s) + T_0 w_1(s))\, ds = 0. \tag{9}$$

It follows by standard arguments that for each integrable function $w_1(t)$ there is a unique solution $v(t)$ of Eq. (8). Moreover, v can be expressed as a convolution product of a suitable matrix kernel and w_1 (see (12)). Indeed, let us define

$$V(t) := \int_0^t e^{A_0(t-s)} B_0 (M_0 v(s) + T_0 w_1(s))\, ds. \tag{10}$$

We have for a.a. $t \in [0, T]$

$$\dot{V}(t) = A_0 V(t) + B_0 M_0 v(t) + B_0 T_0 w_1(t),$$

which implies (see (8))

$$\dot{V}(t) = (A_0 - B_0 M_0 C_0 H_0 A_0)\, V(t) + B_0 T_0 w_1(t).$$

Since $V(0)$ is the null matrix, we obtain by the variation of constants formula

$$V(t) = \int_0^t e^{(A_0 - B_0 M_0 C_0 H_0 A_0)(t-s)} B_0 T_0 w_1(s)\, ds. \tag{11}$$

Thus

$$v(t) = -C_0 H_0 A_0 \int_0^t e^{(A_0 - B_0 M_0 C_0 H_0 A_0)(t-s)} B_0 T_0 w_1(s)\, ds \tag{12}$$

is the (unique) solution of Eq. (8) corresponding to w_1. For this v, according to (10) and (11), we can write

$$P_0 \int_0^t H_0 A_0 e^{A_0(t-s)} B_0 (M_0 v(s) + T_0 w_1(s))\, ds = P_0 H_0 A_0 \int_0^t e^{(A_0 - B_0 M_0 C_0 H_0 A_0)(t-s)} B_0 T_0 w_1(s)\, ds.$$

Thus it is obvious that the existence of a nonzero solution $w(t)$ of Eq. (5) (which is equivalent to system (8) and (9)) is equivalent to the existence of a nonzero solution $w_1(t)$ of the equation

$$\int_0^t H_1 e^{A_1(t-s)} B_1 w_1(s)\, ds = 0, \qquad w_1(t) \in \mathbb{R}^{d_1},$$

where

$$H_1 := P_0 H_0 A_0, \qquad A_1 := (I - B_0 M_0 C_0 H_0)\, A_0, \qquad B_1 := B_0 T_0.$$

Next, assume that $d_1 = \dim\ker(HB) = d = d_0$. Then Eq. (6) reads

$$\int_0^t HA e^{A(t-s)} Bw(s)\, ds = 0, \tag{13}$$

so we can take $w_1 = w$,

$$A_1 := A_0, \qquad B_1 := B_0, \qquad H_1 := H_0 A_0,$$

and we can continue further to construct iteratively matrices A_i, B_i and H_i. Notice that if for some integer i we have $d_i = d_{i+1} = \cdots = d_{i+n-1}$, then

$$H_i B_i = H_i A_i B_i = \cdots = H_i A_i^{n-1} B_i = \mathcal{O},$$

where $\mathcal{O}$ denotes the zero matrix in $M_{m\times d_i}(\mathbb{R})$. Thus the Cayley–Hamilton theorem implies that $H_i A_i^k B_i = \mathcal{O}$ for any integer $k \geq i$. In particular, it follows that $H_i e^{A_i t} B_i = \mathcal{O}$ and there exists a nonzero function $w(t)$ satisfying (5). Moreover, either there exists an integer $i \leq dn$, such that $d_i = 0$ and the iterations terminate or $0 < d_{dn} = d_{dn+1} = \cdots = d_{dn+k} = \cdots$ and there exists a nonzero solution of Eq. (5). □

3. Concluding comments

First of all, it has been brought to our attention that our necessary and sufficient condition formulated in Theorem 1 is a variation of the classical condition described in Property 4.3.6 of Basile and Marro [2, Chapter 4], which involves the geometrical concept of controlled invariant. Indeed, our condition is equivalent to the classical one. This can be shown by comparing the two corresponding algorithms, under the maximal rank condition on B. We do not assume in Theorem 1 that B has maximal rank (i.e., equivalently, $\ker B = \{0\}$), but obviously it is a necessary condition for left invertibility. This equivalence confirms the validity of our result.

While the classical approach is geometrical, our new iterative process relies on simple arguments from linear algebra and the theory of differential equations which allow dimension reduction, as described in the proof of Theorem 1.

If the system is left invertible (i.e., $\ker Q = \{0\}$), one can use an iterative process suggested by the proof of Theorem 1 to solve for $u = u(t)$ the equation $Qu(t) = f(t)$. This operation is nowadays called *deconvolution* since Q is an integral convolution operator. In general u does not depend continuously on f and this makes the problem difficult. Among the existing papers addressing deconvolution methods, we refer the reader to [3–5] and the references therein. We think that our iterative process could generate new efficient deconvolution methods.

The general output equation $y(t) = Hx(t) + Du(t),\ t \in [0, T]$, can also be considered in our framework. Here D is an $m \times d$ matrix with real entries. In this case, instead of Eq. (5), we have the following integral equation:

$$Dw(t) + H \int_0^t \mathrm{e}^{A(t-s)} Bw(s)\,\mathrm{d}s = 0,$$

whose form is similar to Eq. (6). Therefore, one can apply our algorithm described in the proof of Theorem 1 above to derive a necessary and sufficient condition for input identification (or left invertibility). The precise formulation of this condition is left to the reader. In particular, if $\ker D = \{0\}$, then obviously the system input is identifiable.

Acknowledgements

We are indebted to both Luciano Pandolfi and the anonymous referee for their valuable remarks on this work, which have produced improvements in our presentation.

References

[1] N.U. Ahmed, Dynamic Systems and Control with Applications, World Scientific, Hackensack, NJ, 2006.
[2] V. Basile, G. Marro, Controlled and Conditioned Invariants in Linear System Theory, Prentice Hall, Englewood Cliffs, NJ, 1992.
[3] F. Fagnani, L. Pandolfi, Recursive deconvolution: An overview of some recent results, in: V. Barbu, et al. (Eds.), Analysis and Optimization of Differential Systems, Kluwer, Boston, MA, 2003, pp. 145–156.
[4] F. Fagnani, V. Maksimov, L. Pandolfi, A recursive deconvolution approach to disturbance reduction, IEEE Trans. Automat. Control 49 (6) (2004) 907–921.
[5] L. Pandolfi, Adaptive recursive deconvolution and adaptive noise cancellation, Internat. J. Control 80 (3) (2007) 403–415.