



Lyapunov operator inequalities for exponential stability of Banach space semigroups of operators

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ABSTRACT

We obtain continuous-time and discrete-time Lyapunov operator inequalities for the exponential stability of strongly continuous, one-parameter semigroups acting on Banach spaces. Thus we extend the classic result of Datko (1970) [2] from Hilbert spaces to Banach spaces.

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1. Introduction and preliminaries

Let X be a real or complex Banach space and X' its (dual) conjugate space consisting of all bounded and antilinear functionals on X . Also X^* will denote the classic dual space of all bounded and linear functionals on X . If Y is also a Banach space, we will denote by $\mathcal{B}(X, Y)$ the Banach algebra of all linear and bounded operators from X into Y . If $X = Y$, we will simply write $\mathcal{B}(X)$. The norms on X , X' , Y and $\mathcal{B}(X, Y)$ shall be denoted by the symbol $\|\cdot\|$. Also $\mathcal{M}(\mathbb{R}_+, X)$ denotes the space of all strongly measurable functions from \mathbb{R}_+ to X and $L^p(\mathbb{R}_+, X) = \left\{ f \in \mathcal{M}(\mathbb{R}_+, X) : \int_{\mathbb{R}_+} \|f(t)\|^p dt < \infty \right\}$, $p \in [1, \infty)$.

To proceed, we now recall the following definition:

Definition 1.1. A family $\{T(t)\}_{t \geq 0}$ of linear and bounded operators on a Banach space X is said to be a *strongly continuous semigroup* if $T(0) = I$, $T(t+s) = T(t)T(s)$ for all $t, s \in \mathbb{R}_+$, and $\lim_{t \downarrow 0} T(t)x = x$, for all $x \in X$.

Definition 1.2. Let $\{T(t)\}_{t \geq 0}$ be a C_0 -semigroup. Then $\{T(t)\}_{t \geq 0}$ is said to be exponentially stable (e.s.) if there are $N, \nu > 0$ such that $\|T(t)\| \leq Ne^{-\nu t}$, for all $t \geq 0$.

2. Results

The theorem of A.M. Lyapunov establishes that if A is an $n \times n$ complex matrix then A has all its characteristic roots with real parts negative if and only if for any positive definite Hermitian matrix H there exists a unique positive definite Hermitian matrix W satisfying the equation $(L_H)A^*W + WA = -H$ (where $*$ denotes the conjugate transpose of a matrix). The above theorem became widely known and is actually a classic topic in any graduate textbook on dynamical systems. The use of the above Lyapunov operator equation is extended on the infinite-dimensional framework by Daleckij and Krein in [1] for the case of semigroups $T(t) = e^{tA}$, where A is a bounded linear operator. Thus, in [1] the authors prove that $\{e^{tA}\}_{t \geq 0}$,

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with $A \in \mathcal{B}(X)$, is exponentially stable if and only if there exists $W \in \mathcal{B}(X)$, $W \gg 0$ (i.e. there exists $m > 0$ such that $\langle Wx, x \rangle \geq m\|x\|^2$, for any $x \in X$), solution of the Lyapunov equation $A^*W + WA = -I$. This result is extended by Datko in [2], for the general case of C_0 -semigroups as it follows.

Theorem 2.1 ([2], Datko, 1970). *A C_0 -semigroup $\{T(t)\}_{t \geq 0}$ is exponentially stable if and only if there exists $W \in \mathcal{B}(X)$, $W^* = W$, $W \geq 0$ such that $\langle Ax, Wx \rangle + \langle Wx, Ax \rangle = -\|x\|^2$ for all $x \in D(A)$, where A denotes the infinitesimal generator of $\{T(t)\}_{t \geq 0}$.*

See also [3–8] for the Lyapunov operator equations with unbounded A . It is worth noting that an attempt to establish an equivalence between the solvability of the Lyapunov equation and the exponential stability of a C_0 -semigroup in the general context of Banach spaces is presented in [7].

Remark 2.1. It is worth noting that the proof of the above theorem relies on the fact that all trajectories $T(\cdot)x$ (of a C_0 -semigroup $\{T(t)\}_{t \geq 0}$) have an exponential decay as $t \rightarrow \infty$ if and only if, for each vector $x \in X$, the function $t \rightarrow \|T(t)x\|$ lies in $L^2(\mathbb{R}_+, \mathbb{R})$.

Except [7], all the above results are given in the setting of one-parameter semigroups acting on Hilbert spaces. We will try to go more general and find variants of the Lyapunov operator equation for the exponential stability of one-parameter semigroups acting on Banach spaces. This requires to recall some facts about the adjoint of a linear operator on a Banach space.

Definition 2.1. Let X, Y be two Banach spaces and $A \in \mathcal{B}(X, Y)$. Then there exists a unique operator $A^* \in \mathcal{B}(Y', X')$ that satisfies $y(Ax) = A^*y(x)$, for all $x \in X$, $y \in Y'$. A^* will be called the adjoint of A .

It is worth noting that the above notion of the adjoint of a linear and bounded operator between two Banach spaces allows us to create a definition of the adjoint that directly generalizes the definition of the adjoint of an operator on Hilbert spaces. In other words, if X and Y are Hilbert spaces and $A \in \mathcal{B}(X, Y)$ then there is no difference of the adjoint between the adjoint A^* defined by considering X, Y to be Hilbert spaces, and the adjoint A^* defined by considering X, Y to be Banach spaces. If we would choose that $A^* : Y^* \rightarrow X^*$ then we would obtain a different definition compared to the Hilbert space definition. Now, for defining the concept of a self-adjoint operator on a Banach space, we recall that X is isomorphic and isometric with a subspace of X'' .

Definition 2.2. (i) An operator $A \in \mathcal{B}(X, X')$ is self-adjoint if the restriction of A^* to X is A , and therefore $Ay(x) = \overline{Ax(y)}$, for all $x, y \in X$.

(ii) $A \in \mathcal{B}(X, X')$ is positive if A is self-adjoint, and $Ax(x) \geq 0$, for all $x \in X$.

Remark 2.2. It is easy to see that $A \in \mathcal{B}(X, X')$ is positive if and only if $Ax(x)$ is a positive real number, for all $x \in X$.

In the following we will denote $\mathcal{B}^+(X, X') = \{A \in \mathcal{B}(X, X') : A \text{ is positive}\}$.

Theorem 2.2. Let $\{T(t)\}_{t \geq 0}$ be a C_0 -semigroup on the Banach space X and A its infinitesimal generator, and consider $\Gamma \in \mathcal{B}^+(X, X')$ with the property that there exists $\gamma > 0$ such that $\Gamma x(x) \geq \gamma\|x\|^2$, for all $x \in X$. Then $\{T(t)\}_{t \geq 0}$ is exponentially stable if and only if there exists $W \in \mathcal{B}^+(X, X')$ such that $Wx(Ax) + W(Ax)(x) \leq -\Gamma x(x)$ (\mathcal{L}), for all $x \in D(A)$.

Proof. Sufficiency. Let $x \in D(A)$, $t \geq 0$. If $\varphi_x(t) = W(T(t)x)(T(t)x)$ then $\dot{\varphi}_x(t) = W(T(t)Ax)(T(t)x) + W(T(t)x)(T(t)Ax) \leq -\Gamma(T(t)x)(T(t)x) \leq -\gamma\|T(t)x\|^2$, which implies that $\|T(t)x\|^2 \leq -\frac{1}{\gamma}\dot{\varphi}_x(t)$. Then $\int_0^t \|T(\tau)x\|^2 d\tau \leq \frac{1}{\gamma}(\varphi_x(0) - \varphi_x(t)) \leq \frac{1}{\gamma}\varphi_x(0) = \frac{1}{\gamma}Wx(x) \leq \frac{\|W\|}{\gamma}\|x\|^2$, for $x \in D(A)$, $t \geq 0$. For each $x \in X$ there is (x_n) in $D(A)$ with $x_n \rightarrow x$, and by dominated convergence theorem it results that $\int_0^t \|T(\tau)x\|^2 d\tau \leq \frac{\|W\|}{\gamma}\|x\|^2$, $x \in X$, $t \geq 0$. Thus $\int_0^\infty \|T(\tau)x\|^2 d\tau \leq \frac{\|W\|}{\gamma}\|x\|^2$, and by Remark 2.1. we have that $\{T(t)\}_{t \geq 0}$ is e.s.

Necessity. Let $Wx(y) = \int_0^\infty \Gamma(T(\tau)x)(T(\tau)y) d\tau$. Then $|Wx(y)| \leq \frac{\|\Gamma\|}{2v} \|x\| \|y\|$, $x, y \in X$. Then $W : X \rightarrow X'$ is well-defined and $W \in \mathcal{B}(X, X')$. Since

$$\overline{Wy(x)} = \int_0^\infty \overline{\Gamma(T(\tau)y)(T(\tau)x)} d\tau = \int_0^\infty \Gamma(T(\tau)x)(T(\tau)y) d\tau = Wx(y),$$

then W is self-adjoint. We have that $Wx(x) = \int_0^\infty \Gamma(T(\tau)x)(T(\tau)x) d\tau \geq \gamma \int_0^\infty \|T(\tau)x\|^2 d\tau \geq 0$, and therefore W is positive. Since

$$\begin{aligned} Wx(Ax) + W(Ax)(x) &= \int_0^\infty \Gamma(T(\tau)x)(T(\tau)Ax) d\tau + \int_0^\infty \Gamma(T(\tau)Ax)(T(\tau)x) d\tau \\ &= \int_0^\infty \frac{d}{d\tau} \Gamma(T(\tau)x)(T(\tau)x) d\tau = -\Gamma x(x), \quad \text{for all } x \in D(A), \end{aligned}$$

and the inequality (\mathcal{L}) is verified. \square

The above theorem can also be cast in an integral form as follows:

Corollary 2.1. Let $\{T(t)\}_{t \geq 0}$ be a C_0 -semigroup on the Banach space X and $\Gamma \in \mathcal{B}^+(X, X')$ such that there exists $\gamma > 0$ with $\Gamma x(x) \geq \gamma \|x\|^2$, for all $x \in X$. Then $\{T(t)\}_{t \geq 0}$ is exponentially stable if and only if there exists $W \in \mathcal{B}^+(X, X')$ such that:

$$W(T(t)x)(T(t)x) + \int_0^t \Gamma(T(\tau)x)(T(\tau)x) \leq Wx(x), \quad \text{for all } t \geq 0 \text{ and } x \in X.$$

Theorem 2.3. Let $\{T(t)\}_{t \geq 0}$ be a C_0 -semigroup on the Banach space X and $\Gamma \in \mathcal{B}^+(X, X')$ such that there exists $\gamma > 0$ with $\Gamma x(x) \geq \gamma \|x\|^2$, for all $x \in X$. Then $\{T(t)\}_{t \geq 0}$ is exponentially stable if and only if there exists $W \in \mathcal{B}^+(X, X')$ such that

$$W(T(n)x)(T(n)x) + \sum_{k=0}^{n-1} \Gamma(T(k)x)(T(k)x) \leq Wx(x), \quad \text{for all } n \in \mathbb{N}^*, \text{ and } x \in X.$$

Proof. Sufficiency. Since $\sum_{k=0}^{n-1} \Gamma(T(k)x)(T(k)x) \leq Wx(x)$, for all $n \in \mathbb{N}^*$, and $x \in X$, then $\gamma \sum_{k=0}^{n-1} \|T(k)x\|^2 \leq Wx(x) \leq \|W\| \|x\|^2$, for all $n \in \mathbb{N}$. Thus $\sum_{k=0}^{\infty} \|T(k)x\|^2 \leq \frac{\|W\|}{\gamma} \|x\|^2$, for all $x \in X$. If $k := [t]$ (the largest integer less than or equal with t) then $\int_k^{k+1} \|T(t)x\|^2 dt \leq (Me^\omega)^2 \|T(k)x\|^2$. Thus we get that $\int_0^\infty \|T(t)x\|^2 dt \leq (Me^\omega)^2 \sum_{k=0}^\infty \|T(k)x\|^2 < \infty$, for all $x \in X$. Applying Remark 2.1 we obtain that $\{T(t)\}_{t \geq 0}$ is exponentially stable.

Necessity. Let $Wx(y) = \sum_{k=0}^\infty \Gamma(T(k)x)(T(k)y)$. Then $W(T(n)x)(T(n)x) = \sum_{k=0}^\infty \Gamma(T(k+n)x)(T(k+n)x) = Wx(x) - \sum_{k=0}^{n-1} \Gamma(T(k)x)(T(k)x)$. We obtain that $W(T(n)x)(T(n)x) + \sum_{k=0}^{n-1} \Gamma(T(k)x)(T(k)x) = Wx(x)$, for all $n \in \mathbb{N}^*$, $x \in X$. \square

Remark 2.3. In the particular case when X is a Hilbert space endowed with the inner product $\langle \cdot, \cdot \rangle$ the above theorem becomes: “ $\{T(t)\}_{t \geq 0}$ is exponentially stable if and only if there is a positive operator $W \in \mathcal{B}(X)$ such that $\langle W(T(n)x), (T(n)x) \rangle + \sum_{k=0}^{n-1} \|(T(k)x)\|^2 \leq \langle Wx, x \rangle$, for all $n \in \mathbb{N}^*$ and $x \in X$ ”.

Remark 2.4. It is interesting to note that an advantage of having necessary and sufficient conditions for exponential stability in terms of an inequality is that necessary and sufficient conditions for non-exponential stability can also be derived from the inequality. This is indeed the case for the exponential stability and non-exponential stability of Hilbert space contraction semigroups.

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