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Weak geodesic topology and fixed finite subgraph theorems in infinite partial cubes I. Topologies and the geodesic convexity

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Dedicated to Gert Sabidussi on the occasion of his 80th birthday

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ABSTRACT

The weak geodesic topology on the vertex set of a partial cube G is the finest weak topology on $V(G)$ endowed with the geodesic convexity. We prove the equivalence of the following properties: (i) the space $V(G)$ is compact; (ii) $V(G)$ is weakly countably compact; (iii) the vertex set of any ray of G has a limit point; (iv) any concentrated subset of $V(G)$ (i.e. a set A such that any two infinite subsets of A cannot be separated by deleting finitely many vertices) has a finite positive number of limit points. Moreover, if $V(G)$ is compact, then it is scattered. We characterize the partial cubes for which the weak geodesic topology and the geodesic topology (see [N. Polat, Graphs without isometric rays and invariant subgraph properties I. *J. Graph Theory* **27** (1998), 99–109]) coincide, and we show that the class of these particular partial cubes is closed under Cartesian products, retracts and gated amalgams.

1. Introduction

Fixed finite subgraph theorems, which are far-reaching outgrowths of metric fixed point theory, have been a flourishing topic in the literature on metric graph theory. The first of such theorems is Novakowski and Rival's [7] stating that *every self-contraction (i.e. non-expansive self-map) of a graph G fixes a vertex or an edge if and only if G is a rayless tree*. The second is Schmidt's [17] stating that *any rayless graph G contains a non-empty finite set of vertices which is fixed by every automorphism of G*.

Both of these results share the lack of rays as a sufficient condition. Note that a ray in a tree is obviously isometric, which is not true in general. So it was natural to wonder if the lack of isometric rays was a sufficient condition or at least the main part of a sufficient condition of fixed finite subgraph properties for some classes of graphs containing trees as particular instances. Actually, even though this condition turned out to be a good one for several classes of median-like graphs, of partial cubes, for Helly graphs and bridged graphs (see [9–11,14,18]), it is generally inadequate, even for some partial cubes which, in the finite case, are treelike by their very construction (see the infinite treelike partial cubes in [16]). Take for example the subdivision $S(K_{\aleph_0})$ of the infinite complete graph $K_{\aleph_0}$, obtained from $K_{\aleph_0}$ by subdividing each edge of this graph by a single vertex. Then $S(K_{\aleph_0})$ is a partial cube (see Section 2.5) which clearly has no isometric rays, but there exist obvious endomorphisms of this graph which fixes no finite set of vertices.

The lack of isometric rays in a graph G is directly linked to the compactness of a topology, called the *geodesic topology* on $V(G)$ (see [9]). This topology, which is compatible with the geodesic convexity on $V(G)$, is generally finer than the topology generated by the geodesically closed convex sets, that is the geodesic topology is not a weak topology. So, to obtain sufficient conditions for fixed finite subgraph theorems in any graph G, we introduce and study the finest weak topology on $V(G)$ endowed with the geodesic convexity, which we call the *weak geodesic topology* on $V(G)$. In this paper and in the subsequent one [16], in order to pursue the study of partial cubes that we have undertaken in a series of papers, we restrict the study of the weak geodesic topology and of fixed finite subgraph theorems to the class of partial cubes.

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Section 3 is devoted to the study of the weak geodesic topology with emphasis on compactness. We show in particular that the compactness of a weak geodesic space is linked to the lack of divergent rays, rays which are more general than isometric rays. Two of the results of this section are the cornerstones of our proof of the existence of fixed finite subgraphs in partial cubes in [16]: Propositions 3.18 and 3.19. Essentially, if G is a partial cube whose vertex set is compact, then Proposition 3.18 asserts that some particular subsets of $V(G)$, such as the vertex set of any ray of G, has only finitely many limit points, and Proposition 3.19 implies that there always exists a non-empty finite set of special limit points of $V(G)$.

In Section 4, we characterize (Theorem 4.8) the geodesically consistent partial cubes, that is the partial cubes for which the geodesic topology coincides with the weak geodesic topology. These are the partial cubes G for which the sets U_{ab} and U_{ba} are closed with respect to the geodesic topology for each edge ab of G. It turns out that any partial cube whose vertex set is compact is geodesically consistent. The class of the geodesically consistent partial cubes, which is closed under Cartesian products, retracts and gated amalgams, contains in particular some classes of partial cubes which have been intensively studied, such as the ones of median graphs, of cellular bipartite graphs, of benzenoid graphs, and more generally of netlike partial cubes. Finally Section 5 deals with a completeness criterion for geodesically consistent partial cubes which only requires some properties of intervals.

2. Preliminaries

2.1. Graphs

The graphs we consider are undirected, without loops or multiple edges, and may be finite or infinite. Let G be a graph. If $x \in V(G)$, the set $N_G(x) := \{y \in V(G) : xy \in E(G)\}$ is the *neighborhood* of x in G, $N_G[x] := \{x\} \cup N_G(x)$ is the *closed neighborhood* of x in G and $\delta_G(x) := |N_G(x)|$ is the *degree* of x in G. For a set X of vertices of G we put $N_G[X] := \bigcup_{x\in X} N_G[x]$ and $N_G(X) := N_G[X] - X$, we denote by $G[X]$ the subgraph of G induced by X, and we set $G - X := G[V(G) - X]$.

A *path* $P = \langle x_0, \dots, x_n\rangle$ is a graph with $V(P) = \{x_0, \dots, x_n\}$, $x_i \neq x_j$ if $i \neq j$, and $E(P) = \{x_ix_{i+1} : 0 \leq i < n\}$. A path $P = \langle x_0, \dots, x_n\rangle$ is called an *(x_0, x_n)-path*, x_0 and x_n are its *endvertices*, while the other vertices are called its *inner* vertices, $n = |E(P)|$ is the *length* of P. A *ray* (resp. *double ray*) is a one-way (resp. two-way) infinite path, and a graph is *rayless* if it contains no rays. A subgraph of a ray R which is itself a ray is called a *subray* of R.

A *cycle* C with $V(C) = \{x_1, \dots, x_n\}$, $x_i \neq x_j$ if $i \neq j$, and $E(C) = \{x_ix_{i+1} : 1 \leq i < n\} \cup \{x_nx_1\}$, will be denoted by $\langle x_1, \dots, x_n, x_1\rangle$. The non-negative integer $n = |E(C)|$ is the *length* of C, and a cycle of length n is called a *n-cycle* and is often denoted by C_n.

Let G be a connected graph. The usual *distance* between two vertices x and y, that is, the length of any (x, y)-*geodesic* (=shortest (x, y)-path) in G, is denoted by $d_G(x, y)$. A connected subgraph H of G is *isometric* in G if $d_H(x, y) = d_G(x, y)$ for all vertices x and y of H. The *(geodesic) interval* $I_G(x, y)$ between two vertices x and y of G is the set of vertices of all (x, y)-geodesics in G.

2.2. Convexities

A *convexity* on a set X is an algebraic closure system $\mathcal{C}$ on X. The elements of $\mathcal{C}$ are the *convex sets* and the pair $(X, \mathcal{C})$ is called a *convex structure*. See van de Vel [19] for a detailed study of abstract convex structures. Several kinds of graph convexities, that is convexities on the vertex set of a graph G, have already been investigated. We will principally work with the *geodesic convexity*, that is the convexity on $V(G)$ which is induced by the geodesic interval operator I_G. In this convexity, a subset C of $V(G)$ is convex provided it contains the geodesic interval $I_G(x, y)$ for all $x, y \in C$. The *convex hull* $\mathrm{co}_G(A)$ of a subset A of $V(G)$ is the smallest convex set which contains A. The convex hull of a finite set is called a *polytope*. A subset H of $V(G)$ is a *half-space* if H and $V(G) - H$ are convex. A *copoint* at a point $x \in X$ is a convex set K which is maximal with respect to the property that $x \notin K$; x is an *attaching point* of K. We denote by $\mathcal{I}_G$ the pre-hull operator of the geodesic convex structure of G, i.e. the self-map of $\mathcal{P}(V(G))$ such that $\mathcal{I}_G(A) := \bigcup_{x,y\in A} I_G(x, y)$ for each $A \subseteq V(G)$. The convex hull of a set $A \subseteq V(G)$ is then $\mathrm{co}_G(A) = \bigcup_{n\in\mathbb{N}} \mathcal{I}_G^n(A)$. Furthermore we will say that a subgraph of a graph G is *convex* if its vertex set is convex, and by the *convex hull* $\mathrm{co}_G(H)$ of a subgraph H of G we will mean the smallest convex subgraph of G containing H as a subgraph, that is

$$\mathrm{co}_G(H) := G[\mathrm{co}_G(V(H))].$$

2.3. Cartesian products

The *Cartesian product* of a family of graphs $(G_i)_{i\in I}$ is the graph denoted by $\Box_{i\in I} G_i$ (or simply by $G_1\Box G_2$ if $|I| = 2$) with $\prod_{i\in I} V(G_i)$ as vertex set and such that, for all vertices u and v, uv is an edge whenever there exists a unique $j \in I$ with $\mathrm{pr}_j(u)\mathrm{pr}_j(v) \in E\left(G_j\right)$ and $\mathrm{pr}_i(u) = \mathrm{pr}_i(v)$ for every $i \in I - \{j\}$, where pr_i is the *i-th projection* of $\prod_{i\in I} V(G_i)$ onto $V(G_i)$. Connected components of a Cartesian product of connected graphs are called *weak Cartesian products* (see [4]). Clearly, the Cartesian product coincides with the weak Cartesian product provided that I is finite and the factors are connected. In particular, *hypercubes* are the weak Cartesian powers of K_2. For any non-negative integer n, we usually denote by Q_n a hypercube of dimension n.

2.4. Partial cubes

First we will recall some properties of *partial cubes*, that is of isometric subgraphs of hypercubes. Partial cubes are particular connected bipartite graphs.

For an edge ab of a graph G, let

$$W_{ab}^G := \{x \in V(G) : d_G(a,x) < d_G(b,x)\},$$
$$U_{ab}^G := N_G(W_{ba}^G).$$

If no confusion is likely, we will simply denote W_{ab}^G and U_{ab}^G by W_{ab} and U_{ab}, respectively. Note that the sets W_{ab} and W_{ba} are disjoint and that $V(G) = W_{ab} \cup W_{ba}$ if G is bipartite and connected.

Two edges xy and uv are in the Djoković–Winkler relation Θ if

$$d_G(x,u) + d_G(y,v) \neq d_G(x,v) + d_G(y,u).$$

If G is bipartite, the edges xy and uv are in relation Θ if and only if $d_G(x,u) = d_G(y,v)$ and $d_G(x,v) = d_G(y,u)$. The relation Θ is clearly reflexive and symmetric.

Theorem 2.1 (*Djoković [3, Theorem 1] and Winkler [20]*). *Let G be a connected bipartite graph. The following assertions are equivalent:*

(i) *G is a partial cube.*
(ii) *For every edge ab of G, the sets W_{ab} and W_{ba} are convex.*
(iii) *The relation Θ is transitive.*

It follows in particular that *the half-spaces of a partial cube G are the sets W_{ab}, $ab \in E(G)$.*

Lemma 2.2 (*Polat and Sabidussi [15, Proposition 5.6]*). *The copoints of a partial cube G are the sets W_{ab}, $ab \in E(G)$, and the set of all attaching points of W_{ab} is $\mathrm{co}_G(U_{ba})$.*

We recall that the geodesic convexity of a partial cube G has the separation property S_3: if a vertex x does not belong to a convex set $C \subseteq V(G)$, then there is a half-space H which separates x form C, that is $x \notin H$ and $C \subseteq H$. As a matter of fact, the geodesic convexity of a bipartite graph G has the property S_3 if and only if G is a partial cube (see [2, Proposition 2.2]).

We also recall that, if u_0, u_1, u_2 are three vertices of a graph G, then a *median* of the triple (u_0, u_1, u_2) is any element of the intersection $I_G(u_0,u_1) \cap I_G(u_1,u_2) \cap I_G(u_2,u_0)$. Moreover a graph G is a *median graph* if any triple of its vertices has a unique median. Median graphs are particular partial cubes.

Finally note that, because a partial cube is an isometric subgraph of some hypercube, we have the following properties:

- If a triple of vertices of a partial cube has a median, then this median is unique.
- Any partial cube is interval-finite, that is each of its interval is finite.
- Each polytope of a partial cube is finite.

2.5. Examples

We complete this section with two examples of partial cubes that we will use in several parts of this paper.

(1) We call the graph obtained from G by subdividing each edge of G by a single vertex, the *subdivision graph* of G, and we denote it by $S(G)$. For any finite or infinite cardinal α, the subdivision graph $S(K_\alpha)$ is a partial cube. Indeed, each polytope is contained in a $S(K_n)$ for some non-negative integer n, $S(K_n)$ is finite and moreover it is a partial cube by [5, Proposition 2.1]. We use the following notation: we put $V(K_\alpha) = \{x_n : n < \alpha\}$ and, for $n \neq p$, we denote by x_{np} the vertex which subdivides the edge $x_n x_p$ of K_α.
(2) For any cardinal $\alpha \geq 2$, we denote by $S^+(K_\alpha)$ the graph obtained from $S(K_\alpha)$ by joining a new vertex x_α to every element of $V(K_\alpha)$ (see Fig. 1 for $S^+(K_{\aleph_0})$). Note that $S^+(K_2)$ is Q_2, and $S^+(K_3)$ is the partial cube Q_3^-, that is Q_3 minus a vertex. Also note that $S^+(K_\alpha)$ is not a median graph because, for any $n < p < q \leq \alpha$, the triple (x_{np}, x_{pq}, x_{qn}) has no median. On the other hand, let $x_n x_{np} \in E(S^+(K_\alpha))$. The edges of $S^+(K_\alpha)$ which are in relation Θ with $x_n x_{np}$ are the edges $x_q x_{qp}$ for $q \neq p$ and the edge $x_\alpha x_p$. It follows that the relation Θ is clearly transitive, and hence that $S^+(K_\alpha)$ is a partial cube. Note that $S^+(K_\alpha)$ is isomorphic to the subgraph of the α-cube Q_α induced by $\mathcal{I}_{Q_\alpha}(N_{Q_\alpha}[x])$, where x is any vertex of Q_α. In particular $S^+(K_2)$ is a 4-cycle and $S^+(K_3)$ is isomorphic to Q_3^-.

We will use these two examples to illustrate the fact that the separation property S_4 (i.e., if C and D are two disjoint convex subsets of $V(G)$, then there is a half-space H such that $C \subseteq H$ and $D \subseteq V(G) - H$) is not necessarily satisfied by the geodesic convexity of a partial cube G. For example, the geodesic convexity of $S(K_4)$ does not have property S_4, and moreover the pairs of disjoint convex sets which are not separated by a half-space are the pairs $\mathrm{co}_G(x_i,x_j)$, $\mathrm{co}_G(x_k,x_l)$ for all i,j,k,l with $\{i,j,k,l\} = \{0,1,2,3\}$. On the other hand, one can easily check that the geodesic convexity of $S^+(K_4)$ has property S_4; note that, contrary to the case above, the convex sets $\mathrm{co}_G(x_i,x_j)$ and $\mathrm{co}_G(x_k,x_l)$ are not disjoint because $x_4 \in \mathrm{co}_G(x_i,x_j)$ for all i,j with $0 \leq i < j \leq 3$.

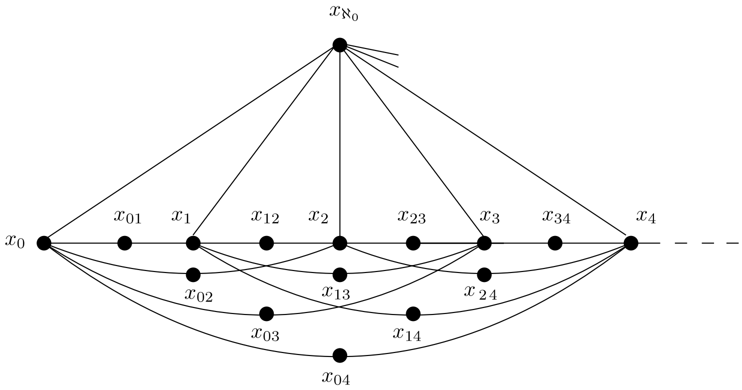

Fig. 1. $S^+(K_{\aleph_0})$.

3. Weak geodesic topology of a partial cube

We recall that a topology on the underlying set of a convex structure is a *weak topology* (see [19]) provided it has a subbase (for closed sets) of convex sets.

Definition 3.1. We call the finest weak topology on the vertex set of a graph G endowed with the geodesic convexity the *weak geodesic topology* on $V(G)$. That is the topology (in terms of closed sets) generated by all convex subsets of $V(G)$ as a subbase.

Throughout this section, unless stated otherwise, G will denote a partial cube, and by a closed set we will always mean a set which is closed for the weak geodesic topology. Because the geodesic convexity on the vertex set of a partial cube G has the property S_3, it follows that each convex set of G is an intersection of half-spaces, so the family of half-spaces of G is a subbase of the weak geodesic topology on $V(G)$.

3.1. Limit points and clusters

In this subsection we will give different characterizations of limit points for the weak geodesic topology of a partial cube which will be used in the proofs of subsequent results of this paper. We first introduce a special infinite set of vertices which will be important for this study.

Definition 3.2. We call an infinite $A \subseteq V(G)$ such that any two infinite subsets of A are not separable by a half-space (that is cannot be contained in complementary half-spaces) a *cluster* of G. In other words A is a cluster if its intersection with any half-space is either a finite or a cofinite subset of A.

Because the half-spaces of G are the sets W_{ab}, $ab \in E(G)$, it follows that the vertex set of a ray R of G is a cluster if and only if, for any edge ab of R, at most finitely many other edges of R are Θ-equivalent to ab. Hence the vertex set of any isometric ray of G or of any ray of $S(K_{\aleph_0})$ and of $S^+(K_{\aleph_0})$ are clusters. Also the neighborhood of any vertex of infinite degree of G is a cluster. Moreover any infinite subset of a cluster is also a cluster.

Proposition 3.3. *Every infinite set A of vertices of G contains a cluster.*

Proof. Let A be an infinite set of vertices of G. Without loss of generality we can suppose that A is countable. Then $\mathrm{co}_G(A)$ is also countable, and thus $E(H)$, where $H := G[\mathrm{co}_G(A)]$, is countable. Hence there exists a family $(a_nb_n)_{n\in\mathbb{N}}$ of pairwise non-Θ-equivalent edges of H such that each edge of H is Θ-equivalent to a_nb_n for some $n \in \mathbb{N}$.

We construct a sequence $A_0, A_1, \ldots$ of infinite subsets of A, and a sequence $x_0, x_1, \ldots$ of elements of A such that $x_n \in A_n$ and $A_{n+1} := A_n \cap W_{a_nb_n}$ or $A_{n+1} := A_n \cap W_{b_na_n}$ for all $n \in \mathbb{N}$. Let $A_0 := A$ and let x_0 be any element of A_0. Suppose that $A_0, \ldots, A_n$ and $x_0, \ldots, x_n$ have already been constructed for some $n \in \mathbb{N}$. Because A_n is infinite, at least one of the sets $A_n \cap W_{a_nb_n}$ or $A_n \cap W_{b_na_n}$ is infinite. Let A_{n+1} be this infinite subset if only one of them is infinite, and any one of them if both of them are infinite; and let x_{n+1} be any element of A_{n+1}.

We will prove that the set $X := \{x_n : n \in \mathbb{N}\}$ is a cluster. To show that no half-space can separate two infinite subsets of X, we only have to take the edges of H into account, and thus the edges a_nb_n, $n \in \mathbb{N}$. Because $\{x_i : i \geq n\} \subseteq A_n$, it follows that either $X \cap W_{a_nb_n}$ or $X \cap W_{b_na_n}$ is finite. Hence X is a cluster. □

For a subset A of vertices of a graph G, we will denote by $M_G(A)$ the set of all vertices belonging to $I_G(a, b)$ for every pair $\{a, b\}$ of distinct elements of A. Note that, *if G is a partial cube and if* $|A| \geq 3$, *then* $|M_G(A)| \leq 1$ because, if $m \in M_G(A)$, then m is the median of any triple of elements of A, and thus is unique.

Theorem 3.4 (*Fundamental Theorem*). *Let $A \subseteq V(G)$ and $m \in V(G)$. The following assertions are equivalent:*

(i) *m is a limit point of A (that is m belongs to the closure of $A - \{m\}$).*
(ii) *$A \cap \bigcap_{0 \le i \le n} H_i \neq \emptyset$ for every finite family $(H_i)_{0 \le i \le n}$ of half-spaces of G containing m.*
(iii) *$A \cap \bigcap_{0 \le i \le n} H_i$ is infinite for every finite family $(H_i)_{0 \le i \le n}$ of half-spaces of G containing m.*
(iv) *m is a limit point of some cluster contained in A.*
(v) *There exists a cluster $B \subseteq A$ such that, for each $w \in V(G)$, there is a cofinite subset C of B such that $m \in I_G(c, w)$ for every $c \in C$.*
(vi) *$m \in M_G(B)$ for some infinite subset B of A.*

Proof. (i) ⇒ (ii): Suppose that m does not satisfy (ii). Then there exists a finite family $(H_i)_{0 \le i \le n}$ of half-spaces of G containing m such that $A \cap \bigcap_{0 \le i \le n} H_i = \emptyset$. Then $(V(G) - H_i)_{0 \le i \le n}$ is a finite family of convex sets whose union contains A but not m. Because the weak geodesic convexity is generated by the set of all convex sets, it follows that m does not belong to the closure of $A - \{m\}$, and thus m is not a limit point of A.

(ii) ⇒ (iii): Suppose that, for some finite family $(H_i)_{0 \le i \le n}$ of half-spaces of G containing m, the set $A \cap \bigcap_{0 \le i \le n} H_i$ is finite, say $A \cap \bigcap_{0 \le i \le n} H_i = \{a_1, \ldots, a_p\}$. Then because any two vertices of a partial cube can be separated by a half-space, it follows that, for $1 \le j \le p$, there exists a half-space H_{n+j} such that $m \in H_{n+j}$ and $a_j \notin H_{n+j}$. Therefore $A \cap \bigcap_{0 \le i \le n+p} H_i = \emptyset$.

(iii) ⇒ (iv): Assume that m satisfies (iii). Then, with almost the same construction as in the proof of Proposition 3.3, we can construct a sequence $A_0, A_1, \ldots$ of infinite subsets of A, and a sequence $x_0, x_1, \ldots$ of elements of A such that $x_n \in A_n$ and $A_{n+1} := A_n \cap W_{a_n b_n}$ or $A_{n+1} := A_n \cap W_{b_n a_n}$ for all $n \in \mathbb{N}$ according to whether m belongs to $W_{a_n b_n}$ or to $W_{b_n a_n}$. This is always possible since m satisfies condition (iii).

By the proof of Proposition 3.3, the set $X := \{x_n : n \in \mathbb{N}\}$ is a cluster with the additional property: for any edge $a_n b_n$, $X \cap W_{a_n b_n}$ (resp. $X \cap W_{b_n a_n}$) is finite if and only if $m \in W_{b_n a_n}$ (resp. $m \in W_{a_n b_n}$), which proves that m is a limit point of X.

(iv) ⇒ (v): Assume that m is a limit point of some cluster contained in A. Without loss of generality, we will suppose that A itself is a cluster. Let $w \in V(G)$, and let $\langle w_0, \ldots, w_n \rangle$ be a (w, m)-geodesic with $w_0 = w$ and $w_n = m$. We will prove by induction that $w_i \in I_G(w, b)$ for all $b \in B_i$, where B_i is a cofinite subset of A (and thus a cluster) such that m is one of its limit points. This is obvious for $i = 0$ with $B_0 := A$. Let i be such that $0 \le i < n$. Suppose that $w_j \in I_G(w, b)$ for all $j \le i$ and $b \in B_j$, where B_j is a cofinite subset of A such that m is one of its limit points. Clearly $w_0 \in W_{w_i w_{i+1}}$ and $m \in W_{w_{i+1} w_i}$. Because m is a limit point of B_i and because B_i is a cluster by the induction hypothesis, it follows that $B_{i+1} := B_i \cap W_{w_{i+1} w_i}$ is a cofinite cluster of B_i, and thus of A. Moreover, because w_{i+1} is the neighbor of w_i in $U_{w_{i+1} w_i}$, it follows that $w_{i+1} \in I_G(w_i, b)$ for every $b \in B_{i+1}$. Hence $w_{i+1} \in I_G(w, b)$ for every $b \in B_{i+1}$ by the induction hypothesis. Finally $m = w_n \in I_G(w, b)$ for every $b \in B_n$.

(v) ⇒ (vi): Suppose that m satisfies (v). As above, without loss of generality, we can assume that A is a cluster. From (v) we infer that, for every infinite $B \subseteq A$ and any $a \in B$, there exist infinitely many $b \in B$ such that $m \in I_G(a, b)$. Hence any infinite $B \subseteq A$ contains a pair $\{a, b\}$ of distinct elements such that $m \in I_G(a, b)$. It follows that, by Ramsey's theorem, A contains an infinite subset B such that $m \in I_G(a, b)$ for every pair $\{a, b\}$ of distinct elements of B, that is such that $m \in M_G(B)$.

(vi) ⇒ (i): Assume that $m \in M_G(B)$ for some infinite $B \subseteq A$. Let $(C_i)_{1 \le i \le n}$ be a finite family of convex sets whose union contains $A - \{m\}$. Since B is infinite, there are two elements b and b' of B which belongs to some C_i. Hence $I_G(b, b') \subseteq C_i$ by the convexity of C_i, and thus $m \in C_i$. Therefore m belongs to the intersection of every finite union of convex sets which contains $A - \{m\}$, and thus belongs to the closure of $A - \{m\}$. Hence m is a limit point of A. □

Note that the set B in the statement of condition (vi) can be chosen to be a cluster. Clearly any vertex of infinite degree is a limit point of its neighborhood. Also, in the graph $S^+(K_{\aleph_0})$, the vertex $x_{\aleph_0}$ is a limit point of the cluster $\{x_n : n \in \mathbb{N}\}$, and more precisely $x_{\aleph_0} \in M_{S^+(K_{\aleph_0})}(\{x_n : n \in \mathbb{N}\})$. In the following proposition we list some particular properties of a limit point of a cluster that we will use frequently.

Proposition 3.5. *Let A be a cluster of G. We have the following properties:*

(i) *A has at most one limit point.*
(ii) *A vertex m is a limit point of A if and only if $A \cap H$ is infinite for every half-space H of G containing m.*
(iii) *If a vertex m is a limit point of A, then m is a limit point of any infinite subset of A.*
(iv) *If a vertex m is a limit point of A, then, for each $w \in V(G)$, there exists a cofinite subset B of A such that $m \in I_G(b, w)$ for every $b \in B$.*

Proof. (i) Suppose that A has two limit points u and v. Then there exists a half-space H of G such that $u \in H$ and $b \in V(G) - H$. Then, because u and v are limit points of A, it follows that $A \cap H$ and $A \cap (V(G) - H)$ are infinite. Therefore A is not a cluster.

(ii) The necessity is clear because of the implication (i) ⇒ (iii) of the Fundamental Theorem. Conversely, suppose that $A \cap H$ is infinite for every half-space H of G containing m. Because A is a cluster, it follows that $A \cap H$ is a cofinite subset of A. Let $(H_i)_{0 \le i \le n}$ be a finite family of half-spaces of G containing m. We infer that $A \cap \bigcap_{0 \le i \le n} H_i = \bigcap_{0 \le i \le n} (A \cap H_i)$ is a cofinite subset of A since so is $A \cap H_i$ for each i, and thus is infinite. Hence m is a limit point of A by the Fundamental Theorem.

(iii) is a consequence of (ii) and of the definition of a cluster. (iv) follows from part (iv) ⇒ (v) of the proof of the Fundamental Theorem. □

Note that if two clusters have a common limit point, then their union is also a cluster.

3.2. Compactness

We will in particular see that the compactness of the weak geodesic space is linked to the lack of special rays.

Definition 3.6. We say that a ray R of G is *divergent* if $V(R)$ has no limit point.

Note that if $R = \langle x_0, x_1, \ldots \rangle$ is a ray of G, then the following assertions are clearly equivalent: (i) R is divergent; (ii) any subsequence of $(x_n)_{n\in\mathbb{N}}$ is divergent in the usual topological sense; (iii) the vertex set of any subray of R is a closed set. For example any ray of the graph $S(K_{\aleph_0})$ is divergent, whereas no ray of $S^+(K_{\aleph_0})$ is divergent because the vertex set of any ray of this graph contains an infinite subset of the cluster $\{x_n : n \in \mathbb{N}\}$, and $x_{\aleph_0}$ is a limit point of this cluster, as we saw above, and thus of any of its infinite subset by Proposition 3.5. We will also see that any isometric ray of a partial cube is divergent (Proposition 3.14).

We now recall some concepts which will be useful. A set A of vertices of a graph G is said to be *fragmented* if its elements are pairwise separated in G by a finite $S \subseteq V(G)$, i.e. any two distinct elements of A belong to distinct components of $G - S$.

Lemma 3.7. *Any infinite fragmented set of G has a limit point.*

Proof. Let A be an infinite fragmented subset of $V(G)$, and let S be a finite subset of $V(G)$ which pairwise separates the elements of A. Because S is finite, there is an $s \in S$ and an infinite $B \subseteq A$ such that $I_G(s, b) - \{s\}$ is contained in the component of $G-S$ containing b, for every $b \in B$. Therefore $s \in M_G(B)$, and hence s is a limit point of A, by the Fundamental Theorem. □

An infinite subset A of $V(G)$ is said to be *concentrated* in G if any two infinite subsets of A cannot be separated by removing finitely many vertices.

For example the vertex set of any ray of a graph G is concentrated. Moreover any infinite subset of a concentrated set is also concentrated. Note that clusters and concentrated sets are unrelated concepts. There are clusters which are not concentrated, the vertex set of $K_{1,\aleph_0}$ for example; and on the other hand, there are concentrated sets which are not clusters, as is shown by the vertex set of a one-way infinite ladder (i.e. the Cartesian product of a ray with K_2).

Proposition 3.8 (*Polat [8, Theorem 3.8]*). *Every infinite set of vertices of a graph contains an infinite subset which is fragmented or concentrated.*

Theorem 3.9 (*Compactness Theorem*). *Let G be a partial cube. The following assertions are equivalent:*

(i) *$V(G)$ is compact.*
(ii) *$V(G)$ is weakly countably compact (i.e. every infinite subset of $V(G)$ has a limit point).*
(iii) *Every cluster of G has a limit point.*
(iv) *Every concentrated set of vertices of G has a limit point.*
(v) *Every concentrated cluster of G has a limit point.*
(vi) *G contains no divergent rays.*
(vii) *Every concentrated set of vertices of G has a finite positive number of limit points.*

Proof. Condition (i) obviously implies conditions (ii)–(vi); the implication (vii) ⇒ (iv) is also obvious. The implication (iii) ⇒ (ii) is a consequence of Proposition 3.3. We will prove the following implications: (vi) ⇒ (v) ⇒ (iv) ⇒ (ii) ⇒ (i). The proof will be completed later by observing that the implication (i) ⇒ (vii) is an immediate consequence of Proposition 3.18 that we will see below.

(ii) ⇒ (i): Assume that $V(G)$ is not compact. Then there exists a nested sequence $(A_\beta)_{\beta<\alpha}$, where α is a limit ordinal, of non-empty close subsets of $V(G)$ such that $\bigcap_{\beta<\alpha} A_\beta = \emptyset$. For any finite $F \subseteq V(G)$, let $\beta(F) := \min\{\beta : F \cap A_\beta = \emptyset\}$. Clearly $\beta(F) < \alpha$ since $\bigcap_{\beta<\alpha} A_\beta = \emptyset$.

We construct inductively a sequence $x_0, x_1, \ldots$ of pairwise distinct vertices of G, and a strictly increasing sequence $\beta_0, \beta_1, \ldots$ of ordinals less than α such that $\mathrm{co}_G(x_0, \ldots, x_{n+1}) \cap A_{\beta_{n+1}} = \emptyset$, in the following way. Let x_0 be any element of A_0 and $\beta_0 := \beta(x_0)$. Suppose that $x_0, \ldots, x_n$ and $\beta_0, \ldots, \beta_n$ have already been constructed for some non-negative integer n. Let x_{n+1} be any element of A_{β_n} and $\beta_{n+1} := \beta(\mathrm{co}_G((x_0, \ldots, x_{n+1})))$ (recall that any polytope in a partial cube is finite). Clearly $\beta_n < \beta_{n+1}$, and thus $\beta_0 < \beta_1 < \cdots < \beta_{n+1}$ by the induction hypothesis.

The set $X := \{x_0, x_1, \ldots\}$ is countably infinite. Suppose that X has a limit point m. Then, by the Fundamental Theorem, $m \in M_G(B)$ for some infinite $B \subseteq X$. Hence $m \in I_G(x_i, x_j)$ for some $i < j$, and thus $m \in \mathrm{co}_G(x_0, \ldots, x_j)$. Therefore $m \notin A_{\beta_{j+1}}$ by the construction. On the other hand, m is a limit point of the set $\{x_{j+1}, x_{j+2}, \ldots\}$, which yields a contradiction.

Consequently G is not weakly countably compact.

(iv) ⇒ (ii): Suppose that we have (iv). Let A be an infinite set of vertices of G. Then, by Proposition 3.8, A contains an infinite subset B which is either fragmented or concentrated. It follows, either by Lemma 3.7 or by (iv) according to whether B is fragmented or concentrated, that B, and thus A, has a limit point.

(v) ⇒ (iv): Assume that (v) is satisfied, and let A be a concentrated set of vertices of G. By Proposition 3.3, A contains a cluster, which is evidently concentrated, and thus which has a limit point m by (v). Hence m is a limit point of A.

(vi) ⇒ (v): Let A be a concentrated cluster of G, and $(a_n)_{n\in\mathbb{N}}$ a sequence of pairwise distinct elements of A. The set $A' := \bigcup_{n\in\mathbb{N}} I_G(a_n, a_{n+1})$ is clearly a cluster.

We claim that it is also concentrated. Let S be a finite subset of $V(G)$, that we can suppose to be convex since any polytope is finite. Because A is concentrated, there is an infinite component X of $G-S$ which contains a cofinite subset B of A. Because S is convex, it follows that $I_G(a,b) \subseteq B \cup S$ for all $a, b \in B$. Hence $X \cap A'$ is cofinite in A'. Therefore A' is concentrated.

It follows that $G[A']$, which is a connected infinite graph, contains a ray R. Suppose that we have (vi). Then $V(R)$ has a limit point, say m. Let uv be an edge of G. Then W_{uv} or W_{vu}, say W_{uv}, contains a cofinite subset of A' because A' is a cluster. It follows on the one hand, that $A \cap W_{uv}$ is a cofinite subset of A, and on the other hand, that there is a subray of R whose vertex set is contained in W_{uv}, which implies that $m \in W_{uv}$. Consequently m is a limit point of A by Proposition 3.5. □

For example the vertex set of the graph $S^+(K_{\aleph_0})$ is compact since it contains no divergent rays as we saw above, whereas the vertex set of $S(K_{\aleph_0})$ is not compact. Moreover, from the equivalence of the conditions (i) and (vi) in the Fundamental Theorem, we obtain:

Corollary 3.10. *The vertex set of any rayless partial cube is compact.*

Proposition 3.11. *If $V(G)$ is compact, then an infinite $A \subseteq V(G)$ is a cluster if and only if A has exactly one limit point.*

Proof. The necessity is Proposition 3.5. Conversely suppose that A is not a cluster. Then there exists $ab \in E(G)$ such that $A_1 := A \cap U_{ab}$ and $A_2 := A \cap U_{ba}$ are infinite. Because $V(G)$ is compact, A_i has a limit point w_i for $i = 1, 2$. Then $w_1 \in U_{ab}$ and $w_2 \in U_{ba}$. It follows that $w_1 \neq w_2$. Therefore A has at least w_1 and w_2 as limit points. □

A vertex x of a connected graph G *geodesically dominates* a subset A of $V(G)$ if, for every finite $S \subseteq V(G-x)$, there exists an $a \in (A - \{x\})$ such that $S \cap I_G(x, a) = \emptyset$.

Proposition 3.12 (*Polat [9, Theorem 3.9]*)**.** *Let G be a graph. The following assertions are equivalent:*

(i) *G contains no isometric rays.*
(ii) *The vertex set of every ray of G is geodesically dominated.*
(iii) *Every concentrated set of G is geodesically dominated.*
(iv) *Every infinite subset of $V(G)$ is geodesically dominated.*

Lemma 3.13. *Any limit point of a subset A of $V(G)$ geodesically dominates A.*

Proof. Let m be a limit point of A. By the Fundamental Theorem, $m \in M_G(B)$ for some infinite $B \subseteq A$. Let S be a finite subset of $V(G-m)$. Because m is the only element of $M_G(C)$ for every infinite $C \subseteq B$, it follows that $S \cap I_G(a,b) = \emptyset$ for infinitely many $a, b \in B$. Therefore $S \cap I_G(m,x) = \emptyset$ for infinitely many $x \in B$, and thus m geodesically dominates A. □

On the other hand, a vertex x which geodesically dominates a set A is not necessarily a limit point of A, even if x is always a limit point of some set, in particular of its neighborhood, as is shown by the following example. The graph $S(K_{\aleph_0})$ has only one end, and contains no isometric rays because its diameter is 3. In this graph, the set $V(K_{\aleph_0})$ is geodesically dominated by any of its elements, but none of these vertices is a limit point of $V(K_{\aleph_0})$.

Proposition 3.14. *Any isometric ray of G is divergent.*

Proof. Let R be an isometric ray of G. By [9, Lemma 3.7], $V(R)$ is not geodesically dominated. Hence $V(R)$ has no limit point by Lemma 3.13, that is R is divergent. □

It follows from the Compactness Theorem and Proposition 3.14 that:

Corollary 3.15. *If the space $V(G)$ is compact, then G contains no isometric rays.*

Note that the converse is false. For example the graph $S(K_{\aleph_0})$ above has no isometric ray, but has divergent rays. Hence the vertex set of this graph is not compact.

Any isometric ray of G is then a divergent ray whose vertex set is a cluster. The converse is not true in general as is shown with the rays of $S(K_{\aleph_0})$. Moreover the vertex set of a divergent ray is generally not a cluster. Take for example an infinite ladder L, i.e. the Cartesian product of a ray with K_2, and let R be a ray of L containing infinitely many rungs of L. Then R, as any ray of L is divergent, but its vertex set is not a cluster since R contains infinitely many edges which are Θ-equivalent. On the other hand, as we will show, any divergent ray is linked to some divergent ray whose vertex set is a cluster.

We recall that the *ends* of a graph G are the classes of the equivalence relation defined on the set of all rays of G as follows: two rays R and R' are said to be *end-equivalent* if and only if there is a ray R'' whose intersections with R and R' are infinite, or equivalently if and only if $V(R)$ and $V(R')$ are infinitely linked in G (i.e. there is an infinite family of pairwise disjoint paths which join $V(R)$ and $V(R')$).

Proposition 3.16. *Any divergent ray of G is end-equivalent to a divergent ray whose vertex set is a cluster.*

Proof. Let R be a divergent ray of G. By Proposition 3.3, $V(R)$ contains a cluster A. If $R = \langle x_0, x_1, \ldots\rangle$, then there exists a strictly increasing sequence $(i_n)_{n\in\mathbb{N}}$ such that $A = \{x_{i_n} : n \in \mathbb{N}\}$. For each $n \in \mathbb{N}$, let P_n be an $(x_{i_n}, x_{i_{n+1}})$-geodesic in G, and let $X := \bigcup_{n\in\mathbb{N}} P_n$. Then X is an infinite connected subgraph of G.

We claim that $V(X)$ is a cluster. Let S be a half-space of G. Because A is a cluster, the intersection of A with S or $V(G) - S$, say with S, is infinite. Then there is a non-negative integer p such that $x_{i_n} \in S$ for all $n \geq p$. It follows that $V(P_n) \subseteq S$ for all $n \geq p$ since S is convex. Hence $V(X)$ is a cluster.

X is locally finite since otherwise, a vertex of infinite degree of X would be a limit point of its neighborhood in X, and thus a limit point of X and thus of $V(R)$, contrary to the fact that $V(R)$ has no limit point since it is divergent. Therefore no subset of X has a limit point, and moreover X contains a ray, say R'. Then R' is divergent since its vertex set has no limit point, and moreover $V(R')$ is a cluster because it is an infinite subset of the cluster X.

Note that R' meets infinitely many P_n. Then, because X is locally finite, it follows that there exists an infinite family of pairwise disjoint paths which join $V(R)$ and $V(R')$. This proves that R and R' are end-equivalent. Hence R' has the required properties. □

Lemma 3.17. *Let A be a cluster of $V(G)$, and w a limit point of A. Then w is the only vertex which geodesically dominates A.*

Proof. Let u be a vertex of G distinct from w, and let $v \in N_G(u) \cap I_G(u, w)$. Then $w \in W_{vu}$, and thus $A \cap U_{uv}$ is finite since A is a cluster and w is a limit point of A. Moreover $v \in I_G(u, x)$ for every $x \in W_{vu}$ and in particular for every $x \in A \cap U_{vu}$. Therefore u does not geodesically dominate A by the definition of the geodesic domination and the fact that $A \cap U_{vu}$ is cofinite in A. □

We recall (see [8, Theorem 3.3]) that an infinite subset A of $V(G)$ is concentrated in G if and only if there exists an end ε such that, for every finite $F \subseteq V(G)$, there are only finitely many elements of A which are not vertices of the unique component of $G - F$ that contains an element of ε. The set A is then said to be *concentrated in* ε.

Proposition 3.18. *Let ε be an end of a partial cube G. If every subset of $V(G)$ which is concentrated in ε has a limit point, then each of these subsets has only finitely many limit points.*

Proof. Assume that every subset of $V(G)$ which is concentrated in ε has a limit point. Let X be a subset of $V(G)$ which is concentrated in ε. Then X has a limit point by the assumption. Suppose that it has infinitely many limit points. Let $d_0, d_1, \ldots$ be a sequence of distinct limit points of X. By Proposition 3.5 and the Fundamental Theorem there exists a family $(A_n)_{n\in\mathbb{N}}$ of pairwise disjoint clusters in X such that $d_n \in M_G(A_n)$ for every $n \in \mathbb{N}$.

(a) Let $m \in V(G)$. We claim that there are a finite $S \subseteq V(G - m)$ and an infinite $N \subseteq \mathbb{N}$ such that $S \cap I_G(m, x)$ is non-empty for every vertex $x \in \bigcup_{n\in N} A_n$.
Suppose that this is not true. Then, for all finite $S \subseteq V(G - m)$ and almost all $n \in \mathbb{N}$ (i.e. all n except finitely many ones), $S \cap I_G(m, x) = \emptyset$ for some $x \in A_n$. It follows that m geodesically dominates A_n for almost all $n \in \mathbb{N}$. By Lemma 3.17, this implies that $m = d_n$ for almost all $n \in \mathbb{N}$, which is evidently impossible.

(b) We construct sequences $m_0, m_1, \ldots, S_0, S_1, \ldots, N_0, N_1, \ldots, G_0, G_1, \ldots$ and $A_n^0, A_n^1, \ldots$ for each $n \in \mathbb{N}$ such that:
- $m_i \in S_i$;
- N_{i+1} is an infinite subset of N_i;
- G_{i+1} is a component of $G_i - S_i$ such that $X \cap V(G_{i+1})$ is infinite;
- A_n^{i+1} is an infinite subset of $A_n^i \cap V(G_{i+1})$ for each $n \in N_{i+1}$;
- S_{i+1} is a finite subset of $V(G_i - m_i)$ such that $S_{i+1} \cap I_{G_i}(m_i, x) \neq \emptyset$ for every $x \in \bigcup_{n\in N_{i+1}} A_n^{i+1}$;
- $m_{i+1} \in I_G(m_j, x)$ for all $x \in \bigcup_{n\in N_{i+1}} A_n^{i+1}$ and j with $0 \leq j \leq i$.

Let m_0 be a vertex of G. Put $N_0 := \mathbb{N}, S_0 := V(G), G_0 := G$ and $A_n^0 := A_n$ for each $n \in \mathbb{N}$. Suppose that $m_0, \ldots, m_i, S_0, \ldots, S_i, N_0, \ldots, N_i, G_0, \ldots, G_i$ and $A_n^0, \ldots, A_n^i$, $n \in \mathbb{N}$, have already been constructed for some $i \geq 0$. The set $\bigcup_{n\in N_i} A_n^i$ is an infinite subset of $X \cap V(G_i)$, and thus it is concentrated since $\bigcup_{0\leq j\leq i} S_j$ is finite and X is concentrated. Therefore, by (a), there exist a finite $S_{i+1} \subseteq V(G_i - m_i)$ and an infinite subset N_i' of N_i such that $S_{i+1} \cap I_{G_i}(m_i, x) \neq \emptyset$ for every vertex $x \in \bigcup_{n\in N_i'} A_n^i$. Since S_{i+1} is finite and $\bigcup_{n\in N_i'} A_n^i$ is concentrated in G_i, there is a component G_{i+1} of $G_i - S_i$, a vertex $m_{i+1} \in S_{i+1}$, an infinite $N_{i+1} \subseteq N_i'$, and for each $n \in N_{i+1}$ an infinite $A_n^{i+1} \subseteq A_n^i \cap V(G_{i+1})$ such that $m_{i+1} \in I_{G_{i+1}}(m_i, x)$ for every $x \in \bigcup_{n\in N_{i+1}} A_n^{i+1}$. By the induction hypothesis, $m_i \in I_G(m_j, x)$ for all $x \in \bigcup_{n\in N_i} A_n^i$ and each j with $0 \leq j < i$. It follows that $m_{i+1} \in I_G(m_j, x)$ for all $x \in \bigcup_{n\in N_{i+1}} A_n^{i+1}$ and each j with $0 \leq j \leq i$.

(c) By the construction, if P_n is an (m_n, m_{n+1})-geodesic for each $n \in \mathbb{N}$, then $R = \bigcup_{n\in\mathbb{N}} P_n$ is an isometric ray of G. Suppose that the set $M := \{m_n : n \in \mathbb{N}\}$ is not concentrated in ε. Then, because X is concentrated in ε, there exists a finite $F \subseteq V(G)$ which separates a cofinite subset X' of X from an infinite subset M' of M. Without loss of generality we can suppose that $X' = X$. Put $M' = \{m_n : n \in P\}$ for some infinite $P \subseteq \mathbb{N}$. Note that we have in particular $F \cap I_G(m_n, x) \neq \emptyset$ for all $n \in P$ and $x \in X$.

Let $j \in P$. By the construction (part (b)), for every $i \geq j$, $m_{i+1} \in I_G(m_j, x)$ for all $x \in \bigcup_{n\in N_{i+1}} A_n^{i+1} \subseteq X$. Hence there are infinitely many $i \geq j$ such that $i + 1 \in P$ and $m_{i+1} \in \bigcup_{u\in F} I_G(m_j, u)$. This yields a contradiction with the fact that F and every interval are finite.

Therefore M is concentrated in ε. It follows that M, and thus $V(R)$, has a limit point by the assumption, contrary to the fact that R is isometric and thus divergent by Proposition 3.14. Consequently the set X has only finitely many limit points. □

From this result we immediately infer that: if $V(G)$ is compact, then the set of all limit points of any concentrated subset of $V(G)$ is non-empty and finite. This proves the implication (i) ⇒ (vii) of the Compactness Theorem, and so completes the proof of this theorem.

Proposition 3.19. *If the space $V(G)$ is compact, then it is scattered (i.e. $V(G)$ contains no non-empty subset A that is dense in itself, that is, such that every element of A is a limit point of A).*

Proof. Assume that $V(G)$ is compact. We have to prove that any infinite set A of vertices of G has an isolated point, that is a vertex $x \in A$ which is not a limit point of A. Suppose that there is an infinite subset A of vertices of G which has no isolated point. Then there exists a closed set A' which is dense in itself, i.e., that is perfect, and which contains A. Without loss of generality we will suppose that $A = A'$. We will construct a sequence of vertices $x_0, x_1, \ldots$ and a sequence of paths $P_0, P_1, \ldots$ such that, for every $n \geq 0$, $x_n \in A$, P_{n+1} is an (x_{n-1}, x_n)-geodesic and $P_0 \cup \ldots \cup P_n$ is an (x_0, x_n)-geodesic.

Let x_0 be any element of A, and let $P_0 := \langle x_0 \rangle$. Suppose that $x_0, \ldots, x_n$ and $P_0, \ldots, P_n$ have already been constructed for some $n \geq 0$. Since x_n is a limit point of A because A is perfect, by the Fundamental Theorem there exists an $x_{n+1} \in A - \{x_n\}$ such that $x_n \in I_G(x_0, x_{n+1})$. Let P_{n+1} be an (x_n, x_{n+1})-geodesic of G. Then $P_0 \cup \ldots \cup P_{n+1}$ is an (x_0, x_{n+1})-geodesic since, by the induction hypothesis, $P_0 \cup \ldots \cup P_n$ is an (x_0, x_n)-geodesic.

Therefore $\bigcup_{n \geq 0} P_n$ is an isometric ray of G, contrary to Corollary 3.15 and the assumption that $V(G)$ is compact. Hence $V(G)$ is scattered. □

If G is a partial cube whose vertex space is compact, then it follows from the Compactness Theorem that $V(G)$ has only finitely many limit points if it is concentrated. This is generally not true if $V(G)$ is not concentrated. Take for example a rayless tree with infinitely many vertices of infinite degree. However, as a consequence of Proposition 3.19 (see [16, Proposition 4.4 and Theorem 4.6]), there always exists a non-empty finite set of special limit points.

3.3. Compactness with respect to subgraphs and operations of partial cubes

The class of partial cubes is closed under faithful subgraphs, retracts, Cartesian products and gated amalgams. In this subsection we will show that compactness is preserved by these particular subgraphs and operations. First we recall some definitions.

Let G be a partial cube. We say that a subgraph H of G is *median-stable* if, for any triple (x, y, z) of vertices of H, if (x, y, z) has a median m in G, then $m \in V(H)$. Note that, if H is isometric in G, then m is the median of (x, y, z) in H. A median-stable isometric subgraph of G is called a *faithful subgraph* of G, or is said to be *faithful* in G. Clearly any convex subgraph of G is faithful.

If G and H are two graphs, then a map $f : V(G) \to V(H)$ is a *contraction* (*weak homomorphism* in [4]) if it is a non-expansive map between the metric spaces $(V(G), d_G)$ and $(V(H), d_H)$, i.e. $d_H(f(x), f(y)) \leq d_G(x, y)$ for all $x, y \in V(G)$. A contraction f of G onto one of its induced subgraphs H is a *retraction*, and H is a *retract* (*weak retract* in [4]) of G, if its restriction to $V(H)$ is the identity. Any retract of a partial cube is clearly a faithful subgraph of this graph.

An induced subgraph H (or its vertex set) of a graph G is said to be *gated* if, for each $x \in V(G)$, there exists a vertex y (the *gate* of x) in H such that $y \in I_G(x, z)$ for every $z \in V(H)$. A graph G is the *gated amalgam* of two graphs G_0 and G_1 if G_0 and G_1 are isomorphic to two intersecting gated subgraphs of G whose union is G.

Theorem 3.20. *Let G and H be two partial cubes. We have the following properties:*

(i) *The space $V(G \Box H)$ is compact if and only if so are the spaces $V(G)$ and $V(H)$.*
(ii) *If H is a faithful subgraph of G, then the space $V(H)$ is compact if so is the space $V(G)$.*
(iii) *If H is a retract of G, then the space $V(H)$ is compact if so is the space $V(G)$.*
(iv) *Let J be the gated amalgam of G and H. Then the space $V(J)$ is compact if and only if so are the spaces $V(G)$ and $V(H)$.*

Proof. We will only prove (i). The proofs of (ii) and (iv) are straightforward and are left to the reader. (iii) is an immediate consequence of (ii).

(i) Assume that the spaces $V(G)$ and $V(H)$ are compact. Denote by pr_G and pr_H the projections of $G \Box H$ onto G and H, respectively. Let A be an infinite subset of $V(G \Box H)$. Then at least one of the sets $\mathrm{pr}_G(A)$ and $\mathrm{pr}_H(A)$ is infinite. Suppose that $\mathrm{pr}_G(A)$ is infinite. Because $V(G)$ is compact, there exist by the Compactness Theorem and the Fundamental Theorem a vertex u and an infinite $B \subseteq \mathrm{pr}_G(A)$ such that $u \in M_G(B)$. Let A' be the subset of A such that $\mathrm{pr}_G(A') = B$. We distinguish two cases.

Case 1. $\mathrm{pr}_H(A')$ is finite.

Then there exist $v \in \mathrm{pr}_H(A')$ and an infinite $A'' \subseteq A'$ such that $\mathrm{pr}_H(A'') = \{v\}$. It follows that $(u, v) \in M_{G \Box H}(A'')$, and thus (u, v) is a limit point of A.

Case 2. $\mathrm{pr}_H(A')$ is infinite.

Because $V(H)$ is compact, there exist by the Compactness Theorem a vertex v and an infinite $C \subseteq \mathrm{pr}_H(A')$ such that $v \in M_H(C)$. Let $A_C \subseteq A'$ be such that $\mathrm{pr}_H(A_C) = C$. By the properties of the distance in a Cartesian product, it follows that $(u, v) \in M_{G \Box H}(A_C)$, and thus (u, v) is a limit point of A.

Consequently the space $V(G \Box H)$ is compact.

Conversely, assume that the space $V(G \Box H)$ is compact. Clearly each G-fiber and each H-fiber of $G \Box H$ are convex subgraphs of $G \Box H$. Hence their vertex spaces are compact by (iii) since $V(G \Box H)$ is compact by assumption. It follows that the spaces $V(G)$ and $V(H)$ are compact as well. □

Note that if H is a convex subgraph of a partial cube G such that the space $V(G)$ is compact, then $V(H)$ is also compact by (ii). However this property is also an immediate consequence of the facts that the subspace $V(H)$ of $V(G)$ is compact because it is closed, and that the induced topology on $V(H)$ coincides with the weak geodesic topology on this set.

3.4. Separation properties

Because the geodesic convexity on $V(G)$ has the separation property S_3 and since every convex set is closed for the weak geodesic topology, it follows that the weak geodesic space $V(G)$ is Hausdorff and has the neighborhood separation property NS_3: if a vertex x does not belong to a convex closed set C, then there is a convex closed neighborhood N of C with $x \notin N$. Therefore, by [19, Proposition 4.7(1)], *the weak geodesic space $V(G)$ is regular*. Furthermore, by [19, Proposition 4.5(3)], if the space $V(G)$ is compact, then it has the neighborhood separation property NS_4: for each pair C, D of disjoint convex closed sets, there is a convex closed neighborhood N of C with $N \cap D = \emptyset$.

As it was mentioned by Tardif in [18], the weak geodesic topology on the vertex set of a median graph G can be shown to be normal. However this does not hold if G is any partial cube, as is shown by the following example. Take the partial cube $S(K_{\aleph_0})$, and let A and B be the convex hull in this graph of two complementary infinite subsets A' and B' of $V(K_{\aleph_0})$. Then A and B are disjoint closed sets of $V(S(K_{\aleph_0}))$ such that $V(S(K_{\aleph_0})) - A \cup B$ is infinite. Moreover, because A' and B' are complementary infinite sets and since the set of convex sets is a subbase, it follows that, for any closed set F containing A (resp. B) and disjoint from B (resp. A), there exist only finitely many n (resp. p) such that $x_n \in A'$, $x_p \in B'$ and $x_{np} \in F$. Therefore, if O_A and O_B are open sets which contain A and B, respectively, then there are infinitely many n, p such that $x_n \in A'$, $x_p \in B'$ and $x_{np} \in O_A \cap O_B$. Hence the weak geodesic topology on $V(S(K_{\aleph_0}))$ is not normal.

4. Geodesic topology

4.1. Geodesically consistent partial cubes

In [9] we endowed the vertex set of a graph G with the topology, called the *geodesic topology*, where a subset A of $V(G)$ is closed if and only if every vertex which geodesically dominates A belongs to A. In particular we proved the following result which completes Proposition 3.12.

Proposition 4.1 (*Polat [9, Theorem 3.9]*). *Let G be a graph. The geodesic space $V(G)$ is compact if and only if G contains no isometric rays.*

The geodesic topology is compatible with the geodesic convexity, that is all polytopes are geodesically closed (i.e. closed for the geodesic topology). Moreover it is coarser than the weak geodesic topology by Lemma 3.13, but it is generally distinct from it. For example take the subdivision $S(K_{\aleph_0})$ of $K_{\aleph_0}$, and let $a \in V(K_{\aleph_0})$. Then $A := V(S(K_{\aleph_0})) - N_{S(K_{\aleph_0})}[a]$ is convex, but it is not geodesically closed since a geodesically dominates A, or in other words a is a limit point of A for the geodesic topology but not for the weak geodesic topology.

More precisely the geodesic topology is generally not a weak topology, that is it is finer than the weak topology $\mathcal{T}_w$ generated by the geodesically closed convex sets. Indeed, in the graph $S(K_{\aleph_0})$, the set $V(K_{\aleph_0})$ is clearly geodesically closed. On the other hand, any finite family $(C_i)_{1 \le i \le n}$ of geodesically closed convex sets whose union contains $V(K_{\aleph_0})$ has an element, say C_i, which is infinite. Hence C_i contains $V(K_{\aleph_0})$ because any vertex in this set geodesically dominates C_i. It follows that $V(S(K_{\aleph_0})) \subseteq C_i$ because C_i is convex, and thus $V(S(K_{\aleph_0})) \subseteq \bigcup_{1 \le i \le n} C_i$. Therefore the closure of $V(K_{\aleph_0})$ for $\mathcal{T}_w$ is $V(S(K_{\aleph_0}))$, which proves that $V(K_{\aleph_0})$, which is geodesically closed, is not closed for $\mathcal{T}_w$.

Definition 4.2. A partial cube G is said to be *geodesically consistent* if the geodesic topology coincides with the weak geodesic topology.

In other words, G is geodesically consistent if the limit points of any set $A \subseteq V(G)$ are the vertices of G which geodesically dominate A. Recall that, by Lemma 3.13, any limit point of such a set A geodesically dominates A, but that the converse is not necessarily true. Before characterizing these graphs, we will give a few elementary properties, one of them (Corollary 4.6) completing the list of the properties equivalent to the compactness of the vertex set of a partial cube.

Lemma 4.3 (*Polat [10, Proposition 4.1]*). *Let G be an interval-finite graph. Then a vertex x of G geodesically dominates a subset A of $V(G)$ if and only if there exists an infinite subset B of A such that $I_G(x, a) \cap I_G(x, b) = \{x\}$ for every pair $\{a, b\}$ of distinct elements of B.*

Proposition 4.4. *If the vertex set of a partial cube G is compact, then G is geodesically consistent.*

Proof. We will show that any vertex which geodesically dominates an infinite subset of $V(G)$ is a limit point of this set. Let x be a vertex which geodesically dominates an infinite $A \subseteq V(G)$. By Lemma 4.3, there exists an infinite subset B of A such that $I_G(x, a) \cap I_G(x, b) = \{x\}$ for every pair $\{a, b\}$ of distinct elements of B. By Proposition 3.3, B contains a cluster C, which is clearly geodesically dominated by x. Because $V(G)$ is compact, this cluster C has a limit point, which is equal to x by Lemma 3.17. Therefore x is a limit point of A.

Consequently G is geodesically consistent. □

We immediately infer the following two corollaries.

Corollary 4.5. *Any rayless partial cube is geodesically consistent.*

Corollary 4.6. *A partial cube G contains no divergent rays if and only if G is geodesically consistent and contains no isometric rays.*

We have a stronger result.

Proposition 4.7. *Let ε be an end of a geodesically consistent partial cube. Then no divergent rays belong to ε if and only if no isometric rays belong to ε.*

Proof. The necessity is clear because any isometric ray is divergent. Conversely, suppose that no isometric rays belong to ε. Then, by [9, Theorem 3.8], any ray in ε is geodesically dominated. Hence, because G is geodesically consistent, the vertex set of any ray in ε has a limit point, and thus is not divergent. □

In order to characterize the partial cubes which are geodesically consistent we will say that a set A of vertices of a graph G is *almost geodesically closed* if every vertex in $\mathrm{co}_G(A)$ which geodesically dominates A belongs to A.

Theorem 4.8. *Let G be a partial cube. The following assertions are equivalent:*

(i) *G is geodesically consistent.*
(ii) *The sets U_{ab} and U_{ba} are geodesically closed for every edge ab of G.*
(iii) *The sets U_{ab} and U_{ba} are almost geodesically closed for every edge ab of G.*

Proof. (i) ⇒ (ii): Suppose that G is geodesically consistent, and let m be a vertex which geodesically dominates U_{ab} for some $ab \in E(G)$. Then $m \in W_{ab}$ because m is a limit point of U_{ab} by (i), and thus, by the implication (i) ⇒ (ii) of the Fundamental Theorem, it cannot be separated from U_{ab} by the half-space W_{ab}. Suppose that $m \notin U_{ab}$. Then m also geodesically dominates U_{ba}, but m is not a limit point of U_{ba} since $U_{ba} \subseteq W_{ba}$ and $m \in W_{ab}$. This yields a contradiction with (i). Hence $m \in U_{ab}$.

(ii) ⇒ (iii) is obvious.

(iii) ⇒ (i): Suppose that U_{ab} and U_{ba} are almost geodesically closed for every edge ab of G. Let A be an infinite subset of $V(G)$ and m a vertex which geodesically dominates A. Suppose that m is not a limit point of A (for the weak geodesic topology). By Lemma 4.3, there exists an infinite subset B of A such that $I_G(m, a) \cap I_G(m, b) = \{m\}$ for all $a, b \in B$. By Proposition 3.3, B contains a cluster, say C. Because m cannot be a limit point of C, it follows from Proposition 3.5(ii) that there is an edge ab of G such that $m \in W_{ab}$ and $C \cap W_{ab}$ is finite. Let K be a copoint at m containing W_{ba}. By Lemma 2.2, $K = W_{uv}$ for some $uv \in E(G)$, and $m \in \mathrm{co}_G(U_{vu})$. Moreover $D := C \cap W_{uv}$ is a cofinite subset of C. Hence m geodesically dominates D. It follows that m geodesically dominates U_{vu}, and thus $m \in U_{vu}$ since U_{vu} is almost geodesically closed. Let m' be the neighbor of m in U_{uv}. Then $m' \in I_G(m, d)$ for every $d \in D$, contrary to the fact that m geodesically dominates D. Consequently m is a limit point of A, and thus the geodesic topology coincides with the weak geodesic topology. □

The following three results show that the class of all geodesically consistent partial cubes is closed under Cartesian products, retracts and gated amalgams.

Theorem 4.9. *Let G and H be two partial cubes. Then $G \Box H$ is geodesically consistent if and only if so are G and H.*

Proof. Assumed that G and H are geodesically consistent. We denote by G_x, $x \in V(H)$, the G-fibers of $F := G \Box H$, and by H_y, $y \in V(G)$, its H-fibers. Any edge of F is either an edge $(a, x)(b, x)$ of G_x for some $x \in V(H)$, or an edge $(y, c)(y, d)$ of H_y for some $y \in V(G)$. Let $uv \in E(F)$. Without loss of generality, we will suppose that $uv = (a, x)(b, x)$ for some $ab \in E(G)$ and $x \in V(H)$.

Clearly an edge Θ-equivalent to uv is of the form $(c, y)(f, y)$, and moreover $(a, x)(b, x)\Theta(c, y)(f, y)$ if and only if $ab\Theta cf$. Hence

$$U^F_{uv} = U^G_{ab} \times V(H) = \bigcup_{y \in V(H)} U^{G_y}_{(a,y)(b,y)}. \tag{1}$$

Besides, by the properties of the Cartesian product,

$$(x, d) \in I_F((c, d), (x, y)) \quad \text{for all } (c, d), (x, y) \in V(F). \tag{2}$$

Let (c, d) be a vertex which geodesically dominates U^F_{uv}. We claim that (c, d) geodesically dominates $U^{G_d}_{(a,d)(b,d)}$ in G_d. Suppose that there is a finite $S \subseteq V(G_d - (c, d))$ such that $S \cap I_{G_d}((c, d), (x, d)) \neq \emptyset$ for every $(x, d) \in U^{G_d}_{(a,d)(b,d)}$. Then, because $(x, d) \in I_F((c, d), (x, y))$ for every $y \in V(H)$ by (2), it follows that $S \cap I_F((c, d), (x', y')) \neq \emptyset$ for every vertex $(x', y') \in U^F_{uv}$ by (1), contrary to the fact that (c, d) geodesically dominates U^F_{uv}. This proves the claim. If follows that $(c, d) \in U^{G_d}_{(a,d)(b,d)} \subseteq U^F_{uv}$ because U^G_{ab}, and thus $U^{G_d}_{(a,d)(b,d)}$, is geodesically closed. Hence U^F_{uv} is geodesically closed. Consequently F is geodesically consistent.

Conversely, if $G \Box H$ is geodesically consistent, then so are G_x and H_y for all $x \in V(H)$ and $y \in V(G)$. It follows that G and H are also geodesically consistent. □

Theorem 4.10. *The class of geodesically consistent partial cubes is closed under retracts.*

Proof. Let f be a retraction of a geodesically consistent partial cube G onto one of its subgraph H. Because f is non-expansive and H is isometric in G, we have:

$$f(I_G(x,y)) = I_H(x,y) \quad \text{for all } x, y \in V(H). \tag{3}$$

Let w be a vertex of H which geodesically dominates in H an infinite $A \subseteq V(H)$. Suppose that w does not geodesically dominate A in G. Then there is a finite $S \subseteq V(G-w)$ such that $S \cap I_G(w,a) \neq \emptyset$ for every $a \in A$. Hence, by (3), $f(S) \cap I_H(w,a) \neq \emptyset$ for every $a \in A$, contrary to the fact that w geodesically dominates A in H. Consequently w geodesically dominates A in G. It follows that w is a limit point of A in G since this graph is geodesically consistent. Hence, by the Fundamental Theorem, $w \in M_G(B)$ for some infinite $B \subseteq A$.

It follows by (3) that $f(w) \in M_H(B)$. Hence $w \in M_H(B)$ since $f(w) = w$ because H is a retract of G. Therefore w is a limit point of B, and thus of A, in H.

Consequently H is geodesically consistent. □

Theorem 4.11. *Let G be the gated amalgam of two geodesically consistent partial cubes G_0 and G_1. Then G is geodesically consistent if and only if so are G_0 and G_1.*

Proof. Assume that G_0 and G_1 are geodesically consistent. Recall that, by the definition of a gated amalgam, G_0 and G_1 can be considered as two intersecting gated (and thus convex) subgraphs of G. Let $A \subseteq V(G)$ and let w be a vertex which geodesically dominates A. Then there exists $i = 0$ or 1, say $i = 0$, such that $w \in V(G_i)$. Let w' be the gate of w in G_1. Then $w' \in I_G(w,x)$ for every $x \in V(G_1)$. Hence $A' := A - V(G_1)$ is infinite, and w geodesically dominates A' in G. It follows that w geodesically dominates A' in G_0 since G_0 is a convex subgraph of G. Hence w is a limit point of A' in G_0 because this graph is geodesically consistent. Then $w \in M_{G_0}(B)$ for some infinite $B \subseteq A'$ by the Fundamental Theorem, and thus $w \in M_G(B)$ since G_0 is a convex subgraph of G. Therefore w is a limit point of A by the Fundamental Theorem.

Consequently G is geodesically consistent. The converse is clear because G_0 and G_1 are convex subgraphs of G. □

4.2. Examples

As we already saw, for any infinite cardinal α, the partial cube $S(K_\alpha)$ is not geodesically consistent; note that the vertex x_p geodesically dominates $U^{S(K_\alpha)}_{x_n x_{np}}$ but does not belong to this set. On the other hand, by Proposition 4.4, for any cardinal $\alpha \geq 2$, the partial cube $S^+(K_\alpha)$ is geodesically consistent since its vertex set is compact.

By Theorem 4.8, there are two classes of partial cubes which are obviously geodesically consistent. These are the class of locally finite partial cubes (note that if G is a locally finite partial cube, then the geodesic space $V(G)$ is compact if and only if G is finite), and the class of median graphs because, by a result of Bandelt [1] (see also [6]), a median graph G is a bipartite graph for which the sets U_{ab} and U_{ba} are convex, and thus closed, for each edge ab of G. There are also classes of partial cubes which are less clearly geodesically consistent, such as the classes of cellular bipartite graphs, of benzenoid graphs and more generally of netlike partial cubes. To prove that, we will recall the definition and some properties of netlike partial cubes.

We denote by $CV(G)$ (resp. $3V(G)$) the set of vertices of a graph G which belong to a cycle of G (resp. whose degree is at least 3). We say that a set $A \subseteq V(G)$ is *$\mathcal{C}$-convex* (resp. (3)-*convex*) if $CV(G[\mathcal{I}_G(A)]) \subseteq A$ (resp. $3V(G[\mathcal{I}_G(A)]) \subseteq A$). The set of $\mathcal{C}$-convex subsets of $V(G)$ and the one of (3)-convex subsets of $V(G)$ are convexities on $V(G)$ which are finer than the geodesic convexity.

Lemma 4.12 (*Polat [13, Corollary 2.7]*). *If A is a $\mathcal{C}$-convex set of a connected graph G, then $\mathcal{I}_G(A)$ is convex.*

By relaxing the type of convexity in Bandelt's characterization of a median graph [1] we obtain what we have called a netlike partial cube (see [12]).

Definition 4.13. We say that a partial cube G is *netlike* if U_{ab} and U_{ba} are $\mathcal{C}$-convex for each edge ab of G.

Thus median graphs are netlike partial cubes. More generally even cycles, benzenoid graphs and cellular bipartite graphs are also netlike partial cubes. Moreover any convex subgraph of a netlike partial cube is a netlike partial cube.

By [12, Theorem 3.8] we have:

Lemma 4.14. *If a partial cube G is netlike, then the sets U_{ab} and U_{ba} are* (3)-*convex for each edge ab of G.*

Proposition 4.15. *Any netlike partial cube is geodesically consistent.*

Proof. By Lemmas 4.12 and 4.14, for each edge ab of a netlike partial cube G, the sets U_{ab} and U_{ba} are almost geodesically closed, which proves, by Theorem 4.8, that G is geodesically consistent. □

5. Intervals and compactness

This last section focuses on compactness properties requiring the use of intervals. We first state some simple facts about isometric rays.

If $\langle x_0, x_1, \ldots\rangle$ is an isometric ray of a partial cube G, then, for every non-negative integer n, $x_i \in W_{x_n x_{n+1}}$ if and only if $i \leq n$, and thus

$$I_G(x_0, x_n) \cap \mathrm{co}_G(\{x_i : n+1 \leq i\}) = \emptyset.$$

It follows in particular that:

$$I_G(x_0, x_n) \subset I_G(x_0, x_{n+1}) \tag{4}$$

and

$$\mathrm{co}_G(\{x_i : n+1 \leq i\}) \subset \mathrm{co}_G(\{x_i : n \leq i\}). \tag{5}$$

Lemma 5.1. *Let G be a partial cube. Then:*

(i) *G contains an isometric ray if and only if there exists a sequence $(I_G(a, b_n))_{n\in\mathbb{N}}$ of intervals of G such that $I_G(a, b_n) \subset I_G(a, b_{n+1})$ and $b_{n+1} \notin I_G(a, b_n)$ for every non-negative integer n;*
(ii) *G contains an isometric double ray if and only if there exists a sequence $(I_G(a_n, b_n))_{n\in\mathbb{N}}$ of intervals of G such that $I_G(a_n, b_n) \subset I_G(a_{n+1}, b_{n+1})$ and $a_{n+1}, b_{n+1} \notin I_G(a_n, b_n)$ for every non-negative integer n.*

Proof. (i) If $R = \langle x_0, x_1, \ldots\rangle$ is an isometric ray of G, then $(I_G(x_0, x_n))_{n\in\mathbb{N}}$ has the required properties by (4). Conversely, suppose that $(I_G(a, b_n))_{n\in\mathbb{N}}$ is such that $I_G(a, b_n) \subset I_G(a, b_{n+1})$ and $b_{n+1} \notin I_G(a, b_n)$ for every non-negative integer n. Let P_0 be an (a, b_0)-geodesic, and P_n a (b_n, b_{n+1})-geodesic for every positive integer n. Then $\bigcup_{n\in\mathbb{N}} P_n$ is an isometric ray of G.

(ii) If $D = \langle \ldots, x_{-1}, x_0, x_1, \ldots\rangle$ is an isometric double ray of G, then, by using (4), we can easily prove that $(I_G(x_{-n}, x_n))_{n\in\mathbb{N}}$ has the required properties. Conversely, suppose that $(I_G(a_n, b_n))_{n\in\mathbb{N}}$ is such that $I_G(a_n, b_n) \subset I_G(a_{n+1}, b_{n+1})$ and $a_{n+1}, b_{n+1} \notin I_G(a_n, b_n)$ for every non-negative integer n. Let P_0 be an (a_0, b_0)-geodesic, and P_n an (a_n, a_{n+1})-geodesic or a (b_n, b_{n+1})-geodesic according to whether n is negative or positive. Then $\bigcup_{n\in\mathbb{Z}} P_n$ is an isometric double ray of G. □

Note that, by the proof above, R is an isometric ray of $G[\bigcup_{n\in\mathbb{N}} I_G(a, b_n)]$, and D an isometric double ray of $G[\bigcup_{n\in\mathbb{N}} I_G(a_n, b_n)]$.

Proposition 5.2. *Let G be a partial cube. If there exists an infinite chain $\mathcal{C}$ (i.e. an infinite set totally ordered by inclusion) of intervals of G, then $G[\bigcup \mathcal{C}]$ contains an isometric ray or an isometric double ray, and thus the space $V(G)$ is not compact.*

Proof. Because $\mathcal{C}$ is infinite, there is an infinite sequence $(I_G(a_n, b_n))_{n\in\mathbb{N}}$ of elements of $\mathcal{C}$ such that $I_G(a_n, b_n) \subset I_G(a_{n+1}, b_{n+1})$ for every non-negative integer n. Clearly a_{n+1} or b_{n+1} does not belong to $I_G(a_n, b_n)$ for every n. We distinguish two cases.

Case 1. There exists an infinite sequence $i_0 < i_1 < \cdots$ of non-negative integers such that $b_{i_{n+1}} \notin I_G(a_{i_n}, b_{i_n})$ for every $n \in \mathbb{N}$.

Without loss of generality we can suppose that $i_n = n$ for all n. We now have two subcases.

Subcase 1.1. There exists an infinite sequence $j_0 < j_1 < \cdots$ of non-negative integers such that $a_{j_{n+1}} \notin I_G(a_{j_n}, b_{j_n})$ for every $n \in \mathbb{N}$.

Without loss of generality we can suppose that $j_n = n$ for all n. Then the sequence of intervals $(I_G(a_n, b_n))_{n\in\mathbb{N}}$ is such that $a_{n+1}, b_{n+1} \notin I_G(a_n, b_n)$. By Lemma 5.1, $G[\bigcup_{n\in\mathbb{N}} I_G(a_n, b_n)]$ contains an isometric double ray.

Subcase 1.2. There exists a non-negative integer n_0 such that $a_n \in I_G(a_{n_0}, b_{n_0})$ for every $n > n_0$.

Because $I_G(a_{n_0}, b_{n_0})$ is finite, there is a vertex $a \in I_G(a_{n_0}, b_{n_0})$ such that $a_n = a$ for infinitely many $n > n_0$. Without loss of generality we can suppose that $n_0 = 0$ and $a_n = a_0$ for every $n \in \mathbb{N}$. Let n be a non-negative integer. Then $b_n \notin I_G(a_0, b_{n+1})$. By Lemma 5.1, $G[\bigcup_{n\in\mathbb{N}} I_G(a_n, b_n)]$ contains an isometric ray.

Case 2. There exists a non-negative integer n_0 such that $b_n \in I_G(a_{n_0}, b_{n_0})$ for every $n > n_0$.

With a proof analogous to that of Case 1, we can show that $G[\bigcup_{n\in\mathbb{N}} I_G(a_n, b_n)]$ contains an isometric ray or an isometric double ray.

It follows, by Corollary 3.15, that the space $V(G)$ is not compact. □

Corollary 5.3. *Let G be a partial cube whose vertex set is compact. Then G has a maximal interval, and moreover each interval of G is contained in a maximal interval.*

Theorem 5.4. *Let G be a geodesically consistent partial cube. Then the space $V(G)$ is compact if and only if every chain of intervals of G is finite.*

Proof. The necessity is a consequence of Proposition 5.2, whereas the sufficiency is a consequence of Lemma 5.1 and the fact that G contains an isometric ray if $V(G)$ is not compact. □

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