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On an Inequality of Ostrowski Type in Three Independent Variables

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In the present paper a new inequality of Ostrowski type in three independent variables is established. The discrete analogue of the main result is also given. © 2000 Academic Press

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1. INTRODUCTION

In 1938, A. Ostrowski proved the following inequality (see [3, p. 468]).

Let f be a differentiable function on (a, b) and on (a, b), $|f'(x)| \leq M$. Then, for every $x \in (a, b)$

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \leq \left[\frac{1}{4} + \frac{(x - (a+b)/2)^2}{(b-a)^2} \right] (b-a) M. \quad (1)$$

A great deal of attention has been given to the above inequality and many papers dealing with numerous variants, generalizations, and extensions have appeared in the literature; see [3, pp. 468–484]. In a recent paper [1] N. S. Barnett and S. S. Dragomir established the following result for real functions of two variables (see also [2, p. 170]).

Let f: $[a, b] \times [c, d] \to R$ be such that $f(x, y)$ is continuous, $f''_{x,y} = \partial^2 f / \partial x\, \partial y$ exist on $(a, b) \times (c, d)$, and

$$\|f''_{s,t}\|_\infty := \sup_{(x,y) \in (a,b) \times (c,d)} \left| \frac{\partial^2 f(x, y)}{\partial x\, \partial y} \right| < \infty.$$

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Then we have the inequality

$$\left|\int_a^b\int_c^d f(s,t)\,ds\,dt - \left[(b-a)\int_c^d f(x,t)\,dt + (d-c)\int_a^b f(s,y)\,ds - (d-c)(b-a)f(x,y)\right]\right|$$
$$\le \left[\frac{1}{4}(b-a)^2 + \left(x-\frac{a+b}{2}\right)^2\right]\left[\frac{1}{4}(d-c)^2 + \left(y-\frac{c+d}{2}\right)^2\right] \times \|f''_{s,t}\|_\infty, \qquad (2)$$

for all $(x,y)\in[a,b]\times[c,d]$.

The main purpose of the present paper is to establish a new inequality of Ostrowski type for real functions of three variables. The discrete analogue of the main result is also given. The analysis used in the proof is elementary and our results provide new estimates on inequalities of this type.

2. MAIN RESULT

In what follows R denotes the set of real numbers and $R^+ = [0,\infty)$. We use the notation $\Delta = [a,k]\times[b,m]\times[c,n]$ for a,b,c,k,m,n in R^+. If $f(r,s,t)$ is a differentiable function defined on Δ, then its partial derivatives are denoted by $D_1 f(r,s,t) = \frac{\partial}{\partial r}f(r,s,t)$, $D_2 f(r,s,t) = \frac{\partial}{\partial s}f(r,s,t)$, $D_3 f(r,s,t) = \frac{\partial}{\partial t}f(r,s,t)$, and $D_3D_2D_1 f(r,s,t) = (\partial^3/\partial t\,\partial s\,\partial r)f(r,s,t)$. We denote by $F(\Delta)$ the class of continuous functions $f\colon \Delta\to R$ for which $D_1 f(r,s,t)$, $D_2 f(r,s,t)$, $D_3 f(r,s,t)$, $D_3D_2D_1 f(r,s,t)$ exist and are continuous on Δ.

Our main result is given in the following theorem.

THEOREM 1. *Let* $f\in F(\Delta)$. *Then*

$$\left|\int_a^k\int_b^m\int_c^n f(r,s,t)\,dt\,ds\,dr - \tfrac{1}{8}(k-a)(m-b)(n-c)[f(a,b,c)+f(k,m,n)] + \tfrac{1}{4}(m-b)(n-c) \times\int_a^k [f(r,b,c)+f(r,m,n)+f(r,m,c)+f(r,b,n)]\,dr\right.$$

$$+\tfrac{1}{4}(k-a)(n-c)$$
$$\times\int_b^m [f(a,s,c)+f(k,s,n)+f(a,s,n)+f(k,s,c)]\,ds$$
$$+\tfrac{1}{4}(k-a)(m-b)$$
$$\times\int_c^n [f(a,b,t)+f(k,m,t)+f(k,b,t)+f(a,m,t)]\,dt$$
$$-\tfrac{1}{2}(k-a)\int_b^m\int_c^n [f(a,s,t)+f(k,s,t)]\,dt\,ds$$
$$-\tfrac{1}{2}(m-b)\int_a^k\int_c^n [f(r,b,t)+f(r,m,t)]\,dt\,dr$$
$$-\tfrac{1}{2}(n-c)\int_a^k\int_b^m [f(r,s,c)+f(r,s,n)]\,ds\,dr\Bigg|$$
$$\le \tfrac{1}{8}(k-a)(m-b)(n-c)$$
$$\times\int_a^k\int_b^m\int_c^n |D_3D_2D_1f(r,s,t)|\,dt\,ds\,dr. \tag{3}$$

Proof. From the hypotheses it is easy to observe that the following identities hold (see [5])

$$\begin{aligned} f(r,s,t) &= f(a,b,c)+f(a,s,t)+f(r,s,c)+f(r,b,t)\\ &\quad -f(a,b,t)-f(a,s,c)-f(r,b,c)\\ &\quad +\int_a^r\int_b^s\int_c^t D_3D_2D_1f(u,v,w)\,dw\,dv\,du, \end{aligned}\tag{4}$$

$$\begin{aligned} f(r,s,t) &= f(r,s,n)+f(a,s,t)+f(r,b,t)+f(a,b,n)\\ &\quad -f(a,b,t)-f(a,s,n)-f(r,b,n)\\ &\quad -\int_a^r\int_b^s\int_t^n D_3D_2D_1f(u,v,w)\,dw\,dv\,du, \end{aligned}\tag{5}$$

$$\begin{aligned} f(r,s,t) &= f(r,m,t)+f(r,s,c)+f(a,m,c)+f(a,s,t)\\ &\quad -f(r,m,c)-f(a,m,t)-f(a,s,c)\\ &\quad -\int_a^r\int_s^m\int_c^t D_3D_2D_1f(u,v,w)\,dw\,dv\,du, \end{aligned}\tag{6}$$

$$\begin{aligned} f(r,s,t) &= f(k,s,t)+f(k,b,c)+f(r,s,c)+f(r,b,t)\\ &\quad -f(k,s,c)-f(k,b,t)-f(r,b,c)\\ &\quad -\int_r^k\int_b^s\int_c^t D_3D_2D_1f(u,v,w)\,dw\,dv\,du, \end{aligned}\tag{7}$$

$$\begin{aligned} f(r,s,t) &= f(r,m,t) + f(r,s,n) + f(a,m,n) + f(a,s,t) \\ &\quad - f(r,m,n) - f(a,m,t) - f(a,s,n) \\ &\quad + \int_a^r \int_s^m \int_t^n D_3 D_2 D_1 f(u,v,w)\, dw\, dv\, du, \end{aligned} \tag{8}$$

$$\begin{aligned} f(r,s,t) &= f(r,m,t) + f(r,s,c) + f(k,s,t) + f(k,m,c) \\ &\quad - f(k,m,t) - f(k,s,c) - f(r,m,c) \\ &\quad + \int_r^k \int_s^m \int_c^t D_3 D_2 D_1 f(u,v,w)\, dw\, dv\, du, \end{aligned} \tag{9}$$

$$\begin{aligned} f(r,s,t) &= f(k,s,t) + f(k,b,n) + f(r,s,n) + f(r,b,t) \\ &\quad - f(k,s,n) - f(k,b,t) - f(r,b,n) \\ &\quad + \int_r^k \int_b^s \int_t^n D_3 D_2 D_1 f(u,v,w)\, dw\, dv\, du, \end{aligned} \tag{10}$$

$$\begin{aligned} f(r,s,t) &= f(k,m,n) + f(k,s,t) + f(r,m,t) + f(r,s,n) \\ &\quad - f(k,m,t) - f(k,s,n) - f(r,m,n) \\ &\quad - \int_r^k \int_s^m \int_t^n D_3 D_2 D_1 f(u,v,w)\, dw\, dv\, du. \end{aligned} \tag{11}$$

From (4)–(11) we observe that

$$\begin{aligned} f(r,s,t) = {} & \tfrac{1}{8}[f(a,b,c) + f(k,m,n)] \\ & - \tfrac{1}{4}[f(r,b,c) + f(r,m,n) + f(r,m,c) + f(r,b,n)] \\ & - \tfrac{1}{4}[f(a,s,c) + f(k,s,n) + f(a,s,n) + f(k,s,c)] \\ & - \tfrac{1}{4}[f(a,b,t) + f(k,m,t) + f(k,b,t) + f(a,m,t)] \\ & + \tfrac{1}{2}[f(a,s,t) + f(k,s,t)] \\ & + \tfrac{1}{2}[f(r,b,t) + f(r,m,t)] \\ & + \tfrac{1}{2}[f(r,s,c) + f(r,s,n)] \\ & + \int_a^r \int_b^s \int_c^t D_3 D_2 D_1 f(u,v,w)\, dw\, dv\, du \\ & - \int_a^r \int_b^s \int_t^n D_3 D_2 D_1 f(u,v,w)\, dw\, dv\, du \end{aligned}$$

$$-\int_a^r \int_s^m \int_c^t D_3 D_2 D_1 f(u, v, w)\, dw\, dv\, du$$

$$-\int_r^k \int_b^s \int_c^t D_3 D_2 D_1 f(u, v, w)\, dw\, dv\, du$$

$$+\int_a^r \int_s^m \int_t^n D_3 D_2 D_1 f(u, v, w)\, dw\, dv\, du$$

$$+\int_r^k \int_s^m \int_c^t D_3 D_2 D_1 f(u, v, w)\, dw\, dv\, du$$

$$+\int_r^k \int_b^s \int_t^n D_3 D_2 D_1 f(u, v, w)\, dw\, dv\, du$$

$$-\int_r^k \int_s^m \int_t^n D_3 D_2 D_1 f(u, v, w)\, dw\, dv\, du. \tag{12}$$

Integrating both sides of (12) on Δ and by making elementary calculations we get the desired inequality in (3) and the proof is complete.

Remark 1. From (12) it is easy to observe that the following inequality also holds

$$\begin{aligned} &\Big| f(r,s,t) - \tfrac{1}{8}[f(a,b,c) + f(k,m,n)] \\ &\quad + \tfrac{1}{4}[f(r,b,c) + f(r,m,n) + f(r,m,c) + f(r,b,n)] \\ &\quad + \tfrac{1}{4}[f(a,s,c) + f(k,s,n) + f(a,s,n) + f(k,s,c)] \\ &\quad + \tfrac{1}{4}[f(a,b,t) + f(k,m,t) + f(k,b,t) + f(a,m,t)] \\ &\quad - \tfrac{1}{2}[f(a,s,t) + f(k,s,t)] \\ &\quad - \tfrac{1}{2}[f(r,b,t) + f(r,m,t)] \\ &\quad - \tfrac{1}{2}[f(r,s,c) + f(r,s,n)]\Big| \\ &\qquad \le \tfrac{1}{8}\int_a^k \int_b^m \int_c^n |D_3 D_2 D_1 f(u,v,w)|\, dw\, dv\, du, \end{aligned} \tag{13}$$

for $(r, s, t) \in \Delta$.

3. DISCRETE ANALOGUE

Let N denote the set of natural numbers, $A = \{1, 2, \ldots, k+1\}$, $B = \{1, 2, \ldots, m+1\}$, $C = \{1, 2, \ldots, n+1\}$ for k, m, n in N, and $E = A \times B \times C$. For a function $f\colon N^3 \to R$ we define the difference operators

$\Delta_1 f(x,y,z) = f(x+1,y,z) - f(x,y,z)$, $\Delta_2 f(x,y,z) = f(x,y+1,z) - f(x,y,z)$, $\Delta_3 f(x,y,z) = f(x,y,z+1) - f(x,y,z)$, $\Delta_1\Delta_2 f(x,y,z) = \Delta_1[\Delta_2 f(x,y,z)]$, and $\Delta_3\Delta_2\Delta_1 f(x,y,z) = \Delta_3[\Delta_2\Delta_1 f(x,y,z)]$. We denote by $G(E)$ the class of functions $f\colon E \to R$ for which $\Delta_1 f(x,y,z)$, $\Delta_2 f(x,y,z)$, $\Delta_3 f(x,y,z)$, and $\Delta_3\Delta_2\Delta_1 f(x,y,z)$ exist on E.

The discrete version of the inequality (3) is established in the following theorem.

THEOREM 2. *Let $f \in G(E)$. Then*

$$\begin{aligned}
&\Bigg| \sum_{u=1}^{k}\sum_{v=1}^{m}\sum_{w=1}^{n} f(u,v,w) - \tfrac{1}{8}[f(1,1,1) + f(k+1,m+1,n+1)] \\
&\quad + \tfrac{1}{4}mn \sum_{r=1}^{k} [f(r,1,1) + f(r,1,n+1) + f(r,m+1,1) \\
&\qquad\qquad + f(r,m+1,n+1)] \\
&\quad + \tfrac{1}{4}kn \sum_{s=1}^{m} [f(k+1,s,n+1) + f(k+1,s,1) + f(1,s,n+1) \\
&\qquad\qquad + f(r,m+1,n+1)] \\
&\quad + \tfrac{1}{4}km \sum_{t=1}^{n} [f(k+1,m+1,t) + f(k+1,1,t) \\
&\qquad\qquad + f(1,m+1,t) + f(1,1,t)] \\
&\quad - \tfrac{1}{2}k \sum_{s=1}^{m}\sum_{t=1}^{n} [f(1,s,t) + f(k+1,s,t)] \\
&\quad - \tfrac{1}{2}m \sum_{r=1}^{k}\sum_{s=1}^{m} [f(r,1,t) + f(r,m+1,t)] \\
&\quad - \tfrac{1}{2}n \sum_{r=1}^{k}\sum_{s=1}^{m} [f(r,s,1) + f(r,s,n+1)] \Bigg| \\
&\qquad \le \tfrac{1}{8}kmn \sum_{u=1}^{k}\sum_{v=1}^{m}\sum_{w=1}^{n} |\Delta_3\Delta_2\Delta_1 f(u,v,w)|. \qquad (14)
\end{aligned}$$

Proof. For $(r,s,t) \in E$, it is easy to observe that the following identities hold (see [4])

$$\begin{aligned}
f(r,s,t) = f(1,1,1) &+ f(1,s,t) + f(r,1,t) + f(r,s,1) \\
&- f(1,1,t) - f(1,s,1) - f(r,1,1) \\
&+ \sum_{u=1}^{r-1}\sum_{v=1}^{s-1}\sum_{w=1}^{t-1} \Delta_3\Delta_2\Delta_1 f(u,v,w), \qquad (15)
\end{aligned}$$

$$\begin{aligned} f(r,s,t) &= f(1,1,n+1) + f(r,s,n+1) + f(1,s,t) + f(r,1,t) \\ &\quad - f(1,s,n+1) - f(r,1,n+1) - f(1,1,t) \\ &\quad - \sum_{u=1}^{r-1} \sum_{v=1}^{s-1} \sum_{w=t}^{n} \Delta_3 \Delta_2 \Delta_1 f(u,v,w), \end{aligned} \tag{16}$$

$$\begin{aligned} f(r,s,t) &= f(r,m+1,t) + f(1,s,t) + f(1,m+1,1) + f(r,s,1) \\ &\quad - f(1,m+1,t) - f(r,m+1,1) - f(1,s,1) \\ &\quad - \sum_{u=1}^{r-1} \sum_{v=s}^{m} \sum_{w=1}^{t-1} \Delta_3 \Delta_2 \Delta_1 f(u,v,w), \end{aligned} \tag{17}$$

$$\begin{aligned} f(r,s,t) &= f(k+1,s,t) + f(r,1,t) + f(r,s,1) + f(k+1,1,1) \\ &\quad - f(k+1,1,t) - f(k+1,s,t) - f(r,1,1) \\ &\quad - \sum_{u=r}^{k} \sum_{v=1}^{s-1} \sum_{w=1}^{t-1} \Delta_3 \Delta_2 \Delta_1 f(u,v,w), \end{aligned} \tag{18}$$

$$\begin{aligned} f(r,s,t) &= f(r,m+1,t) + f(r,s,n+1) + f(1,m+1,n+1) \\ &\quad + f(1,s,t) - f(r,m+1,n+1) - f(1,m+1,t) \\ &\quad - f(1,s,n+1) + \sum_{u=1}^{r-1} \sum_{v=s}^{m} \sum_{w=t}^{n} \Delta_3 \Delta_2 \Delta_1 f(u,v,w), \end{aligned} \tag{19}$$

$$\begin{aligned} f(r,s,t) &= f(r,m+1,t) + f(r,s,1) + f(k+1,s,t) + f(k+1,m,1) \\ &\quad - f(k+1,m+1,t) - f(k+1,s,1) - f(r,m+1,1) \\ &\quad + \sum_{u=r}^{k} \sum_{v=s}^{m} \sum_{w=1}^{t-1} \Delta_3 \Delta_2 \Delta_1 f(u,v,w), \end{aligned} \tag{20}$$

$$\begin{aligned} f(r,s,t) &= f(k+1,s,t) + f(k+1,1,n+1) + f(r,s,n+1) \\ &\quad + f(r,1,t) - f(k+1,1,n+1) - f(k+1,1,t) \\ &\quad - f(r,1,n+1) + \sum_{u=r}^{k} \sum_{v=1}^{s-1} \sum_{w=t}^{n} \Delta_3 \Delta_2 \Delta_1 f(u,v,w), \end{aligned} \tag{21}$$

$$\begin{aligned} f(r,s,t) &= f(k+1,m+1,n+1) + f(k+1,s,t) + f(r,m+1,t) \\ &\quad + f(r,s,n+1) - f(k+1,m+1,t) - f(k+1,s,n+1) \\ &\quad - f(r,m+1,n+1) - \sum_{u=r}^{k} \sum_{v=s}^{m} \sum_{w=t}^{n} \Delta_3 \Delta_2 \Delta_1 f(u,v,w). \end{aligned} \tag{22}$$

From (15)–(22) we get

$$
\begin{aligned}
f(r,s,t) = {} & \tfrac{1}{8}\left[f(1,1,1)+f(k+1,m+1,n+1)\right] \\
& -\tfrac{1}{4}\left[f(r,1,1)+f(r,1,n+1)+f(r,m+1,1)\right. \\
& \qquad \left. +f(r,m+1,n+1)\right] \\
& -\tfrac{1}{4}\left[f(k+1,s,n+1)+f(k+1,s,1)\right. \\
& \qquad \left. +f(1,s,n+1)+f(1,s,1)\right] \\
& -\tfrac{1}{4}\left[f(k+1,m+1,t)+f(k+1,1,t)\right. \\
& \qquad \left. +f(1,m+1,t)+f(1,1,t)\right] \\
& +\tfrac{1}{2}\left[f(1,s,t)+f(k+1,s,t)\right] \\
& +\tfrac{1}{2}\left[f(r,1,t)+f(r,m+1,t)\right] \\
& +\tfrac{1}{2}\left[f(r,s,1)+f(r,s,n+1)\right] \\
& +\sum_{u=1}^{r-1}\sum_{v=1}^{s-1}\sum_{w=1}^{r-1}\Delta_3\Delta_2\Delta_1 f(u,v,w) \\
& -\sum_{u=1}^{r-1}\sum_{v=1}^{s-1}\sum_{w=t}^{n}\Delta_3\Delta_2\Delta_1 f(u,v,w) \\
& -\sum_{u=1}^{r-1}\sum_{v=s}^{m}\sum_{w=1}^{t-1}\Delta_3\Delta_2\Delta_1 f(u,v,w) \\
& -\sum_{u=r}^{k}\sum_{v=1}^{s-1}\sum_{w=1}^{t-1}\Delta_3\Delta_2\Delta_1 f(u,v,w) \\
& +\sum_{u=1}^{r-1}\sum_{v=s}^{m}\sum_{w=t}^{n}\Delta_3\Delta_2\Delta_1 f(u,v,w) \\
& +\sum_{u=r}^{k}\sum_{v=s}^{m}\sum_{w=1}^{t-1}\Delta_3\Delta_2\Delta_1 f(u,v,w) \\
& +\sum_{u=r}^{k}\sum_{v=1}^{s-1}\sum_{w=t}^{n}\Delta_3\Delta_2\Delta_1 f(u,v,w) \\
& -\sum_{u=r}^{k}\sum_{v=s}^{m}\sum_{w=t}^{n}\Delta_3\Delta_2\Delta_1 f(u,v,w).
\end{aligned} \tag{23}
$$

Summing both sides of (23) on E and making elementary calculations we get the desired inequality in (14). The proof is complete.

Remark 2. From (23) it is also easy to obtain the inequality

$$\begin{aligned}\Big| f(r,s,t) &- \tfrac{1}{8}[f(1,1,1)+f(k+1,m+1,n+1)] \\ &+ \tfrac{1}{4}[f(r,1,1)+f(r,1,n+1)+f(r,m+1,1) \\ &\qquad + f(r,m+1,n+1)] \\ &+ \tfrac{1}{4}[f(k+1,s,n+1)+f(k+1,s,1) \\ &\qquad + f(1,s,n+1)+f(1,s,1)] \\ &+ \tfrac{1}{4}[f(k+1,m+1,t)+f(k+1,1,t) \\ &\qquad + f(1,m+1,t)+f(1,1,t)] \\ &- \tfrac{1}{2}[f(1,s,t)+f(k+1,s,t)] \\ &- \tfrac{1}{2}[f(r,1,t)+f(r,m+1,t)] \\ &- \tfrac{1}{2}[f(r,s,1)+f(r,s,n+1)]\Big| \\ &\le \tfrac{1}{8}kmn\sum_{u=1}^{k}\sum_{v=1}^{m}\sum_{w=1}^{n}|\Delta_3\Delta_2\Delta_1 f(u,v,w)|, \qquad (24)\end{aligned}$$

for $(r,s,t)\in E$.

In concluding we note that our result in Theorem 1 is different from the inequalities (1) and (2) and the various results given in [1–3] and we believe that the inequalities given here are new to the literature.

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