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Derived brackets and sh Leibniz algebras

K. Uchino

Science University of Tokyo, 3-14-1 Shinjyuku, Tokyo, Japan

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ABSTRACT

We develop a general framework for the construction of various derived brackets. We show that suitably deforming the differential of a graded Leibniz algebra extends the derived bracket construction and leads to the notion of strong homotopy (sh) Leibniz algebra. We discuss the connections among homotopy algebra theory, deformation theory and derived brackets. We prove that the derived bracket construction induces a map from suitably defined deformation theory equivalence classes to the isomorphism classes of sh Leibniz algebras.

1. Introduction

Let $(V, d, \{,\})$ be a chain complex equipped with a binary bilinear V-valued operation $\{,\}$. The triple $(V, d, \{,\})$ is called a dg Leibniz algebra or a dg Loday algebra by some authors, if the differential is a derivation with respect to the bracket and the bracket satisfies the (graded) Leibniz identity. When the bracket is anti-commutative, the Leibniz identity is equivalent to the Jacobi identity. In this sense, (dg) Leibniz algebras are noncommutative analogues of classical (dg) Lie algebras.

Let $(V, d, \{,\})$ be a dg Leibniz algebra. We define a modified bracket:

$$\{x, y\}_d := (-1)^x\{dx, y\},$$

which is called a *derived bracket*. In Kosmann-Schwarzbach [5], it was shown that the derived bracket satisfies the Leibniz identity. The original idea of the derived bracket goes back at least to Koszul (unpublished). The derived brackets play important roles in modern analytical mechanics (cf. [6]). For instance, a Poisson bracket on a smooth manifold is given as a derived bracket, $\{f, g\} := [df, g]_{SN}$, where f, g are smooth functions on the manifold, $[,]_{SN}$ is a Schouten–Nijenhuis bracket and d is a coboundary operator of Poisson cohomology. It is known that the Schouten–Nijenhuis bracket is also a derived bracket of a certain graded Poisson bracket.

We consider n-fold derived brackets:

$$(\pm)[[\cdots[\delta x_1, x_2]\cdots], x_n],$$

where $[\cdot,\cdot]$ is a Lie bracket, $\pm$ an appropriate sign, and δ a certain derivation, not necessarily of square zero. The n-ary (higher) derived brackets in the category of *Lie* algebras were studied by several authors in various contexts: in an article on Poisson geometry by Roytenberg [15], in a paper on homotopy algebra theory by Voronov [18], in early work of Vallejo [17] who gave a necessary and sufficient condition for the n-ary derived brackets become Nambu–Lie brackets.

The purpose of this note is to complete the theory of higher derived bracket construction in the category of *Leibniz* algebras. To study the higher derived bracket composed of *pure* Leibniz brackets, we apply the theory of *sh Leibniz algebras* (also called Leibniz ∞-algebras, sh Loday algebras or Loday ∞-algebras). Sh Leibniz algebras are Leibniz algebras *up to homotopy* as well as noncommutative analogues of sh Lie algebras. We refer the reader to Ammar and Poncin [1] for the

E-mail addresses: K_Uchino@oct.rikadai.jp, k_uchino_tus@yahoo.co.jp.

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study of sh Leibniz algebras. We give a short survey of sh Leibniz algebras in Section 4.1 below. The main result of this note is Theorem 3.4: Let $(V, \delta_0, \{,\})$ be a dg Leibniz algebra. We consider a deformation of δ_0,

$$\delta_t = \delta_0 + t\delta_1 + t^2\delta_2 + \cdots,$$

where t is a formal parameter and δ_t a differential on $V[[t]]$. We define an i-ary derived bracket as

$$l_i(x_1, \ldots, x_i) := (\pm)\{\{\ldots\{\delta_{i-1}x_1, x_2\}, \ldots\}, x_i\},$$

where $\pm$ is an appropriate sign. We prove that the collection of the higher derived brackets, $\{l_1, l_2, \ldots\}$, yields an sh Leibniz algebra structure. The theorem follows from a universal formula, satisfied by Leibniz brackets, which we establish in Lemma 4.2.

The higher derived bracket construction proposed in this paper is useful to study a relation between homotopy algebra theory and deformation theory. In Proposition 5.1, we will show that if two deformations of δ_0 are gauge equivalent, then the induced sh Leibniz algebras are equivalent; in other words, the higher derived bracket construction is invariant under gauge transformations.

2. Preliminaries

2.1. Notation and assumptions

The base field is a field $\mathbb{K}$ of characteristic zero. The unadorned tensor product denotes the tensor product $\otimes := \otimes_{\mathbb{K}}$ over the field $\mathbb{K}$. We follow the standard Koszul sign convention, for instance, a linear map $f \otimes g : V \otimes V \to V \otimes V$ satisfies

$$(f \otimes g)(x \otimes y) = (-1)^{|g||x|} f(x) \otimes g(y),$$

where $x, y \in V$ and where $|g|, |x|$ are the degrees of g, x. We will denote by s the operator that raises degree by 1 and, likewise, by s^{-1} the operator that lowers degree by 1. The Koszul sign convention for shifting operators is, for instance,

$$s \otimes s = (s \otimes 1)(1 \otimes s) = -(1 \otimes s)(s \otimes 1).$$

We call a derivation of degree 1 a *differential*, if it is of square zero. Given a homogeneous member x of a graded vector space, we denote the sign $(-1)^{|x|}$ simply by $(-1)^x$.

2.2. Leibniz algebras and derived brackets

Let $(V, d, \{,\})$ be a chain complex equipped with a binary bracket. We assume that the degree of the differential is $+1$ (or odd) and the degree of the bracket is 0 (or even). The triple is called a dg (left) Leibniz algebra, or a dg (left) Loday algebra by some authors, if d is a derivation with respect to the bracket and the bracket satisfies a Leibniz identity, i.e.,

$$d\{x, y\} = \{dx, y\} + (-1)^{|x|}\{x, dy\},$$
$$\{x, \{y, z\}\} = \{\{x, y\}, z\} + (-1)^{|x||y|}\{y, \{x, z\}\},$$

where $x, y, z \in V$. A dg Lie algebra can be seen as a special Leibniz algebra of which the bracket is anti-commutative. In this sense, (dg) Leibniz algebras are noncommutative analogues of (dg) Lie algebras.

We recall the classical derived bracket construction in [5,6]. Define a new bracket on the shifted space sV by

$$\{sx, sy\}_d := (-1)^x s\{dx, y\}. \tag{1}$$

This bracket is called a (binary) derived bracket on sV. Eq. (1) is equal to the following tensor identity,

$$\{\cdot, \cdot\}_d(sx \otimes sy) = s\{\cdot, \cdot\}(s^{-1} \otimes s^{-1})(sds^{-1} \otimes 1)(sx \otimes sy).$$

We recall two basic propositions.

- The derived bracket also satisfies the graded Leibniz identity, i.e.,

$$\{sx, \{sy, sz\}_d\}_d = \{\{sx, sy\}_d, sz\}_d + (-1)^{(x+1)(y+1)}\{sy, \{sx, sz\}_d\}_d.$$

We consider the cases of dg Lie algebras.

- Let $(V, d, [,])$ be a dg Lie algebra and let $\mathfrak{g}(\subset V)$ a trivial subalgebra of the Lie algebra. If $s\mathfrak{g}$ is closed under the derived bracket, then $s\mathfrak{g}$ is a Lie algebra, that is, the derived bracket is anti-commutative on $s\mathfrak{g}$.

3. Main results

Let V be a graded vector space and let $l_i : V^{\otimes i} \to V$ be an i-ary multilinear map with the degree $2-i$, for each $i \geq 1$.

Definition 3.1 (*[1]*). The space $(V, l_1, l_2, \ldots)$ with the multilinear maps is called a strong homotopy (sh) Leibniz algebra, if the collection $\{l_i\}_{i\geq 1}$ satisfies (2) below.

$$\sum_{i+j=\mathrm{Const}} \sum_{k=j}^{i+j-1} \sum_{\sigma} \chi(\sigma)(-1)^{(k+1-j)(j-1)}(-1)^{j(x_{\sigma(1)}+\cdots+x_{\sigma(k-j)})}$$
$$l_i(x_{\sigma(1)}, \ldots, x_{\sigma(k-j)}, l_j(x_{\sigma(k+1-j)}, \ldots, x_{\sigma(k-1)}, x_k), x_{k+1}, \ldots, x_{i+j-1}) = 0, \tag{2}$$

where $x_\cdot \in V$, $\sigma \in S_{k-1}$ is a $(k-j, j-1)$-unshuffle [7], i.e.,

$$\sigma(1) < \cdots < \sigma(k-j), \qquad \sigma(k+1-j) < \cdots < \sigma(k-1),$$

and $\chi(\sigma)$ is an anti-Koszul sign, $\chi(\sigma) := \mathrm{sgn}(\sigma)\epsilon(\sigma)$.

An sh Lie algebra can be seen as a special sh Leibniz algebra whose structures $l_{i\geq 2}$ are skewsymmetric.

Let $(V, \{,\})$ be a Leibniz algebra. We define an i-ary bracket associated with the Leibniz bracket as

$$N_i(x_1, \ldots, x_i) := \{\ldots\{\{x_1, x_2\}, x_3\}, \ldots, x_i\}.$$

It is well-known that N_i satisfies an i-ary Leibniz identity, the so-called Nambu–Leibniz identity (cf. [2]). Hence we denote the higher bracket by $N_\cdot$. Let $\mathrm{Der}(V)$ be the space of derivations on the Leibniz algebra. For any $D \in \mathrm{Der}(V)$, we define a multilinear map as

$$N_i D := N_i\left(D \otimes \overbrace{1 \otimes \cdots \otimes 1}^{i-1}\right),$$

or equivalently, $N_i D(x_1, \ldots, x_i) = \{\ldots\{\{D(x_1), x_2\}, x_3\}, \ldots, x_i\}$, in particular, $N_1 D := D$.

Let $\delta_0 \in \mathrm{Der}(V)$ be a differential on the Leibniz algebra. We consider a formal deformation of δ_0,

$$\delta_t := \delta_0 + t\delta_1 + t^2\delta_2 + \cdots.$$

The deformation δ_t is a differential on $V[[t]]$, which is a Leibniz algebra of formal series with coefficients in V. The differential condition $\delta_t^2 = 0$ is equivalent to the following condition,

$$\sum_{i+j=\mathrm{Const}} \delta_i \delta_j = 0. \tag{3}$$

Definition 3.2. We define an i-ary derived bracket on sV as

$$l_i := (-1)^{\frac{(i-1)(i-2)}{2}} s \circ N_i \circ \mathbf{s}^{-1}(i) \circ (s\delta_{i-1}s^{-1} \otimes \mathbf{1}),$$

where $\mathbf{s}^{-1}(i) = \overbrace{s^{-1} \otimes \cdots \otimes s^{-1}}^{i}$, $\mathbf{1} = \overbrace{1 \otimes \cdots \otimes 1}^{i-1}$.

It is obvious that the degree of the i-ary derived bracket is $2-i$ for each $i \geq 1$. We see an explicit expression of the higher derived bracket.

Proposition 3.3. *For each $i \geq 1$, the higher derived bracket has the following form on V,*

$$(\pm)\{\ldots\{\{\delta_{i-1}x_1, x_2\}, x_3\}, \ldots, x_i\} = s^{-1} l_i(sx_1, \ldots, sx_i),$$

where

$$\pm = \begin{cases} (-1)^{x_1+x_3+\cdots+x_{2n+1}+\cdots} & i = \mathit{even}, \\ (-1)^{x_2+x_4+\cdots+x_{2n}+\cdots} & i = \mathit{odd}. \end{cases}$$

Proof.

$$\begin{aligned} l_i(sx_1, \ldots, sx_i) &= (-1)^{\frac{(i-1)(i-2)}{2}} s \circ N_i \circ \mathbf{s}^{-1}(i) \circ (s\delta_{i-1}s^{-1} \otimes \mathbf{1})(sx_1 \otimes \cdots \otimes sx_i) \\ &= (\pm)(-1)^{\frac{(i-1)(i-2)}{2}} s \circ N_i \circ \mathbf{s}^{-1}(i) \circ (s\delta_{i-1}s^{-1} \otimes \mathbf{1}) \circ \mathbf{s}(i)(x_1 \otimes \cdots \otimes x_i) \\ &= (\pm)(-1)^{\frac{(i-1)(i-2)}{2}} s \circ N_i \circ \mathbf{s}^{-1}(i) \circ (s\delta_{i-1} \otimes \mathbf{s}(i-1))(x_1 \otimes \cdots \otimes x_i) \\ &= (\pm)(-1)^{\frac{(i-1)(i-2)}{2}} (-1)^{(i-1)} s \circ N_i \circ \mathbf{s}^{-1}(i) \circ \mathbf{s}(i)(\delta_{i-1}x_1 \otimes \cdots \otimes x_i) \\ &= (\pm)(-1)^{\frac{(i-1)(i-2)}{2}} (-1)^{(i-1)} (-1)^{\frac{i(i-1)}{2}} s \circ N_i(\delta x_1 \otimes \cdots \otimes x_i) \\ &= (\pm)s\{\ldots\{\{\delta_{i-1}x_1, x_2\}, x_3\}, \ldots, x_i\}, \end{aligned}$$

where $\mathbf{s}(i) := s \otimes \cdots \otimes s$ (i-times). $\square$

The main result of this note is as follows.

Theorem 3.4. *The system $(sV, l_1, l_2, l_3, \ldots)$ associated with the higher derived brackets defined in Definition 3.2 forms an sh Leibniz algebra.*

We will give a proof of the theorem in the next section. We consider the cases of dg Lie algebras.

Corollary 3.5. *Assume that in Theorem 3.4 V is a Lie algebra. Let $\mathfrak{g}$ be an abelian subalgebra of the Lie algebra. If $s\mathfrak{g}$ is a subalgebra of the induced sh Leibniz algebra, then $s\mathfrak{g}$ becomes an sh Lie algebra.*

Example 3.6 (*Deformation Theory, cf. [3]*)**.** Let $(V, \delta_0, [,])$ be a dg Lie algebra with a Maurer–Cartan (MC) element $\theta_t := t\theta_1 + t^2\theta_2 + \cdots$, which is a solution of the MC-equation:

$$\delta_0\theta_t + \frac{1}{2}[\theta_t, \theta_t] = 0.$$

We put $\delta_i(-) := [\theta_i, -]$ for each $i \geq 1$. Then the collection $\{\delta_i\}$ satisfies Eq. (3) because θ_t is a solution of the MC-equation. Therefore an algebraic deformation theory admits an sh Leibniz algebra structure, via the higher derived bracket construction.

4. Proof of Theorem 3.4

The theorem is given as a corollary of the key lemma (Lemma 4.2 below). To state the lemma, we recall an alternative definition of sh Leibniz algebra.

4.1. Sh Leibniz algebras (cf. [1])

We recall the notion of dual-Leibniz coalgebra [10,11]. A dual-Leibniz coalgebra is, by definition, a (graded) vector space equipped with a comultiplication, Δ, satisfying the identity below.

$$(1 \otimes \Delta)\Delta = (\Delta \otimes 1)\Delta + ((12) \otimes 1)(\Delta \otimes 1)\Delta,$$

where $(12) \in S_2$. We consider the tensor space over a graded vector space:

$$\bar{T}V := V \oplus V^{\otimes 2} \oplus V^{\otimes 3} \oplus \cdots.$$

Define a comultiplication, $\Delta : \bar{T}V \to \bar{T}V \otimes \bar{T}V$, by $\Delta(V) := 0$ and

$$\Delta(x_1, \ldots, x_{n+1}) := \sum_{i=1}^{n}\sum_{\sigma} \epsilon(\sigma)(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(i)}) \otimes (x_{\sigma(i+1)}, \ldots, x_{\sigma(n)}, x_{n+1}),$$

where $\epsilon(\sigma)$ is a Koszul sign, σ is an $(i, n-i)$-unshuffle and $(x_1, \ldots, x_{n+1}) \in V^{\otimes(n+1)}$. Then the pair $(\bar{T}V, \Delta)$ becomes the cofree nilpotent dual-Leibniz coalgebra over V.

Let $\mathrm{Coder}(\bar{T}V)$ be the space of coderivations on the coalgebra, i.e., $D^c \in \mathrm{Coder}(\bar{T}V)$ satisfies

$$\Delta D^c = (D^c \otimes 1)\Delta + (1 \otimes D^c)\Delta.$$

By a standard argument, we have $\mathrm{Coder}(\bar{T}V) \cong \mathrm{Hom}(\bar{T}V, V)$ (cf. [14]). We recall an explicit formula of the isomorphism. Let $f : V^{\otimes i} \to V$ be an i-ary linear map. It is one of the generators in $\mathrm{Hom}(\bar{T}V, V)$. The coderivation associated with f is defined by $f^c(V^{\otimes n<i}) := 0$ and

$$f^c(x_1, \ldots, x_{n\geq i}) := \sum_{k=i}^{n}\sum_{\sigma} \epsilon(\sigma)(-1)^{|f|(x_{\sigma(1)}+\cdots+x_{\sigma(k-i)})}(x_{\sigma(1)}, \ldots, x_{\sigma(k-i)}, f(x_{\sigma(k+1-i)}, \ldots, x_{\sigma(k-1)}, x_k), x_{k+1}, \ldots, x_n),$$

where σ is a $(k-i, i-1)$-unshuffle. The inverse of the mapping $f \mapsto f^c$ is the (co)restriction.

If $f, g \in \mathrm{Hom}(\bar{T}V, V)$ are i-ary, j-ary multilinear maps respectively, then

$$[f^c, g^c] = (f, g)^c,$$

where $[f^c, g^c]$ is the canonical commutator (Lie bracket) on $\mathrm{Coder}(\bar{T}V)$ and where (f, g) is an $(i+j-1)$-ary multilinear map. Since the mapping $f \mapsto f^c$ is an isomorphism, (f, g) defines a Lie bracket on $\mathrm{Hom}(\bar{T}V, V)$.

In the sequel, we will identify $\mathrm{Coder}(\bar{T}V)$ with $\mathrm{Hom}(\bar{T}V, V)$ as a Lie algebra. We sometimes omit the superscript "c" from f^c.

Given a graded vector space V, an i-ary i-multilinear V-valued operation l_i on V of degree $2-i$ determines a degree 1 element in $\mathrm{Hom}(\bar{T}sV, sV)$. The following proposition provides an alternative definition of sh Leibniz algebras.

Proposition 4.1 ([1]). *Let V be a graded vector space endowed with a system $\{l_i\}_{i\in\mathbb{N}}$ of i-ary i-multilinear V-valued operations, the operation $\{l_i\}_{i\in\mathbb{N}}$ having degree $2-i$ and, for each $i \geq 1$, let*

$$\partial_i := s^{-1} \circ l_i \circ (s \otimes \cdots \otimes s),$$

by construction of degree $+1$, viewed as a member of $\mathrm{Coder}(\bar{T}V)$ via the identifications $\bar{T}V \cong \bar{T}s^{-1}(sV)$. Define the coderivation ∂ by

$$\partial := \partial_1 + \partial_2 + \cdots .$$

The system $(sV, l_1, l_2, \ldots)$ is an sh Leibniz algebra if and only if

$$\frac{1}{2}[\partial, \partial] = 0$$

or equivalently, $\partial\partial = 0$.

4.2. The key lemma

Let $(V, \{,\})$ be a Leibniz algebra. We consider a collection of maps:

$$\mathrm{Der}(V) \to \mathrm{Hom}(\bar{T}V, V) \cong \mathrm{Coder}(\bar{T}V), \qquad D \mapsto N_i D \cong N_i^c D,$$

where $N_i D$ was defined in Section 3 and where $N_\cdot^c D$ is the coderivation associated with $N_\cdot D$. Theorem 3.4 is a consequence of the following

Lemma 4.2. *For any derivations $D, D' \in \mathrm{Der}(V)$ and for any $i, j \geq 1$, the following identity holds.*

$$N_{i+j-1}[D, D'] = (N_i D, N_j D'),$$

or equivalently,

$$N^c_{i+j-1}[D, D'] = [N_i^c D, N_j^c D'].$$

Proof. We show the case of $i = 1$. The general case will be shown in Section 6. We have

$$\begin{aligned} N_j[D, D'] &= N_j([D, D'] \otimes 1 \otimes \cdots \otimes 1) \\ &= N_j(DD' \otimes 1 \otimes \cdots \otimes 1) - (-1)^{DD'} N_j(D'D \otimes 1 \otimes \cdots \otimes 1). \end{aligned}$$

By the derivation property, we have

$$N_j(DD' \otimes 1 \otimes \cdots \otimes 1) = DN_j(D' \otimes 1 \otimes \cdots \otimes 1) - (-1)^{DD'} \sum_{k\geq 2}^{j} N_j(D' \otimes 1 \otimes \cdots \otimes 1 \otimes D^{(k)} \otimes 1 \otimes \cdots \otimes 1).$$

Hence we obtain

$$N_j[D, D'] = DN_j(D' \otimes 1 \otimes \cdots \otimes 1) - (-1)^{DD'} \sum_{k\geq 1}^{j} N_j(D' \otimes 1 \otimes \cdots \otimes 1 \otimes D^{(k)} \otimes 1 \otimes \cdots \otimes 1),$$

which is equal to $N_j[D, D'] = (N_1 D, N_j D')$ because $N_1 D = D$. □

The higher derived brackets are elements in $\mathrm{Hom}(\bar{T}sV, sV)$. Hence they correspond to the coderivations in $\mathrm{Coder}(\bar{T}V)$, via the maps,

$$\mathrm{Hom}(\bar{T}sV, sV) \overset{\text{shift}}{\sim} \mathrm{Hom}(\bar{T}V, V) \cong \mathrm{Coder}(\bar{T}V).$$

Lemma 4.3. *Let ∂_i be the coderivation associated with the i-ary derived bracket. It has the following form,*

$$\partial_i = N_i^c \delta_{i-1}.$$

Proof.

$$\begin{aligned} \partial_i &:= s^{-1} \circ l_i \circ (s \otimes \cdots \otimes s) \\ &= (-1)^{\frac{(i-1)(i-2)}{2}} N_i \circ (s^{-1} \otimes \cdots \otimes s^{-1}) \circ (s\delta_{i-1} \otimes s \otimes \cdots \otimes s) \\ &= (-1)^{\frac{(i-1)(i-2)}{2}} N_i \circ (\delta_{i-1} \otimes s^{-1} \otimes \cdots \otimes s^{-1}) \circ (1 \otimes s \otimes \cdots \otimes s) \\ &= N_i \delta_{i-1}. \end{aligned}$$

Hence $\partial_i = N_i^c \delta_{i-1}$ as a coderivation. □

Now, we give a proof of Theorem 3.4.

Proof. By Lemma 4.3, the differential $\delta_t = \sum t^i \delta_i$ corresponds to the coderivation:

$$\partial := \partial_1 + \partial_2 + \partial_3 + \cdots.$$

By Lemma 4.2, the deformation condition $[d, d] = 0$ corresponds to the homotopy algebra condition,

$$\sum_{i+j=\mathrm{Const}} [\partial_i, \partial_j] = \sum_{i+j=\mathrm{Const}} [N_i^c \delta_{i-1}, N_j^c \delta_{j-1}] = N_{i+j-1}^c \sum_{i+j=\mathrm{Const}} [\delta_{i-1}, \delta_{j-1}] = 0. \quad \square$$

Remark 4.4 (*cf. Lemma 4.2*)**.** We consider the case of the trivial deformation, that is, $\delta_t = t\delta_1$. In this case, the induced sh Leibniz algebra is an ordinary Leibniz algebra. We put $CL^n(sV) := \mathrm{Hom}(V^{\otimes n}, V)$ and $b(-) := (\partial_2, -)$. Then $(CL^*(sV), b)$ is the Leibniz cohomology complex [9]. The key Lemma implies that $\mathrm{Der}(V)$ provides a subcomplex of the Leibniz complex:

$$N_i \mathrm{Der}(V) \subset CL^i(sV),$$

because $(\partial_2, N_i D) = (N_2 \delta_1, N_i D) = N_{i+1}[\delta_1, D]$. If δ_1 is an adjoint representation, i.e., $\delta_1 := ad(\theta) := [\theta, -]$ for some $\theta \in V$, then $N_i ad(V)$ is also a subcomplex,

$$N_i ad(V) \subset N_i \mathrm{Der}(V) \subset CL^i(sV).$$

5. Deformation theory

In this section, we discuss the connection between deformation theory and sh Leibniz algebras. The deformation δ_t is considered to be a differential on $V[[t]]$, which is a Leibniz algebra of formal series with coefficients in V. Let $t\xi_1 \in \mathrm{Der}(V[[t]])$ be a derivation with the degree 0. We consider a transformation,

$$\delta'_t := \exp(X_{t\xi_1})(\delta_t),$$

where $X_{t\xi_1} := [\cdot, t\xi_1]$. By a standard argument, δ'_t is also a deformation of δ_0. We have

$$\begin{aligned}
\delta'_0 &= \delta_0,\\
\delta'_1 &= \delta_1 + [\delta_0, \xi_1],\\
\delta'_2 &= \delta_2 + [\delta_1, \xi_1] + \frac{1}{2!}[[\delta_0, \xi_1], \xi_1],\\
\cdots &\cdots\cdots\\
\delta'_i &= \sum_{n=0}^{i} \frac{1}{(i-n)!} X_{\xi_1}^{i-n}(\delta_n).
\end{aligned}$$

The collection $\{\delta'_i\}_{i\in\mathbb{N}}$ induces an sh Leibniz algebra structure $\partial' = \sum \partial'_i$, via the higher derived bracket construction. From Lemmas 4.2, 4.3, we have

$$\partial'_{i+1} = N_{i+1}^c \delta'_i = \sum_{n=0}^{i} \frac{1}{(i-n)!} X_{N_2^c \xi_1}^{i-n}(\partial_{n+1}).$$

Therefore we obtain

$$\partial' = \exp(X_{N_2^c \xi_1})(\partial),$$

which implies that ∂' is equivalent to ∂. We consider a general case. Let $\xi_t := t\xi_1 + t^2\xi_2 + \cdots$ be a derivation on $V[[t]]$ with degree 0. The transformation (4) below is called a *gauge transformation*.

$$\delta'_t := \exp(X_{\xi_t})(\delta_t). \tag{4}$$

Proposition 5.1. (I) *If two deformations of δ_0 are gauge equivalent, or related via the gauge transformation, then the induced sh Leibniz algebra structures are equivalent to each other, i.e., the codifferential ∂' induced by δ'_t is related with ∂ via the transformation,*

$$\partial' = \exp(X_{\Xi})(\partial), \tag{5}$$

where Ξ is a coderivation,

$$\Xi := N_2^c \xi_1 + N_3^c \xi_2 + \cdots + N_{i+1}^c \xi_i + \cdots.$$

(II) *The exponential of Ξ,*

$$e^{\Xi} := 1 + \Xi + \frac{1}{2!}\Xi^2 + \cdots,$$

is a dg coalgebra isomorphism between $(\bar{T}V, \partial)$ and $(\bar{T}V, \partial')$, namely, (6) *and* (7) *below hold.*

$$\partial' = e^{-\Xi} \cdot \partial \cdot e^{\Xi}, \tag{6}$$

$$\Delta e^{\Xi} = (e^{\Xi} \otimes e^{\Xi})\Delta. \tag{7}$$

The notion of sh Leibniz algebra homomorphism is defined to be a map satisfying (6) and (7). Thus (II) says that e^{Ξ} is an sh Leibniz algebra isomorphism.

Proof. (I) From (4) we have

$$\delta'_n = \delta_n + \sum_{n=i+j}[\delta_i, \xi_j] + \frac{1}{2!}\sum_{n=i+j+k}[[\delta_i, \xi_j], \xi_k] + \cdots.$$

Hence we obtain

$$\begin{aligned}\partial'_{n+1} &= N^c_{n+1}\delta'_n = N^c_{n+1}\delta_n + \sum_{n=i+j} N^c_{n+1}[\delta_i, \xi_j] + \frac{1}{2!}\sum_{n=i+j+k} N^c_{n+1}[[\delta_i, \xi_j], \xi_k] + \cdots \\ &= \partial_{n+1} + \sum_{n=i+j}[\partial_{i+1}, N^c_{j+1}\xi_j] + \frac{1}{2!}\sum_{n=i+j+k}[[\partial_{i+1}, N^c_{j+1}\xi_j], N^c_{k+1}\xi_k] + \cdots.\end{aligned}$$

This gives (5).

(II) The exponential e^{Ξ} is well-defined as an isomorphism on $\bar{T}V$, because e^{Ξ} is finite on $V^{\otimes n}$ for each n. For instance, on $V^{\otimes 3}$,

$$e^{\Xi} \equiv 1 + (N^c_2\xi_1 + N^c_3\xi_2) + \frac{1}{2}(N^c_2\xi_1)^2.$$

By a direct computation, one can prove that

$$\exp(X_{\Xi})(\partial) = e^{-\Xi}\cdot\partial\cdot e^{\Xi}.$$

Thus (6) holds. Since Ξ is a coderivation, e^{Ξ} satisfies (7). □

6. Proof of Lemma 4.2

Claim 6.1. *Let $f : V^{\otimes i} \to V$ be an i-ary linear map. For each n, we define $f^{(k)} : V^{\otimes n} \to V^{\otimes(n-i+1)}$ by*

$$f^{(k)}(x_1, \ldots, x_n) := \sum_{\sigma}\epsilon(\sigma)(-1)^{|f|(x_{\sigma(1)}+\cdots+x_{\sigma(k-i)})}(x_{\sigma(1)}, \ldots, x_{\sigma(k-i)}, f(x_{\sigma(k+1-i)}, \ldots, x_{\sigma(k-1)}, x_k), x_{k+1}, \ldots, x_n).$$

Then the coderivation associated with f decomposes as:

$$f^c = \sum_{k\geq i} f^{(k)}.$$

In Section 4.2, we established the lemma for $i = 1$. We assume the identity of the lemma and prove the case of $i + 1$, i.e., $N_{i+j}[D, D'] = (N_{i+1}D, N_jD')$, or equivalently, $N^c_{i+j}[D, D'] = [N^c_{i+1}D, N^c_jD']$.

We put $\mathbf{x} := (x_1, \ldots, x_{i+j-1})$. From the definition of $N.D$, we have

$$N^c_{i+j}[D, D'](\mathbf{x}, x_{i+j}) = \{N^c_{i+j-1}[D, D'](\mathbf{x}), x_{i+j}\}.$$

The assumption of the induction yields that

$$\begin{aligned}N^c_{i+j}[D, D'](\mathbf{x}, x_{i+j}) &= \{[N^c_iD, N^c_jD'](\mathbf{x}), x_{i+j}\} \\ &= \{N^c_iD\circ N^c_jD'(\mathbf{x}), x_{i+j}\} - (-1)^{DD'}\{N^c_jD'\circ N^c_iD(\mathbf{x}), x_{i+j}\}.\end{aligned}$$

Claim 6.1 derives

$$N^c_jD' = \sum_{k\geq j} N^{(k)}_j D',$$

which gives

$$N^c_{i+j}[D, D'](\mathbf{x}, x_{i+j}) = \sum_{k=j}^{i+j-1}\{N^c_iD\circ N^{(k)}_jD'(\mathbf{x}), x_{i+j}\} - (-1)^{DD'}\{N^c_jD'\circ N^c_iD(\mathbf{x}), x_{i+j}\}. \tag{8}$$

The first term of (8) becomes

$$\begin{aligned}\sum_{k=j}^{i+j-1}\{N^c_iD\circ N^{(k)}_jD'(\mathbf{x}), x_{i+j}\} &= \sum_{k=j}^{i+j-1} N^c_{i+1}D\circ N^{(k)}_jD'(\mathbf{x}, x_{i+j}) \\ &= N^c_{i+1}D\circ N^c_jD'(\mathbf{x}, x_{i+j}) - N^c_{i+1}D\circ N^{(i+j)}_jD'(\mathbf{x}, x_{i+j}),\end{aligned}$$

because the coderivation preserves the position of the most right component x_{i+j}. So it suffices to show that

$$-(-1)^{DD'}\{N^c_jD'\circ N^c_iD(\mathbf{x}), x_{i+j}\} = N^c_{i+1}D\circ N^{(i+j)}_jD'(\mathbf{x}, x_{i+j}) - (-1)^{DD'}N^c_jD'\circ N^c_{i+1}D(\mathbf{x}, x_{i+j}). \tag{9}$$

We need a lemma.

Lemma 6.2. *For any elements in the Leibniz algebra, $A, B, y_1, \ldots, y_n \in V$,*

$$N_{n+2}(A, B, y_1, \ldots, y_n) = -(-1)^{AB}\{B, N_{n+1}(A, y_1, \ldots, y_n)\} + \sum_{a=1}^{n}(-1)^{B(y_1+\cdots+y_{a-1})}N_{n+1}(A, y_1, \ldots, y_{a-1}, \{B, y_a\}, y_{a+1}, \ldots, y_n).$$

Proof. We show the case of $n = 2$. Up to sign,

$$\begin{aligned}\{B, \{\{A, y_1\}, y_2\}\} &= \{\{B, \{A, y_1\}\}, y_2\} + \{\{A, y_1\}, \{B, y_2\}\} \\ &= \{\{\{B, A\}, y_1\}, y_2\} + \{\{A, \{B, y_1\}\}, y_2\} + \{\{A, y_1\}, \{B, y_2\}\} \\ &= -\{\{\{A, B\}, y_1\}, y_2\} + \{\{A, \{B, y_1\}\}, y_2\} + \{\{A, y_1\}, \{B, y_2\}\},\end{aligned}$$

where $-\{\{A, B\}, y_1\} = \{\{B, A\}, y_1\}$ is used. Thus we obtain

$$\{B, N_3(A, y_1, y_2)\} = -N_4(A, B, y_1, y_2) + N_3(A, \{B, y_1\}, y_2) + N_3(A, y_1, \{B, y_2\}). \quad \square$$

We prove (9). By the definition of coderivation,

$$N_i^c D(\mathbf{x}) = \sum_{k=i}^{i+j-1}\sum_{\sigma} E(\sigma, k-i)(x_{\sigma(1)}, \ldots, x_{\sigma(k-i)}, N_i(Dx_{\sigma(k+1-i)}, \ldots, x_{\sigma(k-1)}, x_k), x_{k+1}, \ldots, x_{i+j-1}),$$

where

$$E(\sigma, *) := \epsilon(\sigma)(-1)^{D(x_{\sigma(1)}+\cdots+x_{\sigma(*)})}.$$

Since $N_n(x_1, \ldots, x_n) = \{\{\{x_1, x_2\}, \ldots, \}, x_n\}$,

$$N_n(x_1, \ldots, x_n) = N_{n-i+1}(N_i(x_1, \ldots, x_i), x_{i+1}, \ldots, x_n),$$

which gives

$$S := -(-1)^{DD'}\{N_j^c D' \circ N_i^c D(\mathbf{x}), x_{i+j}\} = -(-1)^{DD'}\sum_{k=i}^{i+j-1}\sum_{\sigma} E(\sigma, k-i)$$

$$N_{i+j-k+2}\big(N_{k-i}(D'x_{\sigma(1)}, \ldots, x_{\sigma(k-i)}), N_i(Dx_{\sigma(k+1-i)}, \ldots, x_{\sigma(k-1)}, x_k), x_{k+1}, \ldots, x_{i+j}\big).$$

We put $A := N_{k-i}(D'x_{\sigma(1)}, \ldots, x_{\sigma(k-i)})$ and $B := N_i(Dx_{\sigma(k+1-i)}, \ldots, x_{\sigma(k-1)}, x_k)$, then from Lemma 6.2,

$$S = T + U, \tag{10}$$

where

$$T := -(-1)^{DD'}\sum_{k=i}^{i+j-1}\sum_{\sigma} E(\sigma, k-i)E_1 N_{i+1}\big(Dx_{\sigma(k+1-i)}, \ldots, x_{\sigma(k-1)}, x_k, N_j(D'x_{\sigma(1)}, \ldots, x_{\sigma(k-i)}, x_{k+1}, \ldots, x_{i+j})\big),$$

$$\begin{aligned}U := &-(-1)^{DD'}\sum_{k=i}^{i+j-1}\sum_{\sigma}\sum_{a=1}^{i+j-k} E(\sigma, k-i)E_2 \\ &\times N_j(D'x_{\sigma(1)}, \ldots, x_{\sigma(k-i)}, x_{k+1}, \ldots, x_{k+a-1}, N_{i+1}(Dx_{\sigma(k+1-i)}, \ldots, x_{\sigma(k-1)}, x_k, x_{k+a}), x_{k+a+1}, \ldots, x_{i+j}),\end{aligned}$$

where E_1 and E_2 are appropriate signs given by the manner in the lemma above.

(I) We show the identity,

$$T = N_{i+1}^c D \circ N_j^{(i+j)} D'(\mathbf{x}, x_{i+j}). \tag{11}$$

We replace σ in T with an unshuffle permutation τ along the table,

$\sigma(k+1-i)$	$\cdots$	$\sigma(k-1)$	k	$\sigma(1)$	$\cdots$	$\sigma(k-i)$
$\tau(1)$	$\cdots$	$\tau(i-1)$	$\tau(i)$	$\tau(i+1)$	$\cdots$	$\tau(k)$

Then the Koszul sign is replaced with $\epsilon(\tau)$:

$$\epsilon(\tau) = \epsilon(\sigma)(-1)^{(x_{\sigma(1)}+\cdots+x_{\sigma(k-i)})(x_{\sigma(k+1-i)}+\cdots+x_{\sigma(k-1)}+x_k)},$$

and then

$$\begin{aligned}E(\sigma, k-i)E_1 &= -\epsilon(\sigma)(-1)^{D(x_{\sigma(1)}+\cdots+x_{\sigma(k-i)})}(-1)^{AB}\\ &= -\epsilon(\sigma)(-1)^{D(x_{\sigma(1)}+\cdots+x_{\sigma(k-i)})}(-1)^{(x_{\sigma(1)}+\cdots+x_{\sigma(k-i)}+D')(x_{\sigma(k+1-i)}+\cdots+x_{\sigma(k-1)}+x_k+D)}\\ &= -\epsilon(\sigma)(-1)^{(x_{\sigma(1)}+\cdots+x_{\sigma(k-i)})(x_{\sigma(k+1-i)}+\cdots+x_{\sigma(k-1)}+x_k)}(-1)^{D'(x_{\sigma(k+1-i)}+\cdots+x_{\sigma(k-1)}+x_k)+DD'}\\ &= -\epsilon(\tau)(-1)^{D'(x_{\sigma(k+1-i)}+\cdots+x_{\sigma(k-1)}+x_k)+DD'}\\ &= -\epsilon(\tau)(-1)^{D'(x_{\tau(1)}+\cdots+x_{\tau(i-1)}+x_{\tau(i)})+DD'} = -E'(\tau, i)(-1)^{DD'}.\end{aligned}$$

Thus T is equal to

$$T' := \sum_{k=i}^{i+j-1}\sum_{\tau} E'(\tau, i)N_{i+1}\big(Dx_{\tau(1)}, \ldots, x_{\tau(i-1)}, x_{\tau(i)=k}, N_j(D'x_{\tau(i+1)}, \ldots, x_{\tau(k)}, x_{k+1}, \ldots, x_{i+j})\big),$$

where τ is an $(i, k-i)$-unshuffle such that $\tau(i) = k$.

Claim 6.3. *$T' = T''$, where*

$$T'' := \sum_{\nu} E'(\nu, i)N_{i+1}\big(Dx_{\nu(1)}, \ldots, x_{\nu(i-1)}, x_{\nu(i)}, N_j(D'x_{\nu(i+1)}, \ldots, x_{\nu(i+j-1)}, x_{i+j})\big),$$

where ν is an $(i, j-1)$-unshuffle.

Proof. We put $k := \nu(i)$ in T''. Since ν is an $(i, j-1)$-unshuffle, $i \le k \le i+j-1$. Replace ν with τ. This replacement preserves the order of variables. Hence $E'(\tau, i) = E'(\nu, i)$, which gives the identity of the claim. □

Since $T'' = N^c_{i+1}D \circ N^{(i+j)}_j D'(\mathbf{x}, x_{i+j})$, we obtain (11).

(II) We show the identity,

$$U = -(-1)^{DD'}N^c_j D' \circ N^c_{i+1}D(\mathbf{x}, x_{i+j}). \tag{12}$$

We replace σ in U with an unshuffle permutation τ along the table,

$\sigma(1)$	$\cdots$	$\sigma(k-i)$	$k+1$	$\cdots$	$k+a-1$
$\tau(1)$	$\cdots$	$\tau(k-i)$	$\tau(k+1-i)$	$\cdots$	$\tau(k+a-1-i)$
$\sigma(k+1-i)$	$\cdots$	$\sigma(k-1)$	k		
$\tau(k+a-i)$	$\cdots$	$\tau(k+a-2)$	$\tau(k+a-1)$		

Then the Koszul sign is replaced with $\epsilon(\tau)$:

$$\epsilon(\tau) = \epsilon(\sigma)(-1)^{(x_{\sigma(k+1-i)}+\cdots+x_{\sigma(k-1)}+x_k)(x_{k+1}+\cdots+x_{k+a-1})},$$

and then

$$\begin{aligned}E(\sigma, k-i)E_2 &= \epsilon(\sigma)(-1)^{D(x_{\sigma(1)}+\cdots+x_{\sigma(k-i)})}(-1)^{B(x_{k+1}+\cdots+x_{k+a-1})}\\ &= \epsilon(\sigma)(-1)^{D(x_{\sigma(1)}+\cdots+x_{\sigma(k-i)})}(-1)^{(x_{\sigma(k+1-i)}+\cdots+x_{\sigma(k-1)}+x_k+D)(x_{k+1}+\cdots+x_{k+a-1})}\\ &= \epsilon(\sigma)(-1)^{(x_{\sigma(k+1-i)}+\cdots+x_{\sigma(k-1)}+x_k)(x_{k+1}+\cdots+x_{k+a-1})}(-1)^{D(x_{\sigma(1)}+\cdots+x_{\sigma(k-i)}+x_{k+1}+\cdots+x_{k+a-1})}\\ &= \epsilon(\tau)(-1)^{D(x_{\tau(1)}+\cdots+x_{\tau(k+a-1-i)})}\\ &= E(\tau, k+a-1-i) = E(\tau, m-i),\end{aligned}$$

where $m := k+a-1$.

Claim 6.4. *$U = U'$, where*

$$\begin{aligned}U' &:= -(-1)^{DD'}\sum_{m=i}^{i+j-1}\sum_{\tau} E(\tau, m-i)\\ &\qquad N_j(D'x_{\tau(1)}, \ldots, x_{\tau(m-i)}, N_{i+1}(Dx_{\tau(m+1-i)}, \ldots, x_{\tau(m)}, x_{m+1}), x_{m+2}, \ldots, x_{i+j}),\end{aligned}$$

where τ is an $(m-i, i)$-unshuffle.

Proof. Let τ be an $(m-i, i)$-unshuffle. We put $k := \tau(m)$ and $a := m+1-\tau(m)$. Then we have

$$\begin{aligned}&(\tau(1), \dots, \tau(m-i); \tau(m+1-i), \dots, \tau(m), m+1, \dots, i+j)\\&\quad = (\tau(1), \dots, \tau(k-i), k+1, \dots, k+a-1; \tau(k+a-i), \dots, \tau(k+a-2), k, k+a, \dots, i+j).\end{aligned}$$

One can replace τ with an unshuffle σ,

$$\begin{aligned}&(\tau(1), \dots, \tau(m-i); \tau(m+1-i), \dots, \tau(m), m+1, \dots, i+j)\\&\quad = (\sigma(1), \dots, \sigma(k-i), k+1, \dots, k+a-1; \sigma(k+1-i), \dots, \sigma(k-1), k, k+a, \dots, i+j),\end{aligned}$$

which gives the table above. Up to this permutation, we obtain

$$\sum_{\tau} = \sum_{(k,a)} \sum_{\sigma}$$

where $(m-i, m)$ is fixed and (k, a) runs over all possible pairs. This gives

$$\sum_{m \ge i} \sum_{\tau} = \sum_{k \ge i} \sum_{a \ge 1} \sum_{\sigma},$$

which implies the identity of the claim. □

Since $U' = -(-1)^{DD'} N_j^c D' \circ N_{i+1}^c D(\mathbf{x}, x_{i+j})$, we obtain (12). From (10)–(12), we get the desired identity (9). The proof is completed.

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