Linear Preservers on Powers of Matrices

Gin-Hor Chan  
Department of Mathematics  
National University of Singapore  
Singapore

and

Ming-Huat Lim  
Department of Mathematics  
University of Malaya  
Malaysia

Submitted by F. Uhlig

ABSTRACT

This paper is concerned with linear maps $L$ on $n \times n$ matrices such that (i) $L(A^k) = L(A)^k$ for all $A$, where $k$ is a fixed integer $\geq 2$; or (ii) $L$ preserves idempotent or tripotent matrices, respectively.

1. INTRODUCTION

Let $F$ be a field. Let $M_n(F)$ and $S_n(F)$ be the vector spaces of all $n \times n$ matrices and symmetric matrices over $F$ respectively. Let $k$ be a fixed integer greater than one. In this paper we characterize linear maps $L$ on $M_n(F)$ that satisfy $L(A^k) = L(A)^k$ for all $A$ when char $F = 0$ or char $F > k$. For linear maps $L$ on $S_n(F)$ we obtain a similar result when $F$ is algebraically closed. For $k = 2, 3$, we need only the weaker hypothesis that $L$ preserves idempotent and tripotent matrices respectively to obtain the structure of $L$. The proofs depend on the structure of linear maps that preserve rank less than or equal to one; see [6, 7]. The result for idempotent preservers on $M_n(F)$ was obtained independently by Beasley and Pullman in [1].
2. LINEAR MAPS THAT COMPUTE WITH THE $k$TH POWER FUNCTION

In this section we consider linear maps $L$ on $M_n(F)$ and $S_n(F)$ such that $L(A^k) = L(A)^k$, where $k$ is a fixed integer greater than one. We shall need the following two simple but interesting results.

**Lemma 1.** Let $L$ be a linear map on $M_n(F)$, where $|F| \geq 3$. If $L$ sends rank one idempotents to rank one or zero matrices, then rank $A \leq 1$ implies that rank $L(A) \leq 1$.

**Proof.** Let $A$ be a rank one matrix which is not a scalar multiple of an idempotent. Then there exists a nonsingular matrix $Q$ such that

$$A = Q \begin{pmatrix} u \\ 0 \\ \vdots \\ 0 \end{pmatrix} Q^{-1},$$

where $u = (\lambda_1, \ldots, \lambda_n)$ is a nonzero vector. Thus $A^2 = \lambda_1 A$, and therefore we must have $\lambda_1 = 0$. Let $B = QE_11Q^{-1}$. Then $B^2 = B$ and hence rank $L(B) \leq 1$. For any nonzero $\lambda \in F$, the matrix $A + \lambda B$ is a scalar multiple of an idempotent. It follows that $L(A) + \lambda L(B) = L(A + \lambda B)$ is of rank one or zero. As $|F| \geq 3$, this is only possible if rank $L(A) \leq 1$. \[\square\]

**Lemma 2.** Let $L$ be a linear map on $S_n(F)$, where $|F| \geq 5$. If $L$ sends rank one idempotents to rank one or zero matrices, then rank $A \leq 1$ implies that rank $L(A) \leq 1$.

**Proof.** Let $A$ be any rank one matrix in $S_n(F)$ which is not a scalar multiple of an idempotent. Then $A = cu^t u$ for some nonzero scalar $c$ and some nonzero row vector $u = (u_1, \ldots, u_n)$ with $uu^t = 0$. We may assume $u_1 \neq 0$ and $c = 1$. Let $0 \neq \lambda \in F$ be such that $2u_1 + \lambda \neq 0$. Let $B_{\lambda} = (\lambda + u_1, u_2, \ldots, u_n)^t (\lambda + u_1, u_2, \ldots, u_n)$. Then $B_{\lambda}^2 = \lambda(2u_1 + \lambda) B_{\lambda}$ and hence $B_{\lambda}$ is a scalar multiple of idempotent. Note that $B_{\lambda} = A + \lambda M_{\lambda}$ for some symmetric matrix $M_{\lambda}$. Hence $L(A) + \lambda L(M_{\lambda})$ is of rank one or zero. By looking at the $2 \times 2$ minors of $L(A) + \lambda L(M_{\lambda})$, we see that every $2 \times 2$ minor of $L(A)$ has determinant equal to zero, since $|F| \geq 5$. Hence rank $L(A) \leq 1$. \[\square\]
An $n \times n$ matrix $A$ is called $k$-potent if $A^k = A$. Throughout this section, we assume that $k \geq 2$.

**Lemma 3.** Let $|F| > k$, and $L$ be a nonzero linear map on $M_n(F)$ preserving $k$-potent matrices. Then $L(E_{ii}) \neq 0$ for all $i$.

**Proof.** Suppose that $L(E_{ii}) = 0$ for some $i$. Then for any $j \neq i$, that $E_{ii} + \lambda E_{ij}$ is $k$-potent implies that $L(E_{ii} + \lambda E_{ij}) = \lambda L(E_{ij})$ is $k$-potent for all $\lambda \in F$. Since $|F| > k$, $L(E_{ij}) = 0$. Similarly, we have $L(E_{ji}) = 0$, since $E_{ii} + \lambda E_{ji}$ is $k$-potent. On the other hand, for each $\beta \in F$,

$$(1 - \beta)E_{ii} + (\beta - \beta^2)E_{ij} + E_{ji} + \beta E_{jj}$$

is idempotent, and hence $\beta L(E_{jj})$ is $k$-potent. Hence $L(E_{jj}) = 0$. This again implies that $L(E_{js}) = 0$ for all $s \neq j$. Hence $L = 0$, a contradiction. This shows that $L(E_{ii}) \neq 0$ for all $i$. $\blacksquare$

If $L$ is a nonzero linear map on $M_n(F)$ such that $L(A^k) = L(A)^k$ for all $A$, where $|F| > k$, then $L$ preserves $k$-potent matrices and hence $L(E_{ii}) \neq 0$ for all $i$. For symmetric matrices, we have the following lemma.

**Lemma 4.** Let $\text{char } F = 0$ or $\text{char } F > k$. Let $L$ be a nonzero linear map from $S_n(F)$ into $M_n(F)$ such that $L(A^k) = L(A)^k$ for all $A$. Then $L(E_{ii}) \neq 0$ for all $i$.

**Proof.** Suppose that $L(E_{ss}) = 0$ for some $s$. For each $j \neq s$, let $B_j = E_{sj} + E_{js}$. Then $E_{ss}B_jE_{ss} = 0$, $B_jE_{ss}B_j = E_{jj}$, $E_{ss}B_j + B_jE_{ss} = B_j$, and

$$B_j^p = \begin{cases} E_{ss} + E_{jj} & \text{if } p \text{ is even}, \\ B_j & \text{if } p \text{ is odd}. \end{cases}$$

Therefore,

$$L\left( (E_{ss} + tB_j)^k \right) = L \left( E_{ss} + t \sum_{p=0}^{k-1} E_{ss}^p B_j E_{ss}^{k-p-1} + t^{k-1} \sum_{p=0}^{k-1} B_j^p E_{ss} B_j^{k-p-1} + \text{sum of other terms} \right)$$

$$= tL(B_j) + t^{k-1}L \left( \sum_{p=0}^{k-1} B_j^p E_{ss} B_j^{k-p-1} \right) + \text{sum of other terms},$$

$$\left[ L(E_{ss} + tB_j) \right]^k = t^k L(B_j)^k.$$
Thus by comparing the coefficients of $t$, we have $L(B_j) = 0$. If $k$ is even, $B_j^k = E_{ss} + E_{jj}$ and

$$L(E_{jj}) = L(E_{ss} + E_{jj}) = L(B_j^k) = L(B_j)^k = 0.$$ 

If $k$ is odd,

$$\sum_{p=0}^{k-1} B_j^p E_{ss} B_j^{k-p-1} = \left[\frac{k + 1}{2}\right] E_{ss} + \left[\frac{k - 1}{2}\right] E_{jj}.$$ 

Hence by comparing the coefficients of $t^{k-1}$, we also have $L(E_{jj}) = 0$. Repeating the same process on $E_{jj}$ for different $p$, we have $L(E_{pp}) = L(E_{jp} + E_{pj}) = 0$ for all $j$ and $p$. Hence $L$ is a zero map, a contradiction.

**Remark 1.** If $F$ is assumed to be algebraically closed and the hypothesis on $L$ is weakened to $L(A^k) = L(A)^k$ for all diagonalizable matrices $A$, then the same conclusion of Lemma 4 holds. One needs only to observe that $E_{ss} + tB_j$ is diagonalizable for infinitely many $t \in F$ in the proof of Lemma 4.

**Remark 2.** Lemma 4 also holds when $L$ is a linear map on the $n \times n$ complex Hermitian matrices. The same proof works if we let $B_j = E_{sj} + E_{js}$ or $i(E_{sj} - E_{js})$.

For convenience, we call two $n \times n$ matrices $X$ and $Y$ orthogonal if $XY = YX = 0$.

**Lemma 5.** Let $\text{char } F = 0$ or $\text{char } F > k$. Let $U$ be a subspace of $M_n(F)$ such that $A^k \in U$ for all $A \in U$, and $L$ a linear map from $U$ into $M_n(F)$ such that $L(A^k) = L(A)^k$ for all $A \in U$. Let $X, Y \in U$ be two orthogonal $k$-potent matrices. Then $L(X)$ and $L(Y)$ are orthogonal.

**Proof.** Consider

$$L\left((X + tY)^k\right) = L(X^k + t^k Y^k) = L(X + t^k Y) = L(X) + t^k L(Y),$$

$$L(X + tY)^k = [L(X) + tL(Y)]^k$$

$$= L(X) + t \sum_{p=0}^{k-1} L(X)^p L(Y) L(X)^{k-p-1} + \text{sum of other terms}.$$
Let \( B = \sum_{p=0}^{k-1} L(X)^p L(Y) L(X)^{k-p-1} \). Then a comparison of the coefficients of \( t \) shows that \( B = 0 \). Furthermore,

\[
L(X)B = BL(X) + L(X)^{k} L(Y) - L(Y) L(X)^{k}.
\]

Thus \( L(X)L(Y) = L(Y)L(X) \) and hence \( L(X)^{k-1} L(Y) = (1/k) B = 0 \). It follows that

\[
L(X)L(Y) = L(X)^{k} L(Y) = L(X)L(X)^{k-1} L(Y) = 0.
\]

This completes the proof.

**COROLLARY 1.** Let \( U \) and \( L \) be as in Lemma 5. Let \( X \) and \( Y \) be orthogonal idempotents in \( U \). Then \( L(X) \) and \( L(Y) \) are orthogonal.

**LEMMA 6.** Let \( L \) be a nonzero linear map as in Lemma 5. If \( U \) is the space \( S_n(F) \) or \( M_n(F) \), then \( L(E_{ii})^{k-1} \), \( i = 1, \ldots, n \), are mutually orthogonal nonzero idempotents and \( L(I_n)^{k-1} = I_n \).

**Proof.** For each \( i \),

\[
\left[ L(E_{ii})^{k-1} \right]^2 = L(E_{ii})^{2k-2} = L(E_{ii})^k L(E_{ii})^{k-2} = L(E_{ii}) L(E_{ii})^{k-2} = L(E_{ii})^k \frac{1}{k}.
\]

Thus \( L(E_{ii})^{k-1} \) is a nonzero idempotent by Lemma 3 and Lemma 4. By Corollary 1, \( L(E_{11}), \ldots, L(E_{nn}) \) are mutually orthogonal and so are \( L(E_{11})^{k-1}, \ldots, L(E_{nn})^{k-1} \). Therefore \( L(E_{11})^{k-1} + \cdots + L(E_{nn})^{k-1} = I_n \). Furthermore

\[
L(I_n)^{k-1} = \left[ L(E_{11}) + \cdots + L(E_{nn}) \right]^{k-1} = I_n.
\]

**THEOREM 1.** Let \( \text{char } F = 0 \) or \( \text{char } F > k \). Let \( L \) be a nonzero linear map on \( M_n(F) \) such that \( L(A^k) = L(A)^k \) for all \( A \). Then there exists a nonsingular matrix \( P \) such that

\[
L(A) = \lambda PAP^{-1} \quad \text{for all } A
\]

or

\[
L(A) = \lambda APA^{-1} \quad \text{for all } A,
\]

where \( \lambda^{k-1} = 1 \).
Proof. Let $A$ be a rank one idempotent. Then there exists a nonsingular matrix $N$ such that $A = NE_{11}N^{-1}$. Let $T$ be a linear map on $M_n(F)$ defined by

$$T(X) = L(NXN^{-1}).$$

Then $T$ satisfies the assumption of the theorem, and $T(E_{11}) = L(A)$. By Lemma 6, $T(E_{11})^k$ is of rank one and so is $L(A) = T(E_{11})^k$. Therefore, by Lemma 1, $L$ preserves rank less than or equal to one, and it follows from a result in [6] that either

$$L(A) = \begin{cases} PAQ & \text{for all } A \\ PA'Q & \text{for all } A \end{cases}$$

for some matrices $P$ and $Q$, or $\text{Im } L$ consists of rank one and zero matrices. Since $L(I_n)^{k-1} = I_n$, $L(A)$ is of form (1). Furthermore, $(PQ)^{k-1} = I_n$ implies that $P$ and $Q$ are nonsingular. We may assume $L(A) = PAQ$ for all $A$.

Let $QP = D = (d_{ij})$. By Lemma 5, for distinct $i$ and $j$,

$$d_{ij}(PE_{ij}Q) = (PE_{ii}Q)(PE_{jj}Q) = 0.\text{ Also we have } d_{ii}^{k-1}(PE_{ii}Q) = (PE_{ii}Q)^k = PE_{ii}Q \neq 0.$$

Thus $d_{ij} = 0$ and $d_{ii}^{k-1} = 1$. Hence $D = \text{diag}(d_{11}, \ldots, d_{nn})$. Now we take

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \oplus 0.$$

Then

$$A^kD = \begin{pmatrix} d_{11} & kd_{22} \\ 0 & d_{22} \end{pmatrix} \oplus 0$$

and

$$(AD)U = \begin{pmatrix} d_{11}^k & \sum_{p=0}^{k-1} d_{11}^p d_{22}^{k-p} \\ 0 & d_{22}^k \end{pmatrix} \oplus 0.$$
Since
\[ A^k D = P^{-1} (PA^k Q) P = P^{-1} (PAQ)^k P = (AD)^k, \]
we have
\[ k-1 \sum_{p=0}^{k-1} d_{11}^p d_{22}^{k-p} = kd_{22}. \]
Then
\[ \sum_{p=0}^{k} d_{11}^p d_{22}^{k-p} = kd_{22} + d_{11}. \]
Multiplying both sides by \( d_{11} - d_{22} \) to obtain
\[ d_{11}^{k+1} - d_{22}^{k+1} = (k - 1)d_{11}d_{22} + d_{11}^2 - kd_{22}^2. \]
As \( d_{ii} = d_{ii} \) for all \( i \), we therefore have \( (k - 1)d_{22}(d_{11} - d_{22}) = 0 \). Hence \( d_{11} = d_{22} \). Similarly we can show that \( d_{ii} = d_{jj} \) for all \( i, j \). This concludes the result.

**Remark 3.** In [5], Hua obtained the structure of bijective additive maps \( L \) on space of all \( n \times n \) matrices over a division ring such that \( L(I_n) = I_n \) and \( L(ABA) = L(A)L(B)L(A) \) for all \( A, B \).

**Theorem 2.** Let \( F \) be algebraically closed of characteristic 0 or greater than \( k \). Let \( L \) be a nonzero linear map on \( S_n(F) \) such that \( L(A^k) = L(A)^k \) for all \( A \). Then there exists an orthogonal matrix \( P \) such that
\[ L(A) = \lambda PAP^t \]
for all \( A \), where \( \lambda^{k-1} = 1 \).

**Proof.** Let \( A \) be a rank one idempotent in \( S_n(F) \). Then there exists an orthogonal matrix \( Q \) in \( M_n(F) \) such that \( QAQ^t = E_{11} \). Hence we may assume that \( A = E_{11} \). By Lemma 6, \( L(A) \) is of rank one, and hence by Lemma 2 and
Corollary 3 in [7], we have either

\[ L(A) = PAP^t \quad \text{for all } A \] (2)

for some matrix \( P \), or

\[ \text{Im } L = \langle M \rangle \]

for some rank one matrix \( M \) in \( S_n(F) \). Since \( L(I_n) \) is nonsingular by Lemma 6, \( L \) is of the form (2) and \( P \) is nonsingular.

Now let \( P^tP = D = (d_{ij}) \). Then for \( s \neq j \), Corollary 1 implies that

\[ d_{sj}(PE_{sj}P^t) = (PE_{ss}P^t)(PE_{jj}P^t) = 0. \]

Also we have

\[ d_{ss}^{-1}(PE_{ss}P^t) = (PE_{ss}P^t)' = PE_{ss}P^t \neq 0. \]

Thus \( d_{sj} = 0 \) and \( d_{ss}^{-1} = 1 \). Hence \( D = \text{diag}(d_{11}, \ldots, d_{nn}) \). Suppose \( d_{11} \neq d_{22} \). Let

\[ A = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} \neq 0, \quad \text{where } i^2 = -1. \]

Then \( A^2 = 0 \) and hence \( I(A)^k = I(A^k) = I(0) = 0 \). On the other hand \( L(A) = PADP^{-1} \), and

\[ AD = \begin{pmatrix} d_{11} & id_{22} \\ id_{11} & -d_{22} \end{pmatrix} \neq 0 \]

is not nilpotent (since \( d_{11} - d_{22} \neq 0 \)), which is a contradiction. Thus \( d_{11} = d_{22} \). Similarly \( d_{11} = d_{jj} \) for every \( j \), and hence \( P^tP = \alpha I_n \) with \( \alpha^{k-1} = 1 \). This completes the proof.

**Remark 4.** Theorem 1 and Theorem 2 are false if \( \text{char } F = p > 0 \) and \( k = p^s \) for some positive integer \( s \). Take \( V = S_n(F) \) or \( M_n(F) \). Let \( B \) be a fixed \( k \)-potent matrix in \( V \). Then the linear map \( L \) on \( V \) defined by

\[ L(A) = A + (\text{tr } A)I_n \quad \text{or} \quad L(A) = \langle \text{tr } A \rangle B \]

has the property that \( L(A^k) = L(A)^k \) for all \( A \) in \( V \), since \( (\text{tr } A)^k = \text{tr}(A^k) \).
3. LINEAR MAPS PRESERVING IDEMPOTENT OR TRIPOTENT MATRICES

In this section we shall characterize all linear maps \( L \) on \( n \times n \) matrices preserving \( k \)-potents, where \( k = 2 \) or 3.

**Lemma 7.** Let \( \text{char } F \neq 2, 3 \) and \( K = \{ A \in M_n(F) | A^3 = A \} \). Then for any \( A, B \in K \), one has \( A \pm B \in K \) if and only if \( AB = BA = 0 \).

**Proof.** Suppose that \( A \pm B \in K \). Then

\[
(A + B)^3 = A^3 + A^2B + BA^2 + ABA + AB^2 + B^2A + BAB + B^3
\]
\[
= A + B,
\]
\[
(A - B)^3 = A^3 - A^2B - BA^2 - ABA + AB^2 + B^2A + BAB - B^3
\]
\[
= A - B.
\]

Thus we have

\[
A^2B + BA^2 + ABA + AB^2 + B^2A + BAB = 0,
\]
\[
- A^2B - BA^2 - ABA + AB^2 + B^2A + BAB = 0.
\]

After adding and subtracting, we get respectively

\[
AB^2 + B^2A + BAB = 0,
\]
\[
A^2B + BA^2 + ABA = 0.
\]

Multiplying by \( X \) on the left and right of (3) for \( X = B \) and \( B^2 \), we have

\[
2BAB + B^2AB^2 = 0,
\]
\[
2B^2AB^2 + BAB = 0.
\]

Thus \( BAB = 0 \), and from (3),

\[
\]
Hence

$$BA = B^3A = B(B^2A) = B(-AB^2) = -BAB B = 0.$$ 

By a similar argument we can show from (4) that $AB = 0$.

The converse is obvious. 

**Lemma 8.** Let $L$ be a nonzero linear map on $M_n(F)$ preserving $k$-potent matrices, where $k = 2, 3$ and $\text{char } F = 0$ or $\text{char } F > k$. Then $L(E_{11}), \ldots, L(E_{nn})$ are mutually orthogonal rank one matrices.

**Proof.** Lemma 2 in [2] and Lemma 7 imply that $L(E_{11}), \ldots, L(E_{nn})$ are mutually orthogonal matrices. The result follows then from Lemma 3 that $L(E_{ii}) \neq 0$ for all $i$. 

**Theorem 3.** Let $L$ be a nonzero linear map on $M_n(F)$ preserving $k$-potent matrices, where $k = 2, 3$ and $\text{char } F = 0$ or $\text{char } F > k$. Then there exists a nonsingular matrix $P$ such that

$$L(A) = \lambda PAP^{-1} \quad \text{for all } A$$

or

$$L(A) = \lambda PA'P^{-1} \quad \text{for all } A,$$

where $\lambda^{k-1} = 1$.

**Proof.** Let $A$ be a rank one idempotent. Then $A$ is similar to $E_{11}$. By Lemma 8 we see that $L(A)$ is of rank one. By a result in [6] and Lemma 1, we have that either

$$L(A) = \begin{cases} PAQ & \text{for all } A \\ PA'Q & \text{for all } A \end{cases}$$

(5)

for some matrices $P$ and $Q$, or $\text{Im } L$ consists of rank one or zero matrices. For $k = 2$, Lemma 8 implies that $L(I_n) = \sum_{i=1}^{n} L(E_{ii}) = I_n$, and hence $L$ is of the form (5) and $PQ = I_n$. 

Let \( k = 3 \). By Lemma 8, \( L(I_n) \) is a nonsingular matrix, and thus \( L \) is of the form (5). Let \( QP = D = (d_{ij}) \). Then for \( i \neq j \),
\[
d_{ij}(PE_{ij}Q) = (PE_{ii}Q)(PE_{jj}Q) = 0,
\]
\[
d_{ii}^2(PE_{ii}Q) = (PE_{ii}Q)^3 = PE_{ii}Q \neq 0.
\]
Thus \( d_{ij} = 0 \) and \( d_{ii}^2 = 1 \). Hence \( D = \text{diag}(d_{11}, \ldots, d_{nn}) \). Suppose that \( d_{11} \neq d_{22} \). Take
\[
A = \begin{pmatrix}
-1 & -2 \\
1 & 2
\end{pmatrix} \oplus 0.
\]
Then \( A^2 = A \) and \( L(A) = PADP^{-1} \) or \( PA^tDP^{-1} \). Since
\[
AD = \pm \begin{pmatrix}
-1 & 2 \\
1 & -2
\end{pmatrix} \oplus 0 \quad \text{and} \quad A^tD = \pm \begin{pmatrix}
-1 & -1 \\
-2 & -2
\end{pmatrix} \oplus 0
\]
are both not 3-potent, we have a contradiction. Hence \( d_{11} = d_{22} \). Similarly we can show that \( d_{ii} = d_{jj} \) for all \( i, j \). This completes the proof.\( \blacksquare \)

When \( k = 2 \), Theorem 3 was obtained independently by Beasley and Pullman in [1]. The result with the additional hypothesis that \( L(I_n) = I_n \) is due to Beasley and Pullman.

**Theorem 4.** Let \( F \) be an algebraically closed field with \( \text{char } F \neq 2 \) or 3. Let \( L \) be a nonzero linear map on \( S_n(F) \) preserving \( k \)-potent matrices, where \( k = 2, 3 \). Then there exists an orthogonal matrix \( P \) such that
\[
L(\lambda A) = \lambda PAP^t
\]
for all \( A \) where \( \lambda^{k-1} = 1 \).

**Proof.** Let \( E_1, \ldots, E_n \) be \( n \) mutually orthogonal rank one idempotents in \( S_n(F) \). Then \( L(E_i) \) is \( k \)-potent for each \( i \). By Lemma 2 in [2] and Lemma 7, \( L(E_1), \ldots, L(E_n) \) are mutually orthogonal. Hence for any \( \lambda_1, \ldots, \lambda_n \) in \( F \),
\[
L\left(\left(\sum_{i=1}^{n} \lambda_i E_i\right)^k\right) = \left[ L\left(\sum_{i=1}^{n} \lambda_i E_i\right)\right]^k.
\]
Let \( A \in S_n(F) \) be a diagonalizable matrix. Then \( A \) is a linear combination of \( n \) mutually orthogonal rank 1 idempotents in \( S_n(F) \). Hence \( L(A^k) = L(A)^k \). Thus by Remark 1, \( L(E_i) \neq 0 \) for each \( i \). This proves that \( L(E_i) \) is of rank one. By Lemma 2, \( L \) preserves matrices of rank \( \leq 1 \). Applying Corollary 3 in [7] and using arguments similar to that in the proof of Theorem 3, we obtain the result.

**Remark 5.** Let \( L \) be a nonzero linear map on the space of \( n \times n \) complex Hermitian matrices preserving \( k \)-potent matrices, where \( k = 2, 3 \). Following the arguments in the proof of Theorem 4 and using a theorem of Loewy on rank 1 preserves in [8], we can show that \( L(A) = \lambda PAP^* \) or \( L(A) = \lambda P^*A^*P^* \) for some unitary matrix \( P \) and some real number \( \lambda \) with \( \lambda^{n-1} = 1 \). For \( k = 2 \), this was obtained in [9] by a different method. The analogous result for real symmetric matrices also holds.

**REFERENCES**


*Received 25 July 1990; final manuscript accepted 22 March 1991*