The weighted Fermat–Torricelli problem for tetrahedra and an “inverse” problem

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**Abstract**

We study the weighted Fermat–Torricelli problem for tetrahedra in \(\mathbb{R}^3\) and solve an “inverse” problem by introducing a method of differentiation. The solution of the inverse problem is the main result which states that: Given the Fermat–Torricelli point \(A_0\) with the vertices lie on four prescribed rays, find the ratios between every pair of non-negative weights of two corresponding rays such that the sum of the four non-negative weights is a constant number. An application of the inverse weighted Fermat–Torricelli problem is the strong invariance principle of the weighted Fermat–Torricelli point which gives some classes of tetrahedra that could be named “evolutionary tetrahedra”.

1. Introduction

The Fermat–Torricelli problem is to find the (unique) point that minimizes the sum of distances from three given points in \(\mathbb{R}^2\). Chapter 2 in [1] and [2] offer an exposition to the subject. A unified approach of the weighted Fermat–Torricelli problem in the plane, two-dimensional sphere and two-dimensional hyperboloid is given in [5]. For a unified approach to geometric properties of the Fermat–Torricelli point of four affinely independent points, we refer to [3] and for \(n\) affinely independent points in [2]. In this paper, we provide a new method, in order to study the weighted Fermat–Torricelli problem for tetrahedra in \(\mathbb{R}^3\) and solve an “inverse” problem by differentiating some geometric relations for tetrahedra with respect to specific dihedral angles. The existence of some classes of tetrahedra named “evolutionary tetrahedra” is a result from the strong invariance principle that possesses the weighted Fermat–Torricelli point. The inverse problem cannot be solved by the numerical method that was introduced by E. Weiszfeld in [4], in order to find the weighted Fermat–Torricelli point for \(n\) non-collinear points, because it calculates the orthogonal coordinates \((x, y, z)\) of the generalized weighted Fermat–Torricelli point.

2. The weighted “Fermat–Torricelli” problem for a tetrahedron

We start by stating the problem for a tetrahedron \(A_1A_2A_3A_4\) in \(\mathbb{R}^3\).

**Problem 1.** Let \(A_1A_2A_3A_4\) be a tetrahedron. Suppose that a weight \(B_i \in \mathbb{R}^+\) corresponds to each vertex \(A_i\) for \(i = 1, 2, 3, 4\), respectively. Find the weighted Fermat–Torricelli point \(A_0\) of \(A_1A_2A_3A_4\) which minimizes the sum of the lengths of the line segments \(a_i\) that connect every vertex with \(A_0\) multiplied by the positive weight \(B_i\):

\[
B_1a_1 + B_2a_2 + B_3a_3 + B_4a_4 = \text{minimum.} \tag{2.1}
\]
Theorem 1. The weighted Fermat–Torricelli point $A_0$ of the tetrahedron $A_1A_2A_3A_4$ exists and is unique.

(i) If

\[ \left\| \sum_{j=1}^{n} B_j \hat{u}(A_i, A_j) \right\| > B_i, \quad i \neq j \]

for $i, j = 1, 2, 3, 4$, then the weighted Fermat–Torricelli point is an interior point of the tetrahedron $A_1A_2A_3A_4$ (floating case).

(ii) If there is some $i$ with

\[ \left\| \sum_{j=1}^{n} B_j \hat{u}(A_i, A_j) \right\| \leq B_i, \quad i \neq j \]

for $i, j = 1, 2, 3, 4$, then the weighted Fermat–Torricelli point is the vertex $A_i$ (absorbed case).

A proof of a more generalized version of Theorem 1 exists in [2].

Solution of Problem 1. The independent variables $a_1, a_2, \alpha$ will be used, in order to find $A_0$, where $\alpha$ is the dihedral angle between the planes $A_0A_1A_2$ and $A_3A_1A_2$ (see Fig. 1). The variables $a_3, a_4$ can be expressed as functions of $a_1, a_2$ and $\alpha$:

\[ a_3 = a_3(a_1, a_2, \alpha), \quad a_4 = a_4(a_1, a_2, \alpha). \] (2.2)

From (2.1) and (2.2) the following equation is obtained:

\[ B_1a_1 + B_2a_2 + B_3a_3(a_1, a_2, \alpha) + B_4a_4(a_1, a_2, \alpha) = \text{minimum}. \] (2.3)

By differentiation of (2.3) with respect to the variables $a_1, a_2$ and $\alpha$ we get

\[ B_1 + B_3 \frac{\partial a_3}{\partial a_1} + B_4 \frac{\partial a_4}{\partial a_1} = 0, \] (2.4)

\[ B_2 + B_3 \frac{\partial a_3}{\partial a_2} + B_4 \frac{\partial a_4}{\partial a_2} = 0, \] (2.5)

\[ B_3 \frac{\partial a_3}{\partial \alpha} + B_4 \frac{\partial a_4}{\partial \alpha} = 0. \] (2.6)
We proceed by calculating Eq. (2.6).
We express $a_3$ as a function of $a_1$, $a_2$ and $\alpha$ by using the following equations:

\[
\cos(\alpha_{102}) = \frac{a_1^2 + a_2^2 - a_{12}^2}{2a_1 a_2},
\]

\[h_{0,12} = \frac{a_1 a_2 \sin(\alpha_{102})}{a_{12}},\]

\[h_{0,123} = h_{0,12} \sin(\alpha),\]

\[x_2^2 = a_2^2 - h_{0,123}^2,\]

\[\sin(\alpha'_{2}) = \frac{h_{0,12} \cos(\alpha)}{x_2},\]

\[x_3^2 = x_2^2 + a_{23}^2 - 2x_2 a_{23} \cos(\alpha_{123} - \alpha'_{2}),\]

\[a_3^2 = x_3^2 + h_{0,123}^2,\]

or

\[a_3^2 = a_2^2 + a_{23}^2 - 2a_{23} \left[ \sqrt{a_2^2 - h_{0,12} \cos(\alpha_{123})} + h_{0,12} \sin(\alpha_{123}) \cos(\alpha) \right],\]

where $h_{0,12}$ is the height of the triangle $\nabla A_0 A_1 A_2$ from $A_0$ to $A_1 A_2$ and $h_{0,123}$ is the distance from $A_0$ to the plane $A_1 A_2 A_3$ (see Fig. 1). We express $a_4$ as a function of $a_1$, $a_2$ and $\alpha$ by using the following equations:

\[h_{0,124} = h_{0,12} \sin(\alpha_g - \alpha),\]

\[x_2'^2 = a_2^2 - h_{0,124}^2,\]

\[\sin(\alpha'_{2}) = \frac{h_{0,12} \cos(\alpha_g - \alpha)}{x_2'},\]

\[x_4^2 = x_2'^2 + a_{24}^2 - 2x_2' a_{24} \cos(\alpha_{124} - \alpha'_{2}),\]

\[a_4^2 = x_4^2 + h_{0,124}^2,\]

or

\[a_4^2 = a_2^2 + a_{24}^2 - 2a_{24} \left[ \sqrt{a_2^2 - h_{0,12} \cos(\alpha_{124})} + h_{0,12} \sin(\alpha_{124}) \cos(\alpha_g - \alpha) \right],\]

where $\alpha_g$ is the given dihedral angle between the planes $A_3 A_1 A_2$ and $A_4 A_1 A_2$ and $h_{0,124}$ is the distance from $A_0$ to the plane $A_1 A_2 A_4$ (see Fig. 2).
We differentiate (2.14), (2.20) with respect to $\alpha$ and we obtain (2.21) and (2.22), respectively.

$$\frac{\partial a_3}{\partial \alpha} = +a_{23}h_{0,12}\sin(\alpha_{123})\sin(\alpha),$$  \hspace{5mm} (2.21)

$$\frac{\partial a_4}{\partial \alpha} = -a_{24}h_{0,12}\sin(\alpha_{124})\sin(\alpha_g - \alpha).$$  \hspace{5mm} (2.22)

The next relation is derived by replacing (2.21) and (2.22), in (2.6):

$$\frac{B_3}{a_3a_{24}\sin(\alpha_{124})h_{0,12}\sin(\alpha_g - \alpha)} = \frac{B_4}{a_4a_{23}\sin(\alpha_{123})h_{0,12}\sin(\alpha)}.$$  \hspace{5mm} (2.23)

We multiply both members of (2.23) by $\frac{1}{a_3^2a_4^2}$ and get

$$\frac{B_3}{a_3\text{Vol}(A_0A_1A_2A_4)} = \frac{B_4}{a_4\text{Vol}(A_0A_1A_2A_3)}.$$  \hspace{5mm} (2.24)

The following relations are derived by working cyclically with variables $a_i, a_j$ and $\alpha(A_0A_iA_j/A_kA_lA_j)$ for $i, j, k = 1, 2, 3, 4, i \neq j \neq k \neq i$ and $k$ is selected such that $k, i, j$ are three sequential numbers. The variable $\alpha(A_0A_iA_j/A_kA_lA_j)$ is the dihedral angle between the planes $A_0A_iA_j$ and $A_kA_lA_j$.

$$\frac{B_3}{a_3\text{Vol}(A_0A_1A_2A_4)} = \frac{B_4}{a_4\text{Vol}(A_0A_1A_2A_3)} = \frac{B_1}{a_1\text{Vol}(A_0A_2A_3A_4)} = \frac{B_2}{a_2\text{Vol}(A_0A_1A_3A_4)} = C,$$  \hspace{5mm} (2.25)

where $C = \frac{\sum_{i=1}^{4} a_i}{\text{Vol}(A_0A_1A_2A_3A_4)}$. Other equations can be expressed from (2.25), like

$$\frac{B_3}{a_i} = \frac{h_{0,kj}}{H_{ikj}}\sum_{j=1}^{4} \frac{B_j}{a_j},$$  \hspace{5mm} (2.26)

where $H_{ikj}$ is the distance from $A_i$ to the plane $A_kA_lA_j$ for $i, j, k, l = 1, 2, 3, 4, i \neq j \neq k \neq l \neq i$.

From (2.25), we divide the two equations for $i = 3$ and $i = 4$, respectively, in order to have the relation:

$$\frac{B_3}{a_3}\frac{H_{3,124}}{h_{0,124}} = \frac{h_{0,123}}{a_4}\frac{\sin(\alpha_{g} - \alpha)}{\sin(\alpha)}.$$  \hspace{5mm} (2.27)

We continue with the calculation of $A_0$ which is clarified by $a_1, a_2$ and $\alpha$.

We replace $a_3 = a_3(a_1, a_2, \alpha), a_4 = a_4(a_1, a_2, \alpha)$ by (2.14), (2.20) and (2.9), (2.15) in (2.27) and square both parts of the equation to get

$$\left(\frac{B_3}{B_4}\right)^2\left(\frac{a_4(a_1, a_2, \alpha)}{a_3(a_1, a_2, \alpha)}\right)^2\left(\frac{H_{3,124}}{H_{4,123}}\right)^2 = \frac{\sin^2(\alpha_g - \alpha)}{\sin^2(\alpha)}.$$  \hspace{5mm} (2.28)

By replacing (2.7), (2.8), (2.15) to the derived equation of (2.25) for $i = 3$, we deduce

$$\frac{B_3}{a_3}\frac{1}{C_0} = \frac{B_1}{a_1} + \frac{B_2}{a_2} + \frac{B_3}{a_3}\left(1 + \frac{B_4}{B_3}\frac{a_3}{a_4}\right),$$  \hspace{5mm} (2.29)

where $C_0 = \frac{4a_3^2(a_1^2 + a_2^2 - a_3^2)^2}{2a_{12}^2h_{124}}\sin(\alpha_g - \alpha)$. We proceed by replacing (2.27) in (2.29):

$$\frac{B_3}{a_3}\left[\frac{1}{C_0} - \frac{H_{3,124}}{H_{4,123}}\frac{\sin(\alpha)}{\sin(\alpha_g - \alpha)}\right] = \left(\frac{B_1}{a_1} + \frac{B_2}{a_2}\right).$$  \hspace{5mm} (2.30)

We square both parts of the derived equation, in order to obtain

$$\frac{B_3^2}{C_0}\left[\frac{1}{C_0} - \frac{H_{3,124}}{H_{4,123}}\frac{\sin(\alpha)}{\sin(\alpha_g - \alpha)}\right]^2 = \left(\frac{B_1}{a_1} + \frac{B_2}{a_2}\right)^2\left(a_3^2(a_1, a_2, \alpha)\right).$$  \hspace{5mm} (2.31)

where $a_3^2(a_1, a_2, \alpha)$ can be replaced by (2.14).

Two equations (2.28), (2.31) are obtained with the independent variables $a_1, a_2, \alpha$, and the third equation will be found from (2.4). We continue by replacing $\frac{a_{23}}{a_{12}}, \frac{a_{24}}{a_{12}}$ in (2.4),

$$\left(\frac{B_1}{a_1}\right) + \left(\frac{B_3}{a_3}\right)\left[\frac{a_{23}}{a_{12}}\cos(\alpha_{123}) - \sin(\alpha_{123})\cos(\alpha)\right] - \frac{a_{23}}{2a_{12}^2h_{0,12}}(-a_1^2 + a_2^2 + a_3^2)$$

$$+ \left(\frac{B_4}{a_4}\right)\left[\frac{a_{24}}{a_{12}}\cos(\alpha_{124}) - \sin(\alpha_{124})\cos(\alpha_g - \alpha)\right] - \frac{a_{24}}{2a_{12}^2h_{0,12}}(-a_1^2 + a_2^2 + a_3^2) = 0.$$  \hspace{5mm} (2.32)
and weights that correspond to the vertices $B_i$, non-negative weights $a_i$ gives Theorem 1) and the corresponding weights satisfy some weighted inequalities. For extreme cases, we refer to [3] and [2] (absorbed case of Theorem 1).

**Example 1.** Given a tetrahedron $A_1A_2A_3A_4$ with vertices $A_1 = (0, -3, 0), A_2 = (2, 2, 0), A_3 = (0, 4, 0), A_4 = (0, 0, 7)$, which gives $a_{12} = \sqrt{29}, a_{23} = 2 \sqrt{2}, a_{24} = \sqrt{57}, H_3,124 = -\frac{96}{145} = 0.663, H_4,123 = 7, \cos(\alpha_{123}) = -\frac{3}{\sqrt{58}}, \cos(\alpha_{124}) = \frac{14}{\sqrt{1653}}, \alpha_g = 1.41295$ rad and weights that correspond to the vertices $B_1 = 3, B_2 = 2.5, B_3 = 2, B_4 = 2.7$, we will calculate the weighted Fermat–Torricelli point $A_0$. Consider the three equations (2.36), (2.31), (2.28) which are functions of $a_1, a_2$ and $\alpha$ and choose three starting values for $a_1^0 = 3.6, a_2^0 = 3.9$ and $\alpha^0 = 0.7$, which satisfy the inequalities:

$$\alpha^0 < \alpha_g, \quad |a_i^0 - a_g^0| < a_{1i} < a_i^0 + a_g^0, \quad a_1^0 a_2^0 \cos(\alpha^0) < a_{12} a_{23} \sin(\alpha_{123}), \quad a_i^0 a_g^0 \cos(\alpha_g - \alpha^0) < a_{1i} a_{24} \sin(\alpha_{124}).$$

The Newton method gives $a_1 = 4.17195, a_2 = 1.81605$ and $\alpha = 0.979839$ rad which coincides with the result derived by the Weiszfeld algorithm that approximates $A_0 = (0.87743, 0.95973, 0.97762)$, with 5-digit precision.

Special cases are tetrahedra with equal non-negative weights and regular tetrahedra with unequal non-negative weights.

### 3. The inverse weighted Fermat–Torricelli problem

**Problem 2.** Given the Fermat–Torricelli point $A_0$ with the vertices lie on four prescribed rays, find the ratios between the non-negative weights $\frac{B_i}{B_i}$, $i, j = 1, 2, 3, 4$ such that:

$$\sum_{i=1}^{4} B_i = \text{constant}.$$ 

This is the inverse weighted Fermat–Torricelli problem for tetrahedra in $\mathbb{R}^3$. 

\[ \frac{B_1}{a_1} + \frac{B_3 a_{23}}{a_3 a_{12}} \cos(\alpha_{123}) + \frac{B_4 a_{24}}{a_4 a_{12}} \cos(\alpha_{124}) = \frac{B_3}{a_3} \cos(\alpha) \sin(\alpha_{123}) a_{23} \sqrt{a_2^2 - h_{0,12}^2} + \frac{B_4}{a_4} \cos(\alpha_g - \alpha) \sin(\alpha_{124}) a_{24} \sqrt{a_2^2 - h_{0,12}^2} \] 

or

\[ \frac{B_1}{a_1} + \frac{B_3 a_{23}}{a_3 a_{12}} \cos(\alpha_{123}) + \frac{B_4 a_{24}}{a_4 a_{12}} \cos(\alpha_{124}) = \frac{B_3}{a_3} H_{4,123} \frac{\sqrt{a_2^2 - h_{0,12}^2}}{a_{12} h_{0,12}} \sin(\alpha_{123}) a_{23} \sin(\alpha_g) \sin(\alpha_g - \alpha) \] 

We apply (2.27) in (2.34), in order to obtain:

\[ \frac{B_1}{a_1} + \frac{B_3 a_{23}}{a_3 a_{12}} \cos(\alpha_{123}) + \frac{B_4 a_{24}}{a_4 a_{12}} \cos(\alpha_{124}) = \frac{B_1}{a_1} \frac{a_{23}}{a_{12}} \cos(\alpha_{123}) - \frac{H_{3,124}}{H_{4,123}} \sin(\alpha) \frac{a_{24}}{a_{12}} \cos(\alpha_{124}) \] 

The three equations (2.36), (2.31), (2.28) depend on $a_1, a_2$ and $\alpha$ and can be solved numerically.
Proposition 1. The weighted Fermat–Torricelli point of a tetrahedron \( A_1 A_2 A_3 A_4 \) remains the same for any tetrahedron \( A'_1 A'_2 A'_3 A'_4 \) if the floating case occurs for values of \( B_i \) that correspond to any vertex \( A_i, i = 1, 2, 3, 4 \) and verify the relation (2.25) (strong invariance principle).

Proof. The volume formulas for \( A_0 A_1 A_2 A_4 \) and \( A_0 A_1 A_2 A_3 \) are used by applying the orthogonal projection of \( a_3 \) and \( a_4 \) on the plane \( A_0 A_1 A_2 \) (see Fig. 3).

\[
\begin{align*}
    a_3 \text{VOL}(A_0 A_1 A_2 A_4) &= \frac{1}{6} a_3 a_1 a_2 a_4 \sin(\alpha_{012}) \sin(\alpha_{4102}), \\
    a_4 \text{VOL}(A_0 A_1 A_2 A_3) &= \frac{1}{6} a_4 a_1 a_2 a_3 \sin(\alpha_{012}) \sin(\alpha_{3102})
\end{align*}
\]

and we place them into (2.24), which gives

\[
\frac{B_3}{\sin(\alpha_{4102})} = \frac{B_4}{\sin(\alpha_{3102})}. \tag{3.1}
\]

We denote by \( \alpha_{i,j,k,l} \) the angle that is formulated by the line segment that connects \( A_0 \) with the trace of the orthogonal projection of \( A_i \) to the plane \( A_j A_k A_l \) with \( a_i \), for \( i, j, k, l = 1, 2, 3, 4, i \neq j \neq k \neq i \).

Similarly, by applying the volume formula for tetrahedra \( A_0 A_i A_k A_l \) in (2.25) and using the orthogonal projection of \( a_i, a_j \) on the plane \( A_0 A_k A_l \) we get

\[
\frac{B_i}{\sin(\alpha_{j,k,l})} = \frac{B_j}{\sin(\alpha_{i,k,l})}, \tag{3.2}
\]

for \( i, j, k, l = 1, 2, 3, 4 \).

Solution of Problem 2. Given the angles \( \alpha_{i,j} \) of the weighted Fermat–Torricelli point \( A_0 \), we will calculate the angles \( \alpha_{i,j,k,l} \), for \( i, j, k = 1, 2, 3, 4, i \neq j \neq k \neq i \). We express the unit vectors \( \vec{a}_i \) for \( i = 1, 2, 3, 4 \) in a parametric form

\[
\begin{align*}
    \vec{a}_1 &= (1, 0, 0), \\
    \vec{a}_2 &= (\cos(\alpha_{102}), \sin(\alpha_{102}), 0), \\
    \vec{a}_3 &= (\cos(\alpha_{3102}) \cos(\alpha_{3102}), \cos(\alpha_{3102}) \sin(\omega_{3102}), \sin(\alpha_{3102})).
\end{align*}
\]
\[ a_i = (\cos(\alpha_{4,102}) \cos(\omega_{4,102}), \cos(\alpha_{4,102}) \sin(\omega_{4,102}), \sin(\alpha_{4,102})) \]  \hfill (3.6)

such that: \(|\vec{a}_i| = 1\). The inner product of \(\vec{a}_i, \vec{a}_j\) is
\[ \vec{a}_i \cdot \vec{a}_j = \cos(\alpha_{i0j}). \]  \hfill (3.7)

We take into consideration (3.7), for \(i, j = 1, 2, 3, 4\), in order to find the angles \(\alpha_{3,102}\) and \(\alpha_{4,102}\). The angles \(\alpha_{i,j,k}\) can be derived by working cyclically with \(\vec{a}_i\) and choosing similar parametrization with respect to (3.3)–(3.6), regarding the plane \(A_jA_0A_k\) for \(i, j, k = 1, 2, 3, 4, i \neq j \neq k \neq i\).

For example, we will calculate the ratio \(\frac{B_3}{B_4}\), by taking into account the following inner products:
\[ \vec{a}_1 \cdot \vec{a}_3 = \cos(\alpha_{103}) = \cos(\alpha_{3,102}) \cos(\omega_{3,102}). \]  \hfill (3.8)

\[ \vec{a}_2 \cdot \vec{a}_3 = \cos(\alpha_{203}) = \cos(\alpha_{102}) \cos(\alpha_{3,102}) \cos(\omega_{3,102}) + \sin(\alpha_{102}) \cos(\alpha_{3,102}) \sin(\omega_{3,102}). \]  \hfill (3.9)

From (3.8), we replace \(\cos(\alpha_{3,102})\) and \(\sin(\alpha_{3,102})\) in (3.9) and obtain the equation
\[ \cos^2(\alpha_{3,102}) = \frac{\cos^2(\alpha_{203}) + \cos^2(\alpha_{103}) - 2 \cos(\alpha_{203}) \cos(\alpha_{103}) \cos(\alpha_{3,102})}{\sin^2(\alpha_{102})}. \]  \hfill (3.10)

Similarly, we obtain the equation for \(\alpha_{4,102}\):
\[ \cos^2(\alpha_{4,102}) = \frac{\cos^2(\alpha_{204}) + \cos^2(\alpha_{104}) - 2 \cos(\alpha_{204}) \cos(\alpha_{104}) \cos(\alpha_{4,102})}{\sin^2(\alpha_{102})}. \]  \hfill (3.11)

We replace (3.10), (3.11) in (3.1) of Proposition 1 and square both parts of the derived equation, in order to calculate the ratio \(\frac{B_3}{B_4}\):
\[ \left( \frac{B_3}{B_4} \right)^2 = \frac{\sin^2(\alpha_{4,102})}{\sin^2(\alpha_{3,102})} = \frac{1 - \cos^2(\alpha_{4,102})}{1 - \cos^2(\alpha_{3,102})} \]

or
\[ \left( \frac{B_4}{B_3} \right)^2 = \frac{\sin^2(\alpha_{102}) - \cos^2(\alpha_{203}) - \cos^2(\alpha_{103}) + 2 \cos(\alpha_{203}) \cos(\alpha_{103}) \cos(\alpha_{4,102})}{\sin^2(\alpha_{102}) - \cos^2(\alpha_{204}) - \cos^2(\alpha_{104}) + 2 \cos(\alpha_{204}) \cos(\alpha_{104}) \cos(\alpha_{4,102})}. \]  \hfill (3.12)

Similarly, by applying (3.2), the ratio \(\frac{B_i}{B_j}\) is derived:
\[ \left( \frac{B_i}{B_j} \right)^2 = \frac{\sin^2(\alpha_{k0m}) - \cos^2(\alpha_{m0i}) - \cos^2(\alpha_{k0i}) + 2 \cos(\alpha_{m0i}) \cos(\alpha_{k0i}) \cos(\alpha_{k0m})}{\sin^2(\alpha_{k0m}) - \cos^2(\alpha_{m0j}) - \cos^2(\alpha_{k0j}) + 2 \cos(\alpha_{m0j}) \cos(\alpha_{k0j}) \cos(\alpha_{k0m})}. \]  \hfill (3.13)

for \(i, j, k, m = 1, 2, 3, 4\) and \(i \neq j \neq k \neq m \neq i\). Finally, we would like to note that every ratio \(\frac{B_i}{B_j}\) depends on 5 given angles: \(\alpha_{k0m}, \alpha_{m0i}, \alpha_{k0i}, \alpha_{m0j}, \alpha_{k0j}\).

References