Boundary and Interior Transition Layer Phenomena for Pairs of Second-Order Differential Equations

PAUL C. FIFE

Mathematical Institute, University of Oxford, Oxford, England, and Department of Mathematics, University of Arizona, Tucson, Arizona 85721

Submitted by Norman Levinson

1. INTRODUCTION

In this paper we investigate the asymptotic behavior, for small $\varepsilon$, of solutions of two-point boundary value problems for autonomous systems of the form

$$\varepsilon^2 u'' = f(u, v), \quad (1.1a)$$
$$v'' = g(u, v), \quad (1.1b)$$

and (as a particular case), single equations of the form

$$\varepsilon^2 u'' = f(u, x). \quad (1.2)$$

Such systems arise, for example, in the study of steady-state configurations of a mixture of chemically reacting and diffusing substances.

Principal attention will be given to layer-type qualitative features of the solution; that is, to the fact that families of solutions with $\varepsilon$ as parameter commonly exist which approach discontinuous functions of $x$ as $\varepsilon \to 0$. The solution when $\varepsilon$ is small but nonzero, being smooth, exhibits an abrupt but continuously differentiable transition at the location of the limit discontinuity. An interval where such an abrupt change takes place is loosely called a "layer"—a "boundary layer" when it is adjacent to the boundary. It is possible for internal layers also to exist, and we give particular attention to such in Sections 4–6.

If a family $(u_\varepsilon, v_\varepsilon)$ of solutions exists with a limit in some sense as $\varepsilon \to 0$, then one expects the limiting pair $(U, V)$ to be a solution of the reduced problem $f(U, V) = 0; V'' = g(U, V)$, with boundary conditions imposed

* Supported by N.S.F. Grant GP-42739, and an S.R.C. Fellowship.
upon \( V \). In a \( V \)-interval in which the first equation may be solved for \( U = h(V) \), the single equation

\[
V'' - g(h(V), V) = 0
\]  

result. If this equation, together with the boundary conditions for \( V \), has a solution \( V(x) \), then the pair \((U(x), V(x))\), where \( U(x) = h(V(x)) \), serves as a likely candidate for a limiting configuration of exact solutions. That is to say, there may exist a family \((u_\epsilon, v_\epsilon)\) with \( u_\epsilon \to U, v_\epsilon \to V \), as \( \epsilon \to 0 \), for \( x \) in the interior of the interval under consideration. On the other hand, there may exist a second solution \( U = h_1(V) \) of \((U, V) = 0\), in which case there may be a second family \((u_\epsilon, v_\epsilon)\) approaching a different limit. Uniqueness is certainly not to be expected.

In some cases, however, it is not possible to solve \( f(U, V) = 0 \) for \( U = h(V) \) with \( h \) continuous and defined in a \( V \)-interval large enough for (1.3) to be solvable, with the required boundary conditions. Consider the example \( f(U, V) = -V - 3U + U^3 \). The zero contour \( f = 0 \) is a cubic in the \( U-V \) plane (Fig. 1) which has one solution branch \( U = h_0(V) \) defined for \( V < 2 \), and another: \( U = h_1(V) \) defined for \( V > -2 \). (Also, of course, a third defined for \(-2 < V < 2\), which will not be used.) If the boundary conditions imposed upon \( V \) are \( V(0) = \beta_0 < -2 \) and \( V(1) = \beta_1 > 2 \), then no single function \( h(V) \) may be used in (1.3) for the entire range of \( V \), since that range must include at least the interval \([\beta_0, \beta_1]\). In this case, if \( g \) satisfies suitable conditions, it is possible to piece the two functions \( h_1 \) together at an

\[ \text{Fig. 1. The dashed line represents the discontinuous function } U = L(V) \text{ used in constructing the initial approximation.} \]
appropriate value of $V$ (it turns out to be $V = 0$) and obtain a family $(u_\epsilon, v_\epsilon)$ such that $u_\epsilon$ approaches the discontinuous function

$$U(x) = h_0(V(x)), \quad V(x) < 0$$

$$= h_1(V(x)), \quad V(x) > 0,$$

where the function $V$ is a solution of the boundary value problem

$$I'' = \begin{cases} g(h_0(V'), V'), & V < 0 \\ g(h_1(V'), V'), & V > 0 \end{cases} \quad I'(0) = \beta_0; \quad I'(1) = \beta_1.$$

Thus, an internal transition layer is formed.

Finally, in some cases there are solution families with internal layers, and other solution families for the same problem without internal layers. Or, there may be families with many internal layers. These possibilities are explored for a particular example in Sect. 6.

Two-point boundary value problems for systems including those treated here were studied by Hoppensteadt [5], Vasil'eva and Butuzov [9], and others, but internal layers for such systems have not been previously studied.

Single equations of the type (1.2) were treated in [1–3, 7–10]. In particular, when the equation is autonomous, the possible layer (internal and boundary) structure of the solutions was brought out clearly in [7, 10]. When the equation is nonautonomous, theories were given in [1, 3, 8, 9]. In [4], internal transition layers were studied for a partial differential equation of the form $\varepsilon^2 \Delta u = f(u, x)$.

The present paper uses an implicit function argument akin to that in [2], and a patching argument as in [8]. However, the fact that we work with a system necessitates obtaining a stronger estimate for the inverse of the appropriate Frechet derivative than has been obtained previously. This estimate, which in fact yields results for the single equation under weaker differentiability requirements than those known before (see Section 3), is given in Section 2. The patching argument used for the transition layer is dependent upon another implicit function theorem for a pair of equations in three real variables. The lemma we develop for the purpose (Theorem 4.3) is interesting in its own right in that it does not require differentiability, and when the functions are differentiable, does not require their Jacobian to be invertible. (See the note added in proof at the end of this paper.)

2. A USEFUL LEMMA

Here we consider the boundary value problem

$$\varepsilon^2 u'' = f(u, x), \quad 0 < x < 1,$$  \hspace{1cm} (2.1)

$$u(0) = \alpha_0, \quad u(1) = \alpha_1.$$  \hspace{1cm} (2.2)
This problem is a particular case of (3.1) treated later (namely, \( g = 0; \ v = x \)). Furthermore, it has been treated in previous papers [1, 2, 7–10]. Therefore the existence theory will not be given. Nevertheless, we need certain results connected with the problem which are stronger in some sense than those obtained before, for later use in Section 3.

We assume that there exists a continuous function \( h \) satisfying

\[
\begin{align*}
  f(h(x), x) &> 0, \quad 0 \leq x \leq 1 \\
  f_u(h(x), x) &> 0,
\end{align*}
\]

and also that for \( i = 0, 1, \)

\[
\int_{h(i)}^{k} f(u, i) \, du > 0 \quad \text{for} \quad k \neq h(i) \text{ in the closed interval between } h(i) \text{ and } \alpha_i. \quad (2.5)
\]

We also assume \( f \) to be continuously differentiable for \( x \in [0, 1] \) and for all \( u \).

In the following we restrict \( \epsilon \) so that \( 0 < \epsilon < 1 \).

A first approximation \( U_0 \) to the solution may be constructed as follows.

Let \( \tilde{z}_0(\eta) \) be the unique monotone solution of

\[
\begin{align*}
  \tilde{z}_0'' &= f(h(0) + \tilde{z}_0, 0), \quad 0 < \eta < \infty; \\
  \tilde{z}_0(0) &= \alpha_0 - h(0); \quad \tilde{z}_0(\infty) = 0.
\end{align*}
\]

Such a unique solution was shown, for example in [2], to exist, and to satisfy \( \tilde{z}_0'(\eta) \neq 0 \) for \( \eta \geq 0 \), unless \( \tilde{z}_0 \equiv 0 \).

Furthermore, it was shown there to decay exponentially as \( \eta \to \infty \).

Let \( \xi(t) \) be a \( C^\infty \) cutoff function satisfying \( \xi \equiv 1 \) for \( 0 < t < \frac{1}{4} \); \( \xi \equiv 0 \) for \( t \geq \frac{1}{2} \). Let

\[
z_0(x, \epsilon) \equiv \tilde{z}_0(x/\epsilon) \xi(x). \quad (7.7a)
\]

Similarly, let \( \tilde{z}_1(\eta) \) be monotone and satisfy

\[
\begin{align*}
  \tilde{z}_1'' &= f(h(1) + \tilde{z}_1, 1); \quad \tilde{z}_1(0) = \alpha_1 - h(1); \quad \tilde{z}_1(\infty) = 0,
\end{align*}
\]

and

\[
z_1(x, \epsilon) \equiv \tilde{z}_1((1 - x)/\epsilon) \xi(1 - x). \quad (7.7b)
\]

Finally, let

\[
U_0(x, \epsilon) \equiv h(x) + z_0(x, \epsilon) + z_1(x, \epsilon). \quad (7.8)
\]

Let \( C'(a, b) \) denote the space of functions with derivatives to order 1.
continuous on $[a, b]$, and $C^2_0(a, b)$ the subspace of such functions with $l = 2$
vanishing at $x = a$, and $x = b$. When $(a, b) = (0, 1)$, we write $C'$ and $C^2_0$ and
use the symbol $| \cdot |_2$ for their norms.

For $\epsilon > 0$, we define $| u |^3_2 = | u |_0 + \epsilon | u' |_0 + \epsilon^2 | u'' |_0$, and denote by
$C^2_\epsilon$ and $C^2_\epsilon, 0$ the Banach spaces of functions in $C^2_0$ and $C^2_0$, respectively,
endowed with this norm.

We define a linear operator $L_\epsilon$, mapping $C^2_\epsilon, 0$ into $C^0$, by

$$L_\epsilon u = \epsilon^2 u'' - f(u, x, \epsilon) u; \quad u(0) = u(1) = 0.$$  

**Lemma 2.1.** The operator $L_\epsilon$ has an inverse defined on all of $C^0$ bounded
independently of $\epsilon$, for sufficiently small positive $\epsilon$.

**Proof.** For the proof, it suffices to show the existence of a constant $K$,
independent of $\epsilon$, such that for any continuous function $F$ with $| F |_0 \leq 1$
and any sufficiently small positive $\epsilon$, there exists a solution $u_\epsilon(x)$ of

$$L_\epsilon u_\epsilon = F, \quad 0 \leq x \leq 1, \quad (2.9)$$

$$u_\epsilon(0) = u_\epsilon(1) = 0, \quad (2.10)$$

satisfying

$$| u_\epsilon |_2^3 \leq K. \quad (2.11)$$

This is done by constructing supersolutions $u\bar{\epsilon}$ and subsolutions $u_\epsilon$. By
definition, a supersolution satisfies $L_\epsilon u\bar{\epsilon} \leq F$; $u_\epsilon(0) \geq 0$, $u_\epsilon(1) \geq 0$. Sub-
solutions satisfy the opposite inequalities. If a positive supersolution can be
constructed, then we merely take $u_\epsilon \equiv -u\bar{\epsilon}$. Then by a theorem of Nagumo
[6], we know that there exists an exact solution (2.9), (2.10) with $| u_\epsilon |_0 < | u\bar{\epsilon} |_0$. By use of this inequality and Eq. (2.9) itself, together with an inter-
polation inequality relating $| u'' |_0$, $| u' |_0$, and $| u |_0$, we in fact obtain that

$$| u_\epsilon |_2^3 \leq C | u\bar{\epsilon} |_0.$$ 

Thus the proof reduces to constructing supersolutions $u\bar{\epsilon}$ with $| u\bar{\epsilon} |_0$
bounded independently of $\epsilon$.

By (2.4), there exists a constant $\beta > 0$ such that $f_\epsilon(h(x), x) \geq 2\beta$ for
$0 \leq x \leq 1$. Furthermore, since $\hat{z}_\epsilon(\eta)$ decays exponentially as $\eta \to \infty$, there
is an $\eta_0$ such that

$$f_\epsilon(h(x) + \pi_\epsilon(x, \epsilon) + \pi_\epsilon(x, \epsilon), x) \geq \beta \quad (2.17)$$

for $\epsilon \eta_0 < x < 1 - \epsilon \eta_0$.

Let $\phi_\epsilon(\eta) \equiv | \hat{z}_\epsilon(\eta) |; \phi_\epsilon(\eta) \equiv | \hat{z}_\epsilon(\eta) |$. Since $\hat{z}_\epsilon(\eta)$ are monotone and the
derivatives do not vanish for finite $\eta$, we have that $\phi_i(\eta) > 0$. By differen-
tiating (2.6a) we find that for all $\eta$,

$$\phi_0'(\eta) - f_u(h(0) + \hat{z}_0(\eta), 0) \phi_0(\eta) = 0; \quad (2.13)$$
similarly,

$$\phi_1'(\eta) - f_u(h(1) + \hat{z}_1(\eta), 1) \phi_1(\eta) = 0. \quad (2.14)$$

Define the functions $w_0(\eta)$ and $w_1(\eta)$, $0 \leq \eta \leq \eta_0$, as the unique solutions of the following initial value problems ($i = 0, 1$):

$$w_i'' - f_u(h(i) + \hat{z}_i(\eta), i) w_i = -1, \quad 0 \leq \eta \leq \eta_0; \quad (2.15)$$

$$w_i(\eta_0) = 0; \quad w_i'(\eta_0) = -\phi_i'(\eta_0) \phi_i(\eta_0). \quad (2.16)$$

Let $M$ be a constant large enough that

$$w_i(\eta) + M \phi_i(\eta) > 0 \quad \text{for } \eta \in [0, \eta_0], \quad i = 0, 1. \quad (2.17)$$

Finally, we define

$$(2.18)$$

First, we show that $v \in C^1$. For this we need only show the continuity of $v$ and $v'$ at $x = \epsilon \eta_0$. But this is guaranteed by the initial conditions (2.16) imposed on $w_0$ and $w_1$.

Next, we show that $v$ is a supersolution. For $x \in (0, \epsilon \eta_0)$, use (2.15), (2.13) to obtain

$$L_x v = w_0(x/\epsilon) - f_u(h(x) + z_0(x, \epsilon), x) w_0 + M(\phi_0(x/\epsilon) - f_u(\cdots) \phi_0)$$

$$= -1 + [f_u(h(0) + z_0, 0) - f_u(h(x) + z_0, x)](w_0 + M \phi_0) \leq -\frac{1}{2} \quad (2.19)$$
for small enough $\epsilon$, by continuity of $f_u$ and $h$. The same inequality holds for $x \in (1 - \epsilon \eta_0, 1)$. For $x \in [\epsilon \eta_0, \frac{1}{2}]$, we have by (2.12),

$$L_\epsilon \psi = \left( M - \frac{1}{\phi_0(\eta_0)} \right) \left( \phi_0(x' \epsilon) - f_u(\cdots) \phi_0(x' \epsilon) \right) - f_u(h(x) + z_0(x, \epsilon), x)$$

and

$$\leq -\beta + \left( M - \frac{1}{\phi_0(\eta_0)} \right) [f_u(h(0) + z_0, 0) - f_u(h(x) - z_0, x)] \phi_0(x \epsilon).$$

Since $z_0$ decays exponentially, the quantity in brackets is bounded in absolute value by a function of the form $C_1 e^{-k x/\epsilon}$, $k > 0$; and also by $\omega(x)$, where $\omega$ is a modulus of continuity such that $|f_u(h(x) + k, x) - f_u(h(0) + k, 0)| \leq \omega(x)$ for all $k$ in the range of $z_0$. Hence it is bounded by the minimum of the two, which approaches zero uniformly in $x$ as $\epsilon \to 0$.

Therefore for small enough $\epsilon$ we have

$$L_\epsilon \psi \leq -\beta/2 \quad (2.20)$$

for $x \in (\epsilon \eta_0, \frac{1}{2})$. For simplicity, assume $\beta < 1$. Then similar arguments show (2.20) to hold for other values of $x$ as well; hence (2.20) is valid in all of $(0, 1)$.

Our desired supersolution is

$$\bar{u}_\epsilon(x) = (2/\beta) \psi(x, \epsilon) > 0,$$

for then $L_\epsilon \bar{u}_\epsilon = (2/\beta) L_\epsilon \psi \leq -1 \leq F(x)$.

By our construction, $|\bar{u}_\epsilon|_0$ is bounded independently of $\epsilon$. This completes the proof.

### 3. A Boundary Value Problem

We now consider a system of two equations of the form

$$\epsilon^2 u'' = f(u, v), \quad (3.1a)$$

$$v'' = g(u, v), \quad (3.1b)$$

together with boundary conditions

$$u(0) = a_0, \quad u(1) = a_1, \quad v(0) = \beta_0, \quad v(1) = \beta_1. \quad (3.2)$$

We assume there exists a function $h(v)$, defined for $v$ in some interval $I$ containing $\beta_0$ and $\beta_1$, which satisfies

$$f(h(v), v) = 0, \quad (3.3)$$

$$f_u(h(v), v) > 0. \quad (3.4)$$
We also assume that for \( i = 0, 1, \)

\[
\int_{h(\beta_i)}^{\alpha_i} f(u, \beta_i) \, du > 0 \quad \text{for} \quad k \neq h(\beta_i) \quad \text{in the closed interval between} \quad h(\beta_i) \quad \text{and} \quad \alpha_i.
\]  

(3.5)

Regarding \( g \), we assume that the problem

\[
V'' = g(h(V), V), \quad 0 < x < 1; \quad V(0) = \beta_0; \quad V(1) = \beta_1
\]

has a solution \( V(x) \) with range in \( I \), such that

\[
V'(x) \neq 0.
\]

(3.7)

Sufficient conditions for this can easily be established, for example, by phase-plane analysis.

**Theorem 3.1.** Let \( f, g \) and \( h \) be twice continuously differentiable for all \( u \) and \( v \), and let \( f, g, h, V, \alpha_i, \) and \( \beta_i \) satisfy (3.3)-(3.7).

Let \( \pi = \sup_x |g_s(h(V(x)), V(x))| \). There exist constants \( \pi_0, \epsilon_0 \) [\( \pi_0 \) depending on \( f \) and on the bound for the operator \( I_{-1}^{-1} \) defined below] such that if \( \pi < \pi_0 \), then there exists a family \( u(x, \epsilon), v(x, \epsilon) \) of solutions of (3.1), (3.2), defined for \( 0 < \epsilon < \epsilon_0 \), satisfying

\[
\lim_{\epsilon \to 0} u(x, \epsilon) = h(V(x)) \quad \text{uniformly for } x \in (\kappa, 1 - \kappa) \quad \text{for every } \kappa > 0; 
\]

(3.8)

\[
\lim_{\epsilon \to 0} v(x, \epsilon) = V(x) \quad \text{uniformly in } [0, 1];
\]

(3.9a)

\[
\lim_{\epsilon \to 0} v_s(x, \epsilon) = V'(x) \quad \text{uniformly in } [0, 1];
\]

(3.9b)

\[
\lim_{\epsilon \to 0} \epsilon u_s(0, \epsilon) = \mp \left( 2 \int_{h(\beta_0)}^{\alpha_0} f(u, \beta_0) \, du \right)^{1/2}; 
\]

(3.10a)

\[
\lim_{\epsilon \to 0} \epsilon u_s(1, \epsilon) = \pm \left( 2 \int_{h(\beta_1)}^{\alpha_1} f(u, \beta_1) \, du \right)^{1/2}; 
\]

(3.10b)

\[
|u(\cdot, \epsilon)|_2^2 + |v(\cdot, \epsilon)|_2 \text{ is bounded independently of } \epsilon.
\]

(3.11)

In (3.10) the upper sign is chosen when the upper limit of integration surpasses the lower.

**Proof.** Let \( z_0(x, \epsilon) \) and \( z_1(x, \epsilon) \) be defined as in (2.6), (2.7), except that \( h(0) \) and \( h(1) \) are to be replaced there by \( h(\beta_0) \) and \( h(\beta_1) \), respectively. Let

\[
U_0(x, \epsilon) = h(V(x)) + z_0(x, \epsilon) + z_1(x, \epsilon).
\]

(3.12)

It clearly satisfies the boundary conditions (3.2) required of \( u \).
We seek correction terms \( r(x, \epsilon) \) and \( s(x, \epsilon) \) which vanish at \( x = 0 \) or \( 1 \), and such that the pair

\[
  u(x, \epsilon) = U_0(x, \epsilon) + r(x, \epsilon), \quad v(x, \epsilon) = V(x) + s(x, \epsilon)
\]

is an exact solution of (3.1). Substituting these into (3.1), we form the operators

\[
  R(r, s, \epsilon) = \epsilon^2(U_{0xx} + r_{xx}) - f(U_0 + r, V + s),
  S(r, s, \epsilon) = V_{xx} + s_{xx} - g(U_0 + r, V + s).
\]

Let \( t = (r, s) \) and \( T(t, \epsilon) = (R(r, s, \epsilon), S(r, s, \epsilon)) \). For each small positive \( \epsilon \), \( T \) is a differentiable mapping from \( C^2_{\alpha, 0} \times C^2_{\beta, 0} \) into \( C^0 \times C^0 \). Its derivative \( T, \epsilon \) has modulus of continuity bounded independently of \( \epsilon \) and \( t \), for \( \epsilon \) and \( t \) in bounded sets.

**Lemma 3.2.** If \( \pi \) is small enough, the operator \( T_\pi(0, \epsilon) = L_\epsilon \) has an inverse defined on all of \( C^0 \times C^0 \), which is bounded independently of \( \epsilon \).

**Proof.** Let \( F = (F_1, F_2) \in C^0 \times C^0 \). The equation \( T_\pi(0, \epsilon) t = F \) can be written as

\[
  R_\pi(0, 0, \epsilon) r + R_\pi(0, 0, \epsilon) s = F_1 \tag{3.14a}
  S_\pi(0, 0, \epsilon)r + S_\pi(0, 0, \epsilon) s = F_2. \tag{3.14b}
\]

The operator \( R_\pi(0, 0, \epsilon) \) satisfies all hypotheses of the operator \( L_\epsilon \) of Lemma 2.1, so it has a bounded inverse. Thus (3.14) becomes

\[
  r = R_{\pi}^{-1}F_1 - R_{\pi}^{-1}R_\pi s,
  s = F_2 - S_{\pi}R_{\pi}^{-1}F_1 + S_{\pi}R_{\pi}^{-1}R_\pi s. \tag{3.15}
\]

It follows from (3.7) that the operator \( L_1 \) defined by \( L_1 s = s'' - G'(V(s)) s \), \( s(0) = s(1) = 0 \), is invertible, where \( G(V) = g(h(V), V') \). To see this, we first observe by differentiating (3.6) that \( w(x) = V'(x) \) satisfies \( w'' - G'(V) w = 0 \). Hence if there existed a nontrivial solution \( \phi \) of \( L_1 \phi = 0 \), \( \phi(0) = \phi(1) = 0 \), then by the interweaving property of the zeros of solutions, we would have that \( w = 0 \) for some \( x \in [0, 1] \). This contradicts (3.7), so there can be no such \( \phi \), and hence \( L_1 \) is invertible.

Now

\[
  S_\pi(0, 0, \epsilon) = s'' - g_\pi(U_0, V) s = L_1 s + g_\pi(h(V'), V') h'(V) s - (\Delta g_\pi) s,
\]

where \( \Delta g_\pi = g_\pi(U_0, V') - g_\pi(h(V'), V') \).
Therefore, applying $L_1^{-1}$ to (3.15), we write it in the form
\[ s = F_3 + L_2 s, \]  
where $F_3 = L_1^{-1}(F_2 - S, R^{-1}F_1)$, and
\[ L_2 s = L_1^{-1}(S, R^{-1}R - g_u(h, V) h'(V) + Ag_v)s. \]  

By expressing $L_1^{-1}$ as an integral operator with bounded Green's functional kernel, we see that for some constant $C_1$,
\[ |L_1^{-1}w|_0 \leq C_1 |w|_{\mathcal{G}(0,1)}. \]  

By the construction of $U_0$, we know that $|\Delta g_v|_{\mathcal{G}(0,1)} \to 0$ as $\epsilon \to 0$, so that by (3.17), the operator $L_1^{-1}\Delta g_v$ approaches 0 in the norm of operators from $C^0$ to $C^0$, as $\epsilon \to 0$.

Furthermore, $S, r$ is simply the operator of multiplication by $-g_u(U_0, V)$. We may express this as $-g_u(h(V), V) - \Delta g_u$, and observe again that $|\Delta g_u|_{\mathcal{G}(0,1)} \to 0$. It follows that $L_2$ can be made small in operator norm by requiring $\pi$ and $\epsilon$ to be small enough. We choose $\pi_0$ so that $|\lambda| \leq \frac{1}{2}$ for small $\epsilon$; then (3.16) can be solved for $s$, completing the solution of (3.14) and the proof of the lemma.

**Lemma 3.3.** $\lim_{\epsilon \to 0} |T(0, \epsilon)| = 0$.

**Proof.** This can be checked by direct calculation.

The proof of Theorem 3.1 will be completed with the aid of the following implicit function theorem.

**Theorem 3.4.** Let $\mathcal{N}$ be a neighborhood of the origin in $R^m$, and $\epsilon_0$ a positive number. Let $X$ and $Y$ be Banach spaces, and for each $\epsilon, \gamma$ with $\epsilon \in (0, \epsilon_0)$, $\gamma \in \mathcal{N}$, let $F(\cdot, \epsilon, \gamma)$ be a continuously differentiable mapping from $X$ into $Y$. Let $p_i(u), i = 1, \ldots, q$, be seminorms on $X$ such that $\sum p_i(u)$ is a norm. Let $m_\epsilon(\epsilon)$ be functions, continuously differentiable in $[0, \epsilon_0]$, positive for $\epsilon > 0$, and let $|u|_{X, \epsilon} = \sum m_\epsilon(\epsilon) p_i(u)$. Assume that $F(u, \epsilon, \gamma)$ has derivative (with respect to $u$) with modulus of continuity relative to the norm $|\cdot|_{X, \epsilon}$ independent of $\epsilon$ and $\gamma$, and that the derivative $F_u(0, \epsilon, \gamma)$ has inverse bounded with respect to this norm, independently of $\epsilon$ and $\gamma$. Finally, assume $\lim_{\epsilon \to 0} |F(0, \epsilon, \gamma)| = 0$, uniformly in $\gamma$. Then there exists a function $u(\epsilon, \gamma) \in X$ defined for small enough $\gamma$ and small enough positive $\epsilon$, satisfying $F(u(\epsilon, \gamma), \epsilon, \gamma) \equiv 0$ and $\lim_{\epsilon \to 0} |u(\epsilon, \gamma)|_{X, \epsilon} = 0$. Furthermore $u(\epsilon, \gamma)$ is continuous in $\epsilon$ and $\gamma$ relative to the norm $|u|_{X, \epsilon}$ uniformly in $\epsilon$ and $\gamma$. In other words, for each $\kappa > 0$, there exists a $\delta$ such that
\[ |u(\epsilon_1, \gamma_1) - u(\epsilon_2, \gamma_2)|_{X, \epsilon_1} \leq \kappa \quad \text{for} \quad |\epsilon_1 - \epsilon_2| + |\gamma_1 - \gamma_2| < \delta; \]
\[ \epsilon_1 > 0. \]  

(3.18)
Proof. A common proof of the standard implicit function theorem is based on a contractive mapping principle. With some refinements, that proof carries over to the present situation, to yield the existence of \( u(\epsilon, \gamma) \) with \( \lim_{\epsilon \to 0} |u(\epsilon, \gamma)|_{X, \epsilon} = 0 \). Moreover, the continuity of \( u \) with respect to \( \epsilon \) and \( \gamma \), for \( \epsilon > 0 \), is proved by the standard technique. Its uniformity follows from the uniformity assumptions in the theorem. See [4] for an implicit function theorem which includes this one, except for the final continuity assertion.

In view of Lemmas 3.2 and 3.3, Theorem 3.4 with no \( \gamma \)-dependence may now be applied to the operator \( T \), with \( X = C_0^2 \times C_0^2 \) and \( \|x, \epsilon\| = \|r\|_2 + \|s\|_2 \). We thus obtain a solution \( t(x, \epsilon) = (r(x, \epsilon), s(x, \epsilon)) \) of \( T(t, \epsilon) = 0 \), satisfying

\[
\lim_{\epsilon \to 0} (\|r\|_2 + \|s\|_2) = 0.
\]

In view of (3.13), (3.12), and the properties of the \( z_i \), we obtain (3.8), (3.9), and (3.11), as well as the fact that

\[
\lim_{\epsilon \to 0} e \alpha(0, \epsilon) = \hat{z}_0(0).
\]

The construction of \( \hat{z}_0 \) (see, for example, [2]) yields the fact that \( \hat{z}_0'(0) \) is just the right-hand side of (3.10a); this equation is thereby established. A similar argument yields (3.10b). This completes the proof of the theorem.

We shall need to examine problems slightly more general than (3.1), in that they may depend on additional real parameters \( \gamma = (\gamma_1, \ldots, \gamma_m) \).

**Theorem 3.5.** Let \( a(\gamma), b(\gamma), \alpha_i(\gamma), \beta_i(\gamma), i = 0, 1 \), be continuous functions of \( X \in \mathbb{R}^m \) for \( |\gamma| < 1 \), satisfying \( a(0) = 0, b(0) = 1 \). Let \( f(u, v, \gamma), g(u, v, \gamma) \) be continuously differentiable in \( u, v \), and continuous in \( \gamma \), and let \( h(v, \gamma) \) (continuously differentiable) satisfy \( f(h(v, \gamma), v, \gamma) = 0 \). For each \( |\gamma| < 1 \) assume that \( f, g, h, \alpha_i, \beta_i \), satisfy the hypothesis (3.4), (3.5) uniformly in \( \gamma \); and that (3.6), with 0 and 1 replaced by \( a(\gamma) \) and \( b(\gamma) \), respectively, has a solution \( V(x, \gamma) \) satisfying (3.7), with \( V \) and \( V_x \) continuous in \( \gamma \). Assume \( \pi = \sup_{x, \gamma} |g_s(h(V(x, \gamma), \gamma), V(x, \gamma), \gamma)| \) is small enough. Then for some \( \epsilon_0 > 0 \), \( \gamma_0 > 0 \), and all \( \epsilon \) and \( \gamma \) with \( 0 < \epsilon \leq \epsilon_0 \), \( |\gamma| \leq \gamma_0 \), there exists a solution \( u(x, \epsilon, \gamma), v(x, \epsilon, \gamma) \) of

\[
\begin{align*}
\epsilon^2 u'' &= f(u, v, \gamma), & a(\gamma) \leq x \leq b(\gamma), \quad (3.19) \\
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
u'' = g(u, v, \gamma), & a(\gamma) \leq x \leq b(\gamma) \\
u(a(\gamma)) = \alpha_0(\gamma), & u(b) = \alpha_1, & v(a) = \beta_1, & v(b) = \beta_1, \quad (3.20)
\end{cases}
\]

satisfying

\[
\lim_{\epsilon \to 0} u(x, \epsilon, \gamma) = h(V(x, \gamma), \gamma) \text{ uniformly for } x \in [a(\gamma) + \kappa, b(\gamma) - \kappa](\kappa > 0); \quad (3.21)
\]

\[
\lim_{\epsilon \to 0} \|v - V\|_{C^1(a(\gamma), b(\gamma))} = 0; \quad (3.22)
\]
and the analogs of (3.10), and (3.11). Here all these bounds and limit processes are uniform in γ, for small γ. Furthermore, v and v_α are continuous uniformly for x ∈ [a(γ), b(γ)], 0 < ε ≤ ε_1, |γ| ≤ γ_1; and εu_α(a(γ), ε, γ), εu_α(b(γ), ε, γ) are continuous uniformly for 0 < ε ≤ ε_1, |γ| ≤ γ_1.

Proof. First, observe that it suffices to consider the case a(γ) = 0, b(γ) = 1. In fact, a linear change of independent variable x = φ(x, γ) may be performed to map the interval [a(γ), b(γ)] onto [0, 1]. This change of variable does not affect the essential properties of the system (3.19).

Following the proof of Theorem 3.1, we define an operator T(t, ε, γ) which, with its derivative T_t, is continuous in t, ε, and γ. The invertibility of T_t(O, ε, γ) follows from Lemma 3.2, and the continuity of the inverse with respect to γ follows from that of T_t. Therefore (T_t(O, ε, γ))^{-1} is bounded independently of ε and γ, for small ε and γ. Theorem 3.4 may now be applied as before. It follows that there exist numbers ε_0, γ_0 such that for 0 < ε ≤ ε_0, |γ| ≤ γ_0, there exists a family t(x, ε, γ) ∈ C_ε,0^2 satisfying

\[ T(t, ε, γ) = 0, \quad \lim_{γ → 0} |t| = 0. \]

This family will be continuous in ε and γ, relative to the norm \( r \| r \|_2 + |s|_2 \), uniformly for ε and γ in the domain specified. As before, this implies the analogs of (3.8)–(3.11). Since the norm in C^2 is used for s, we have that v and v_α are continuous uniformly in the given domain of x, ε, and γ.

There only remains to prove the statement about εu_α. The argument above shows that εr_α is continuous uniformly for x, ε, and γ in the given domain. It therefore suffices to prove the uniform continuity of εu_α(a(γ), ε, γ), εU_p(b(γ), ε, γ), where \( U_0 = h(V(x, γ), γ) + z_0(x, ε, γ) + z(x, ε, γ) \). When one calculates the derivative, the part arising from the first term h is seen to be continuous in γ because of the continuity of V and h. The contributions from z_i can be written explicitly (the analogs of the right-hand sides of (3.10)), and directly verified to be continuous. This completes the proof.

4. The Transition Layer

We again consider the system (3.1), (3.2), but this time look for solutions such that as ε → 0, u approaches a function with discontinuity in the interior of the interval (0, 1). For simplicity we assume β_0 < β_1; this is not essential. Our assumptions are as follows.

I. The equation \( f(u, v) = 0 \) has two distinct solutions \( u = h_0(v) \), \( u = h_1(v) \), for \( v ∈ I_0 \) and \( v ∈ I_1 \), respectively, where \( I_i \) are open overlapping intervals with \( β_i ∈ I_i \). On \( I_i \),

\[ f_u(h_i(v), v) > 0. \]
Again for simplicity we assume \( h_0(\psi) < h_1(\psi) \).

Let \( J(\psi) = \int_{h_0(\psi)}^{h_1(\psi)} f(u, \psi) \, du \), defined on \( I_0 \cap I_1 \). We assume that

II. \( J \) has an isolated zero at some value \( \psi^* \in I_0 \cap I_1 \), and \( J \) changes sign as \( \psi \) passes through \( \psi^* \). Furthermore,

\[
\int_{h_0(\psi^*)}^{k} f(u, \psi^*) \, du > 0 \quad \text{for} \quad k \in (h_0(\psi^*), h_1(\psi^*)).
\] (4.2)

Remark. Consider the case of a function \( f \) whose zero contour has the general shape depicted in Fig. 1, so that there exist exactly three solutions \( u = h_i(\psi) \) of \( f(u, \psi) = 0 \) for \( \psi \in I_0 \cap I_1 \). Then it is seen immediately that \( J > 0 \) for \( \psi \) at the left endpoint of \( I_0 \), and \( J < 0 \) for \( \psi \) at the right endpoint of \( I_0 \). Therefore there will certainly be a value \( \psi^* \in I_0 \cap I_1 \) at which \( J = 0 \), and it can be checked that (4.2) will hold for any such zero. Therefore if we further assume that there is only one zero of \( J \) in the interval, then Hypothesis II is automatically fulfilled.

Let

\[
G(\psi) = \begin{cases} 
  g(h_0(\psi), \psi), & \psi \in I_0 \cap \{ \psi \leq \psi^* \}, \\
  g(h_1(\psi), \psi), & \psi \in I_1 \cap \{ \psi > \psi^* \}.
\end{cases}
\]

III. The boundary value problem

\[
V'' = G(V), \quad 0 < x < 1; \quad V(0) = \beta_0; \quad V(1) = \beta_1.
\] (4.3)

has a solution \( V(x) \) with \( V'(x) \neq 0 \), such that

\[
V'(x^*) = \psi^*.
\] (4.4)

for some value \( x^* \in (0, 1) \).

IV. For \( i = 0, 1 \),

\[
\int_{h_i(\beta_i)}^{t} f(u, \beta_i) \, du > 0
\]

for \( t \neq h_k(\beta_i) \) in the closed interval between \( h_k(\beta_i) \) and \( x_i \). Here \( k = 0 \) if \( V(x) < \psi^* \) for \( x \) in a neighborhood of the boundary point \( i \), and \( k = 1 \) if \( V(x) > \psi^* \) in this neighborhood.

We define the constant \( \pi \) by

\[
\pi = \sup_{t_0} |g_0(h_0(\psi), \psi)| + \sup_{t_1} |g_0(h_1(\psi), \nu)|.
\] (4.5)

Theorem 4.1. Under the above assumptions, there exist constants \( \pi_0, \epsilon_0 \),
such that if $\pi < \pi_0$, then there exists a family $(u(x, \epsilon), v(x, \epsilon))$ of solutions of (3.1), (3.2), defined for $0 < \epsilon < \epsilon_0$, satisfying (for any $\lambda > 0$)

$$\lim_{\epsilon \to 0} u(x, \epsilon) = h_0(V(x)) \text{ uniformly for } x \in (\lambda, x^\ast - \lambda)$$

$$= h_1(V(x)) \text{ uniformly for } x \in (x^\ast + \lambda, 1 - \lambda); \quad (4.6)$$

$$\lim_{\epsilon \to 0} v(x, \epsilon) = V(x) \text{ uniformly for } x \in [0, 1]. \quad (4.7)$$

**Note.** The construction of these functions yields much more specific information about the functions $u(x, \epsilon), v(x, \epsilon)$ than that stated in the theorem; specifically, it gives information about their layer behavior.

**Proof.** Let $\delta$ and $\omega$ be small real numbers, for the moment arbitrary. We set $\gamma = (h_0(v^\ast) + h_1(v^\ast))/2$.

We separate the interval $[0, 1]$ into two parts, $[0, x^\ast + \delta]$ and $[x^\ast + \delta, 1]$, and consider a boundary value problem on each separately:

$$\begin{cases}
\varepsilon^2 u_0^\ast - f(u_0, v_0) = 0, & x \in (0, x^\ast + \delta), \\
v_0(0) = \alpha_0, & u_0(x^\ast + \delta) = \gamma, \quad v_0(0) = \beta_0, \quad v_0(x^\ast + \delta) = v^\ast + \omega;
\end{cases} \quad (4.8a)$$

$$\begin{cases}
\varepsilon^2 u_1^\ast - f(u_1, v_1) = 0, & x \in (x^\ast + \delta, 1), \\
v_1(0) = \alpha_1, & u_1(x^\ast + \delta) = \gamma, \quad v_1(x^\ast + \delta) = v^\ast + \omega, \quad v_1(1) = \beta_1.
\end{cases} \quad (4.8b)$$

We seek solutions such that $u_i \to h_i(V(x))$ as $\epsilon \to 0$. To show the existence of solutions $u_i, v_i$, we shall need the following lemma; which is proved in Appendix B.

**Lemma 4.2.** For $\delta$ and $\omega$ small enough, there exist monotone solutions $V_0(x; \delta, \omega), V_1(x; \delta, \omega)$ of the problems

$$V_0^\ast = g(h_0(V_0), V_0), \quad x \in (0, x^\ast + \delta), \quad V_0(0) = \beta_0; \quad V_0(x^\ast + \delta) = v^\ast + \omega; \quad (4.9)$$

$$V_1^\ast = g(h_1(V_1), V_1), \quad x \in (x^\ast + \delta, 1), \quad V_1(x^\ast + \delta) = v^\ast + \omega; \quad V_1(1) = \beta_1. \quad (4.10)$$

Let

$$\hat{V}(x; \delta, \omega) \equiv V_0(x), \quad x \in [0, x^\ast + \delta],$$

$$= V_1(x), \quad x \in [x^\ast + \delta, 1].$$
Then

$$\lim_{\delta, \omega \to 0} |V - V_1| = 0. \quad (4.11)$$

We may now apply Theorem 3.5, with \( \gamma = (\delta, \omega) \), to problems (4.8) to obtain the existence of solutions \((u_{0}(x; \epsilon, \delta, \omega), v_{0}(x; \epsilon, \delta, \omega))\), and \((u_{1}, v_{1})\).

In the case of (4.8a) we take \( a(\gamma) = 0, b(\gamma) = x^* + \delta \). For (4.8b) we take \( a(\gamma) = x^* + \delta, b(\gamma) = 1 \). These solutions satisfy limit relations (3.21), (3.22), with subscripts 0 and 1 adjoined to \( u, v, h, \) and \( V \). They also satisfy conditions analogous to (3.10). For example, for \( i = 1, 2, \)

$$\lim_{\epsilon \to 0} \varepsilon u_{ij}(x^* + \delta; \epsilon, \delta, \omega) = + \left( 2 \int_{h_{j}(x^* + \omega)}^{x^*} f(u, v^* + \omega) \, du \right)^{1/2}. \quad (4.12)$$

Finally, the last assertion of Theorem 3.4 implies that \( \varepsilon u_{ij}(x^* + \delta; \epsilon, \delta, \omega) \) and \( v_{ij}(x^* + \delta; \epsilon, \delta, \omega) \) are uniformly continuous functions of \( \epsilon, \delta, \) and \( \omega \).

We define two functions

$$\Phi(\epsilon, \delta, \omega) = [\varepsilon u_{0j}(x^* + \delta; \epsilon, \delta, \omega)]^2 - [\varepsilon u_{1j}(x^* + \delta; \epsilon, \delta, \omega)]^2, \quad (4.13)$$

$$\Psi(\epsilon, \delta, \omega) = v_{1j}(x^* + \delta; \epsilon, \delta, \omega) - v_{0j}(x^* + \delta; \epsilon, \delta, \omega). \quad (4.14)$$

As we have just seen, \( \Phi \) and \( \Psi \) are uniformly continuous.

For the proof of Theorem 4.1, it will suffice to prove the existence of functions \( \delta(\epsilon), \omega(\epsilon) \) with

$$\lim_{\epsilon \to 0} \delta(\epsilon) = \lim_{\epsilon \to 0} \omega(\epsilon) = 0 \quad (4.15)$$

and

$$\Phi(\epsilon, \delta(\epsilon), \omega(\epsilon)) = \Psi(\epsilon, \delta(\epsilon), \omega(\epsilon)) = 0. \quad (4.16)$$

For then we define

$$u(x, \epsilon) = u_{0}(x; \epsilon, \delta(\epsilon), \omega(\epsilon)), \quad x \in [0, x^* + \delta(\epsilon)],$$

$$= u_{1}(x; \epsilon, \delta(\epsilon), \omega(\epsilon)), \quad x \in [x^* + \delta(\epsilon), 1],$$

with a similar definition for \( v(x, \epsilon) \). By virtue of (4.16), the derivatives of \( u \) and \( v \) match at \( x^* + \delta \), and \((u, v)\) will be a solution of our basic problem (3.1), (3.2). Moreover, the limit relations (4.6), (4.7) follow immediately from the limit relations (3.21), (3.22) satisfied by \( u \) and \( v \), from (4.11), and from (4.15).

Thus, our problem reduces to showing the existence of a solution of (4.16). This will be based on the following implicit function theorem.
THEOREM 4.3. Let \( F(x, y, z) \) and \( G(x, y, z) \) be functions of three real variables, continuous in some one-sided neighborhood of the origin \( \{0 < x < a, |y| + |z| < a\} \), and satisfying \( F(0, y, z) = F(0, 0, x) \). Also assume that the functions \( F_0(z) \equiv F(0, 0, z) \) and \( G_0(y) \equiv G(0, y, 0) \) have isolated zeros at the origin, and change sign as their respective individual variables pass through the origin. Then there exist functions \( y(x), z(x) \), defined for small enough positive \( x \), satisfying

\[
\lim_{x \to 0^+} y(x) = \lim_{x \to 0^+} z(x) = 0 \tag{4.17}
\]

and

\[
F(x, y(x), z(x)) = G(x, y(x), z(x)) = 0. \tag{4.18}
\]

This lemma will be proved in Appendix C.* We apply it to the functions \( \Phi \) and \( \Psi \) (in place of \( F \) and \( G \)). The uniformity of the continuity of \( \Phi \) and \( \Psi \) imply they may be extended by continuity to be defined for \( \epsilon \to 0 \) as well, so that after extension, are continuous for \( 0 < \epsilon \leq \epsilon_1, |\delta| + |\omega| \leq \delta_1 \), for some \( \epsilon_1 > 0, \delta_1 > 0 \). We calculate the following, using (4.12):

\[
\Phi_0(\omega) = \Phi(0, \delta, \omega)
= 2 \int_{h_0(v^* + \omega)}^{h_1(v^* + \omega)} f(u, v^* + \omega) \, du
- 2 \int_{h_0(v^* + \omega)}^{h_1(v^* + \omega)} f(u, v^* + \omega) \, du
= 2 \int_{h_0}^{h_1} f \, du = 2J(v^* + \omega).
\]

Hypothesis II assures us that \( \Phi_0 \) has an isolated zero at the origin, and changes sign there. Moreover \( \Phi(0, \delta, \omega) = \Phi(0, 0, \omega) \).

We now consider the function \( \Psi \). From (3.22) with subscripts attached, we know that

\[
\Psi_0(\delta) = \Psi(0, \delta, 0) = V_{1x}(x^* + \delta; \delta, 0) - V_{0x}(x^* + \delta; \delta, 0), \tag{4.19}
\]

where \( V_0 \) and \( V_1 \) are given by Lemma 4.2 (in this case, with \( \omega = 0 \)).

LEMMA 4.4. \( \Psi_0'(0) \neq 0 \).

Proof. We apply Lemma A (in Appendix A) to the function \( V_0(x; \delta, \omega) \), noting that \( V_0(x; 0, 0) = V(x) \). Formulas (A.3) and (A.4) yield

\[
\frac{d}{d\delta} V_0(x^* + \delta; \delta, 0) \bigg|_{\delta=0} = -\frac{V'(0)}{V_0(x^*)} = -\left[ V'(x^*) \int_0^{x^*} \frac{dt}{(V'(t))^2} \right]^{-1}. \]

* See the note added in proof at the end of this paper.
with a similar formula for \( (d/d\delta) V_1 \). Thus,

\[
\Psi_0'(0) = -[V'(x^*)]^{-1} \left[ \left( \int_0^{x^*} \frac{dt}{(V')^2} \right)^{-1} - \left( \int_1^{x^*} \frac{dt}{(V')^2} \right)^{-1} \right]
\]

\[
= \int_0^1 \frac{dt}{(V')^2} \left[ V'(x^*) \int_0^{x^*} \frac{dt}{(V')^2} \int_1^{x^*} \frac{dt}{(V')^2} \right]^{-1} = 0.
\]

Therefore, \( \Psi_0 \) also has an isolated zero at 0, and changes sign there. The requirements of Theorem 4.3 are fulfilled, and we conclude that there exist functions \( \delta(\epsilon), \omega(\epsilon) \) satisfying (4.15) and (4.16); this completes the proof of Theorem 4.1.

5. Extensions

The hypotheses of Theorem 4.1 may be weakened in several ways.

First, the assumption of monotonicity of \( F'(s) \) in Hypothesis III can be removed, provided certain inequalities are satisfied. Specifically, the following hypothesis, which is implied by Hypothesis III, is sufficient.

III'. \textit{Problem (4.3) has a solution \( V(x) \) assuming the value \( v^* \) at exactly one point \( x^* \).}

Let \( F_i(V) = g(h_i(V), V), i = 0, 1 \). Let \( w_i(x) \) be defined by

\[
w_i - F'_i(V(x)) w_i = 0, \quad w_i(0) = 0, \quad w'_i(1) = 1.
\]

Then

\[
w_i(x^*) \neq 0, \quad (5.1)
\]

and

\[
V'(0)/w_0(x^*) \neq V'(1)/w_1(x^*). \quad (5.2)
\]

If this hypothesis is made, then Lemma 4.2 can be proved with the aid of Lemma A, and the proof of Theorem 4.1 completed as before. In fact, from Lemma A we obtain

\[
\Psi_0'(0) = (V'(0)/w_0(x^*)) - (V'(1)/w_1(x^*)),
\]

which by assumption is not zero. We thus obtain

\textbf{Theorem 5.1.} \textit{The conclusions of Theorem 4.1 remain valid when Hypothesis III is replaced by Hypothesis III'.}
Another possible generalization would be to replace Hypothesis III by the following:

III*. For each \( y \in (0, 1) \), there exists a solution \( V_y(x) \) of

\[
V_y = \begin{cases} 
  g(h_0(V_y), V_y), & 0 < x < y \\
  g(h_1(V_y), V_y), & y < x < 1,
\end{cases}
\]

\[ V_y(0) = \beta_0, \quad V_y(1) = \beta_1, \]

and the equation \( V_y(y) = v^* \) can be solved for \( y = x^* \).

Let \( V(x) = V_{x^*}(x) \). Then inequalities (5.1), (5.2) hold.

The difference here is that \( V \) may now assume values both greater and less than \( v^* \) for \( x < x^* \).

Finally, Hypothesis II may be replaced by

II'. \( J(v^*) = 0 \) for some \( v^* \in I_0 \cap I_1 \). There are values of \( v \) arbitrarily close to \( v^* \) for which \( J < 0 \), and values arbitrarily close at which \( J > 0 \).

The proof under this hypothesis involves a refinement of Theorem 4.3.

**Theorem 5.2.** The conclusions of Theorem 4.1 remain valid when Hypothesis III is replaced by Hypothesis III*, and/or Hypothesis II by Hypothesis II'.

We come now to the question as to whether there exist solutions with many transition layers. This turns out to be possible, when the equation \( V(x) = v^* \) has many solutions, or when the equation \( f(u, v) = 0 \) can be solved for many functions \( u = h_i(v) \). We explore only the first possibility. Our third hypothesis now assumes the following form.

III**. The problem (4.3) has a solution \( V(x) \). The function \( V(x) \) assumes the value \( v^* \) at exactly \( q \) points \( x_n^* \):

\[
0 < x_1^* < x_2^* < \cdots < x_q^* < 1,
\]

with \( V(x) - v^* \) alternately positive and negative in intervals between the \( x_n^* \).

Let \( x_0^* = 0, \quad x_{q+1}^* = 1 \). For \( n = 1, \ldots, q \), define \( w_n(x) \) for \( x \in [x_{n-1}^*, x_{n+1}^*] \) by the initial-value problem

\[
\begin{align*}
  w_n'' - G'(V(x)) w_n &= 0, & x &\in (x_{n-1}^*, x_n^*) \cup (x_n^*, x_{n+1}^*), \\
  w_n(x_n^*) &= 0; & w_n'(x_n^*) &= 1.
\end{align*}
\]

(Note that \( G \) is differentiable for \( V \neq v^* \), and that \( V \neq v^* \) for \( x \neq x_n^* \).)

Then for \( n = 1, \ldots, q \),

\[
w_n(x_{n+1}^*) \neq 0, \quad (5.3)
\]

and

\[
V'(x_{n+1}^*)/w_{n-1}(x_n^*) \neq V'(x_{n+1}^*)/w_{n+1}(x_n^*). \quad (5.4)
\]
THEOREM 5.3. Let the assumptions of Theorem 4.1 be fulfilled, with Hypothesis III replaced by Hypothesis III". Then the conclusions of that theorem are true with the following change: the limit of \( u(x, \epsilon) \) as \( \epsilon \to 0 \) is \( h_0(V(x)) \) wherever \( V(x) < \nu^* \), and is \( h_1(V(x)) \) where \( V(x) > \nu^* \).

Sketch of proof. The solution is constructed in stages. First, for arbitrary small \( \delta_2, \omega_2 \), Theorem 5.2 gives the existence of a solution of (3.1) on the interval \([0, x_2^* + \delta_2]\) satisfying \( u(0) = \alpha_0 \), \( u(x_2^* + \delta_2) = \gamma \), \( v(0) = \beta_0 \), \( v(x_2^* + \delta_2) = \nu^* + \omega_2 \) with \( u \) approaching \( h_1(V(x)) \) as \( \epsilon \to 0 \) for \( x \in (0, x_1^*) \) (if \( \beta_0 < \nu^* \)) and \( h_1(V(x)) \) for \( x \in (x_1^*, x_2^* + \delta_2) \). To check this, one need only verify that the appropriate inequalities of types (5.1), (5.2) hold for this problem. In the case \( \delta_2 = \omega_2 = 0 \), they follow immediately from (5.3), (5.4) with \( n = 1 \). But since \( \nu_n \) and \( V' \) vary continuously with \( \delta_2 \) and \( \omega_2 \), they remain valid for small \( \delta_2 \) and \( \omega_2 \).

Secondly, for all small \( \delta_3, \omega_3 \), Theorem 3.1 yields the existence of a solution of (3.1) on \([x_2^* + \delta_2, x_3^* + \delta_3]\) satisfying the above conditions at \( x_2^* + \delta_2 \), similar conditions at \( x_3^* + \delta_3 \), and approaching \( h_0(V(x)) \) as \( \epsilon \to 0 \).

Next, for small \( \delta_3, \omega_3 \), one shows the existence of functions \( \delta_2(\epsilon), \omega_2(\epsilon) \) such that the two solutions constructed above have matching derivatives at \( x_2^* + \delta_2(\epsilon) \). The matching is accomplished as in the proof of Theorem 4.1, by solving certain equations \( \Phi(\epsilon, \delta_2, \omega_2) = \Psi(\epsilon, \delta_2, \omega_2) = 0 \). Exactly as in the former proof, one sees that \( \Phi \) satisfies the appropriate hypothesis of Theorem 4.3. To verify that \( \Psi \) does also, one shows that \( \Psi(0, 0, 0) \neq 0 \). This follows from (5.4) using the same train of thought as before, slightly complicated this time by the fact that when \( \epsilon \neq 0, \delta_1 \) and \( \omega_1 \) depend in an unknown way on \( \delta_2 \) and \( \omega_2 \). However, this dependence vanishes when \( \epsilon \to 0 \).

This establishes a solution with two transition layers, at \( x_1^* \) and \( x_2^* \). The same procedure is continued, and the solution extended to include layers at the other points.

6. An Example

When \( g \) does not depend on \( u, v = V \) is known a priori. We then have a problem of type (2.1), and examples with any number of transition layers can be constructed. We give here a less trivial example, in which the existence of transition layers depends on the coupling between the two equations of (3.1).

Consider the problem

\[
\varepsilon^2 u'' = (u - v)(u^2 - 1), \quad u(0) = \alpha_0, \quad u(1) = \alpha_1 \\
v'' = -au, \quad v(0) = \beta_0, \quad v(1) = \beta_1.
\]
The reduced equation has two solutions, \( u = h_0(v) = -1 \) and \( u = h_1(v) = 1 \). It is easily checked that \( v^* = 0 \). The equation for \( V \) then becomes

\[
V^* = \begin{cases} 
  a, & V < 0 \\
-a, & V > 0; 
\end{cases} \quad V(0) = \beta_0; \quad V(1) = \beta_1.
\]

We are interested in constructing solutions with many transition points \( x_n^* \), as in Theorem 5.3. In intervals between the points \( x_n^* \), \( V \) must be of the form

\[
V = \pm \frac{m^2}{2a} + m(x - x_n^*), \tag{6.1}
\]

the \( \pm \) sign holding when \( V < 0 \). The continuity of \( V' \) requires that the same constant \( m \) be chosen for all intervals. There results a function, periodic in \( x \), parabolic in sections, with amplitude \( m^2/2a \) and spacing \( 2m/a \) between zeros. In fact, any such function, which is of the form (6.1) between zeros, with any value of \( m \), is a solution of the differential equation for \( V \). The boundary conditions may often be satisfied by an appropriate choice of \( m \), together with an appropriate translation of the independent variable of the resulting function. For example, if \( \beta_0 < 0 < \beta_1 \), we may always find a solution \( V \) with exactly one zero. Furthermore, a solution with \( q \) zeros can be found, provided that \( |\beta_0| \) and \( |\beta_1| \) are less than some number depending only on \( q \) and \( a \). In the particular case \( \beta_0 = \beta_1 = 0 \), there exist an infinite number of possible functions \( V \).

According to Theorem 5.3, there is a solution family such that \( u(x, \epsilon) \) has a
transition layer at each of the $x_n^*$, if the hypotheses are met. Hypotheses I and II certainly hold. Hypothesis IV constitutes two rather unrestrictive inequalities relating the $\alpha_i$ and $\beta_i$. As for Hypothesis III", we need only check that (5.3) and (5.4) hold. The first is always satisfied, because $w_n(x) = x - x^*$; the second will be satisfied except in special cases. Finally, the condition that $\pi$ be small enough can be insured by choosing $a$ to be small.

The graph of $u$ and $v$ are as in Fig. 2.

The case of $\beta_0 = \beta_1 = 0$ is particularly interesting, because there exist functions $V$ with arbitrarily many zeros. Hypothesis IV is simply: $\alpha_0 < 1$, $\alpha_1 > -1$. Of course by interchanging the roles of $h_0$ and $h_1$, we may also allow data $\alpha_i$ satisfying $\alpha_i > -1$, $\alpha_i < 1$. Under these conditions, there exist an infinite number of solution families to the problem; in fact there exists a family with any given number of transition layers.

**APPENDIX A**

**Lemma A.** Consider the problem

$$u'' = f(u); \quad u(a) = \beta_0, \quad u(b + \delta) = \beta_1 + \omega,$$  \hspace{1cm} (A.1)

where $f$ continuously differentiable. Suppose it has a solution $u_0(x)$ when $\delta = \omega = 0$. Let $w$ be the solution of

$$w'' - f'(u_0(x))w = 0; \quad w(a) = 0; \quad w'(a) = 1.$$  \hspace{1cm} (A.2)

(i) If $w(b) \neq 0$, then for all small enough numbers $(\delta, \omega)$, there exists a solution $u(x; \delta, \omega)$ of (A.1). Let $\psi(\delta) = u_0(b + \delta; \delta, 0)$. Then

$$\psi'(0) = -u_0'(a)/w(b).$$  \hspace{1cm} (A.3)

(ii) If $u_0'(x) \neq 0$ for all $x \in [a, b]$, then

$$w(x) = u_0'(a) u_0'(x) \int_a^x (dt; (u_0'(t))^2).$$  \hspace{1cm} (A.4)

In particular, $w(b) \neq 0$, so the conclusions in (i) hold.

**Proof.** Most of the lemma is known, but the proof will be given for completeness. Let $u(x; \eta)$ be the solution of

$$u'' = f(u); \quad u(a) = \beta_0; \quad u'(a) = k + \eta,$$

where $k = u_0'(a)$. Thus $u_0(x) = u(x; 0)$. The theory of differential equations yield that for some small $\delta_1$, $u(x; \eta)$ exists for $x$ in a $\delta_1$-neighborhood of $b$,
provided \( \eta \) is small enough. Moreover, \( u(x; \eta) \) is a differentiable function of \( \eta \). Setting \( z(x) = \frac{d}{d\eta} u(x; 0) \), we may check that \( z \) satisfies (A.2), so \( z \in \mathcal{W} \).

Let \( F(\eta, \delta, \omega) = u(b + \delta; \eta) - (\beta_1 + \omega) \). Then \( F(0, 0, 0) = 0 \), and \( F_\delta(0, 0, 0) = \frac{d}{d\eta} u(b, 0) = \omega(b) \). If \( \omega(b) \neq 0 \), we may then solve the equation \( F(\eta, \delta, \omega) = 0 \) for \( \eta = \eta(\delta, \omega) \) when \( \delta \) and \( \omega \) are small enough, and thus obtain the desired solution \( u(x; \eta(\delta, \omega)) \).

For \( \omega \equiv 0 \), we have

\[
0 = F_\delta(0, 0, 0) \frac{\partial \eta}{\partial \delta}(0, 0) + F_\delta(0, 0, 0) = \omega(b) \frac{\partial \eta}{\partial \delta} + u_0'(b). \quad (A.5)
\]

We calculate

\[
\psi'(0) = \frac{d}{d\delta} u_x(b + \delta; \eta(\delta, 0)) \bigg|_{\delta=0} = u_{xx}(b, 0) + \eta_\delta(0, 0) u_{x\eta}(b; 0).
\]

Setting \( W(x) = u_0'(x) \), we have from this expression and (A.5)

\[
\psi'(0) = W'(b) - (W(b)w(b)) w'(b) = Q(b)/w(b),
\]

where \( Q(x) = wW' - w'W \). But \( W \) is also a solution of (A.2) with different initial conditions, so the Wronskian \( Q \) is constant, equal to \( Q(0) = -u_0'(a) \). This establishes (A.3).

If \( W(x) \neq 0 \), the function \( v(x) = W(a) W(x) \int_0^x \frac{dt}{w(t)} \) also satisfies (A.2), so \( v \equiv w \). This establishes part (ii), and completes the proof.

**APPENDIX B: PROOF OF LEMMA 4.2**

The existence of \( V_0 \) and \( V_1 \) follows from Lemma A (Appendix A) or, alternatively, from a phase-plane analysis. In fact, from such an analysis, we have the representation

\[
V_0' = \left( 2 \int_{\delta_0}^{\eta_0} g(h_0(t), t) \, dt + K^2(\delta, \omega) \right)^{1/2}, \quad (B.1)
\]

where \( K(\delta, \omega) = V_0'(0; \delta, \omega) \). Although \( V_0 \) was defined on \([0, x^* + \delta] \), which does not extend to \( x^* \) when \( \delta < 0 \), it may be continued as a solution of the same differential equation for \( x \) in some neighborhood of \( x^* \). The same applies to \( V_1 \). Note that \( V_0(x; 0, 0) = V(x) \), for \( x \in [0, x^*] \). As \( (\delta, \omega) \to (0, 0) \), we have \( K(\delta, \omega) \to K(0, 0) \), so by (B.1), \( V_0' \to V' \) uniformly on \([0, x^*] \).

Hence \( V_0 \to V \) uniformly on the same interval. Thus \( |V_0 - V|_{C^1(0, x^*)} \to 0 \). The same considerations apply to \( V_1 \) on the interval \([x^*, 1] \). Now consider
the function \( \dot{V} \). Suppose for the sake of simplicity that \( \delta < 0 \); the other cases may be treated similarly. Then

\[
\| \dot{V} - V \|_{1} \leq \| V_{0} - V \|_{C^{1}(0,x^{*}+\delta)} + \| V_{1} - V \|_{C^{1}(x^{*}-\delta,x^{*})} + \| V_{1} - V \|_{C^{1}(x^{*},1)}.
\]

We have seen that the first and third terms approach 0 as \( (\delta, \omega) \to (0,0) \). The second term, on the other hand, is bounded by

\[
\sup_{x \in [x^{*}-\delta,x^{*}]} (\| V_{1}(x) - V_{1}(x^{*}) \| + \| V(x) - V(x^{*}) \|) + \sup_{x \in [x^{*}-\delta,x^{*}]} (\| V_{1}'(x) - V_{1}'(x^{*}) \| + \| V'(x) - V'(x^{*}) \|) + \| V_{1}'(x^{*}) - V(x^{*}) \| + \| V_{1}'(x^{*}) - V'(x^{*}) \|
\]

The continuity of \( V, V_{1}, V' \), and \( V_{1}' \) show that the two suprema approach zero; and the above considerations show that the last two terms do also. This completes the proof.

**APPENDIX C: PROOF OF THEOREM 4.3**

For the sake of definiteness, we assume that \( zF_{0}(z) > 0 \) and \( yG_{0}(y) > 0 \), for \( |y| \) and \( |z| \) small enough. The other cases are treated similarly. Let \( f(z) \) and \( g(y) \) be continuous strictly increasing functions satisfying \( zf(z) < zF_{0}(z) \) and \( yg(y) < yG_{0}(y) \) for \( y, z \neq 0 \), \( 0 < |y| < a_{1} \), \( 0 < |z| < a_{1} \), where \( a_{1} \) is some positive number <\( a \). Such functions exist. For example, let us choose \( a_{1} \) so that \( F_{0} \) and \( G_{0} \) are nonzero in the range \( 0 < |y| < a_{1} \), \( 0 < |z| < a_{1} \). Then for each positive integer \( n \), there is a linear function \( \phi_{n}(z) \) with positive slope satisfying \( \phi_{n}(1/n) = 0 \), \( \phi_{n}(z) < F_{0}(z) \) for \( z \in [0,a_{1}] \). We could then define \( f(z) = \sup_{n} \phi_{n}(z) \) for \( 0 < |z| < a_{1} \), with a similar definition when \( z < 0 \), and a similar one for \( g(y) \). Being strictly increasing, \( f \) and \( g \) have inverses \( f^{-1} \) and \( g^{-1} \).

Let \( f_{1}(x) \) be a continuous increasing function satisfying \( f_{1}(0) = 0 \), \( f_{1}(x) \geq |F(x,y,z) - F_{0}(z)| \); the assumption on \( F \) guarantees its existence. Finally, let \( g_{1}(x,z) \) be a continuous function, increasing in both variables, satisfying \( g_{1}(0,0) = 0 \), \( g_{1}(x,z) \geq |G(x,y,z) - G_{0}(y)| \).

Given small enough \( x_{1} > 0 \), let \( z_{1} = f^{-1}(f_{1}(x)) \) and \( y_{1} = g^{-1}(g_{1}(x_{1},z_{1})) \). Thus

\[
\lim_{x_{1} \to 0} z_{1} = \lim_{x_{1} \to 0} y_{1} = 0. \tag{C.1}
\]

Let \( S \) be the rectangle in the \((y,z)\)-plane defined by \( \{y \leq y_{1}, |z| \leq z_{1}\} \), and let \( L_{0}, L_{1} \) be the lateral sides: \( L_{0} = \{(y,z): y = -y_{1}, |z| \leq z_{1}\} \),
$L_1 = \{(y, z): y = y_1, |z| \leq z_1\}$. For each $x \in (0, x_1)$, let $\Omega(x) = \{(y, z) \in S: F(x, y, z) = 0\}$.

For each $\eta > 0$, let $\Omega(x, \eta)$ be the union of a finite number of open balls of radius $\eta$, covering $\Omega(x) \cup L_0$. We require that two of the balls be centered at $(-y_1, -z_1)$ and $(y_1, z_1)$ (the endpoints of $L_0$). Let $K$ be the component of the boundary of $\Omega(x, \eta)$ containing the point $(-y_1, z_1 + \eta)$. $K$ consists of a chain of arcs of circles. This chain does not terminate; otherwise the terminal arc would be bounded on both sides by a ball, which is impossible. For a similar reason, the chain cannot bifurcate, so it is closed curve. Since it contains points with $y < -y_1$ as well as some with $y > -y_1$, $K$ must cross the line $y = -y_1$ at another point besides $(-y_1, z_1 + \eta)$. The only possibility is $(-y_1, -z_1 - \eta)$. Therefore $K$ joins these latter two points. Let $K_+$ be the the portion of $K$ with $y > -y_1$, (that is, to the right of $L_0$).

For $z = z_1$, $F(x, y, z) = F_0(z_1) + [F(x, y, z_1) - F_0(z_1)] > f(z_1) - f_1(x) \geq f(z_1) - f_1(x_1) = 0$, by the definition of $z_1$, and the increasing nature of $f_1$. Therefore $\Omega(x)$ does not touch the top of $S$, and for small enough $\eta$, $K_+$ will have exactly one point $(-y_1 + \eta, z_1)$ on the top boundary of $S$. At that point, $F > 0$. A similar argument shows that for small enough $\eta$, $K_+$ has one point on the bottom boundary of $S$, at which point $F < 0$. Therefore $F = 0$ at some intermediate point $p$ on $K_+$. If $p \in S$, then $p \in \Omega(x)$. However, $K_+$ is on the boundary of an open set containing $\Omega(x)$, so $K_+ \cap \Omega(x) = \emptyset$ and $p \notin \Omega$. Therefore $p \notin S$. Since $p$ is to the right of $L_0$, it must in fact be to the right of $L_1$. Therefore $\Omega(x, \eta)$ contains points of $L_1$. Hence there exists a curve $\Gamma(x, \eta)$ connecting $L_0$ to $L_1$, and contained within $\Omega(x, \eta)$. By definition of $\Omega(x, \eta)$, each point of $\Gamma$ is at a distance at most $\eta$ from $\Omega(x)$. Represent $\Gamma$ by $y = y_{z,n}(t)$, $z = z_{z,n}(t)$, these being continuous functions of $t$ for $t \in [0, 1]$, with $y_{z,n}(0) = -y_1$ and $y_{z,n}(1) = y_1$.

Let $h(x, t, \eta) = G(x, y_{z,n}(t), z_{z,n}(t))$. It is a continuous function of $t$. Now $h(x, 1, \eta) = G(x, y_1, z_{z,n}(1)) = G_0(y_1) + [G(x, y_1, z_{z,n}(1)) - G_0(y_1)] > g(y_1) - g_1(x_1, z_1) = 0$. Similarly, $h(x, 0, \eta) < 0$. Therefore, $h(x, t^*, \eta) = 0$ for some $t^* \in (0, 1)$. Let $\rho(x, \eta) = (y_{z,n}(t^*), z_{z,n}(t^*))$, so that $G(x, \rho(x, \eta)) = 0$ and $\rho \in \Omega(x, \eta)$. As $\eta \to 0$, there exists a subsequence $\rho(x, \eta_k) \to \rho(x) = (y(x), z(x))$. By continuity of $F$ and $G$, we have $F(x, y(x), z(x)) = F(x, y(x), z(x)) = 0$, establishing (4.18). The limit relation (4.17) follows from (C.1). This completes the proof.

REFERENCES


