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Radial and Non-radial Solutions of $-\Delta u = \lambda f(u)$, on an Annulus of $\mathbb{R}^n$, $n \geqslant 3$

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1. INTRODUCTION

We recall here some results concerning the following problem:

Let Ω be a bounded open subset of $\mathbb{R}^n$, we want to find $u \in \mathscr{C}^{2,\alpha}(\Omega, \mathbb{R})$ such that

$$\begin{cases} -\Delta u(x) = \lambda e^{u(x)} & \text{on } \Omega \\ u(x) = 0 & \text{on } \partial\Omega \\ u(x) > 0 & \text{on } \Omega. \end{cases} \qquad (1)$$

First, let us note that every regular solution of (1) is strictly positive on Ω, this is just a consequence of the maximum principle [8]. Moreover it is very easy to obtain the lemma:

LEMMA 1. *There exists $\lambda^*(\Omega)$ depending only on Ω such that* (1) *has no regular solution whenever $\lambda \in \mathbb{R} \setminus [0, \lambda^*]$.*

The proof of this lemma is straightforward. Just multiplying the equation by u and integrating by parts we obtain

$$\int_\Omega |\nabla u(x)|^2 \, dx = \lambda \int_\Omega u(x) \, e^{u(x)} \, dx.$$

So λ cannot be negative. Next we multiply (1) by φ_1, the first eigenfunction of the Laplacian on Ω

$$\begin{cases} -\Delta \varphi_1(x) = \lambda_1 \varphi_1(x) & \text{on } \Omega \\ \varphi_1(x) = 0 & \text{on } \partial\Omega \\ \varphi_1(x) > 0 & \text{on } \Omega, \end{cases}$$

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and we obtain

$$\int_\Omega \{\varphi_1(x)\,\Delta u(x) - \Delta\varphi_1(x)\,u(x)\}\,dx = 0 = \int_\Omega \varphi_1(x)\{\lambda_1\,u(x) - \lambda e^{u(x)}\}\,dx.$$

A necessary condition for (1) to have solutions is that the function $\psi(x) = \lambda_1 x - \lambda e^x$ changes sign on $\mathbb{R}^+$ and this is equivalent to $\lambda_1/\lambda > e$. This proves the above lemma.

We can note that Lemma 1 is still valid if e^u is replaced by $f(u)$ in (1), with f satisfying the following:

There exists $\alpha > 0$ such that $f(x) > \alpha \sup(x, 1)$ for all $x \in \mathbb{R}^+$. In particular $f(0) > 0$.

We now state a well-known theorem of M. G. Crandall and P. H. Rabinowitz [5]:

THEOREM 1 [5]. *There exists a number $\bar\lambda(\Omega) \in [0, +\infty)$ depending only on Ω and a continuous map $\lambda \to u$ from $[0, \bar\lambda)$ into $\mathscr{C}_0^{2,\alpha}(\Omega, \mathbb{R})$ satisfying*:

(i) $-\Delta u = \lambda e^u$ *on* Ω,

(ii) *the least eigenvalue of the linearized operator* $Lw = -\Delta w - \lambda e^u w$ *is strictly positive for* $\lambda \in [0, \bar\lambda)$ *so* L *is invertible from* $\mathscr{C}_0^{2,\alpha}(\Omega, \mathbb{R})$ *into* $\mathscr{C}^{0,\alpha}(\Omega, \mathbb{R})$.

When Ω is a ball, $\Omega = B^n = \{x \in \mathbb{R}^n / |x| < 1\}$, D. D. Joseph and T. S. Lungren [9] have studied all the regular solutions of (1) which are radially symmetric. It is enough to study the set of solutions for a ball of radius one because if u_0 is such a solution, with $\lambda = \lambda_0$, then $u(x) = u_0(x/R)$ is a solution of (1) on a ball of radius R, with $\lambda = \lambda_0/R^2$. The result of B. Gidas, W. M. Ni, and L. Nirenberg [7] ensures that all regular solutions of (1) on a ball are radially symmetric, so, following [9], we can give a complete description of the set of regular solutions of (1) when Ω is a ball.

THEOREM 2 [9]. *Let $n - 2 > 0$, and Ω be a ball, the solutions of* (1) *have the following properties*:

- *There exists $\bar\lambda > 0$ such that there is no solution when $\lambda > \bar\lambda$.*
- *For $n \geqslant 10$ there exists a unique solution for each $\lambda \in [0, \bar\lambda)$.*
- *For $n < 10$ and $\lambda = \bar\lambda$ there exists a unique solution.*
- *For $n < 10$ and $\lambda = 2(n-2)$ there are infinitely many solutions.*
- *For $n < 10$ and $\lambda \neq 2(n-2)$ there exists at least one solution.*

We must also mention the work of F. Mignot and J. P. Puel on related topics [13]. In another work [14] the same authors have proven that, on a ball, there was a unique radial solution singular at the origin if

$\lambda = 2(n-2)$ and no such a solution when $\lambda \neq 2(n-2)$. We give here a brief summary of their technique for the sake of completeness. They look for radial solutions, so that the original equation is reduced to the ordinary differential equation:

Find u from $(0, 1]$ into $\mathbb{R}^+$ satisfying

$$\begin{cases} \dfrac{d^2u}{dr^2}(r) + \dfrac{n-1}{r}\dfrac{du}{dr}(r) = -\lambda e^{u(r)} \\ u(1) = 0. \end{cases} \tag{2}$$

After the suitable change of variables, $s = \log r$, if we set $v = du/ds$ and $w = -\lambda e^{2s} e^u$, we see that (2) is equivalent to:

Find v, w from $(-\infty, 0]$ into $\mathbb{R}$ satisfying

$$\begin{cases} \dfrac{dv}{ds}(s) = w(s) - (n-2)\, v(s) \\ \dfrac{dw}{ds}(s) = w(s)(v(s) + 2) \\ w(0) = -\lambda. \end{cases}$$

According to the dimension of the space, the branch of regular solutions has a different behavior (see [14] for a precise statement). In the (v, w) space, the point $(0, 0)$ is always an hyperbolic fixed point independently of the dimension. For $n \in \{3, ..., 9\}$, the other stationary point $(-2, -2(n-2))$ is a spiral attractor, whereas it is an attractor for $n \geqslant 10$. Moreover $W^u(0, 0)$ the unstable manifold of the point $(0, 0)$ is tangent to the w-axis and agrees with $W^s(-2, -2(n-2))$ the stable manifold of the point $(-2, -2(n-2))$.

From this point of view, the regular solutions of (1) are in one-to-one correspondence with the trajectories $(v(s), w(s))$ included in $W^u(0, 0)$, such that $(v(s), w(s))$ tends to $(0, 0)$ when s tends to $-\infty$ and $w(0) = -\lambda$.

We are going to formulate all the former results in a more convenient way for our purpose. To begin, just observe that, with our notations, we have

$$v(s) = \frac{du}{ds}(s) = e^s \frac{du}{dr}(e^s).$$

In particular $v(0) = (du/dr)(1)$ and $w(0) = -\lambda$.

If we define $\mathscr{P}(\lambda_0)$ to be the set of values of the modulus of the normal derivatives on the boundary, for all the radial (regular and singular) solutions of (1), with $\Omega = B^n$ and $\lambda = \lambda_0$, we find that

$$\mathscr{P}(\lambda_0) = \{v \in \mathbb{R}^+ / (-v, -\lambda) \in W^u(0, 0)\}.$$

From what we have just seen, we can derive that $\mathscr{P}(\lambda)$ is bounded independently of $\lambda > 0$ and empty for λ large enough.

The purpose of this paper is to study (1) when Ω is an annulus of $\mathbb{R}^n$, $n \geq 3$. Most of the work we have done has been motivated by the study of (1), which is done for the two dimensional case, in papers of T. Susuki and K. Nagasaki [15, 24] and also S. S. Lin [11].

In the preceding works, the existence of a continuous curve of radial solutions for (1), on an annulus, and also the existence of infinitely many symmetry breaking points along this curve is proved.

2. Some Basic Definitions

Let Ω be a bounded open of $\mathbb{R}^n$ and $M(u, \lambda)$ be a smooth operator defined from $\mathscr{C}_0^{2,\alpha}(\Omega, \mathbb{R}) \times \mathbb{R}^+$ into $\mathscr{C}^{0,\alpha}(\Omega, \mathbb{R})$. We consider the following problem (P_0):

Find $(u, \lambda) \in \mathscr{C}_0^{2,\alpha}(\Omega, \mathbb{R}) \times \mathbb{R}^+$ such that $M(u, \lambda) = 0$.

Definition 1. We will say that $\Gamma \subset \mathscr{C}_0^{2,\alpha}(\Omega, \mathbb{R}) \times \mathbb{R}^+$ is a continuous (resp. $\mathscr{C}^k$) branch of solutions for (P_0) if there exists a continuous (resp. $\mathscr{C}^k$) map ϕ from $\mathbb{R}^+$ into $\mathscr{C}_0^{2,\alpha}(\Omega, \mathbb{R}) \times \mathbb{R}^+$ satisfying $M(\phi(\theta)) = 0$ and $\Gamma = \{\phi(\theta)/\theta \in R^+\}$.

When a group G is acting on both $\mathscr{C}_0^{2,\alpha}(\Omega, \mathbb{R})$ and $\mathscr{C}^{0,\alpha}(\Omega, \mathbb{R})$, we introduce the notion of symmetry:

Definition 2. We say that M is equivariant under the action of a group G if $M(g \cdot u, \lambda) = g \cdot M(u, \lambda)$ for all $g \in G$, $u \in \mathscr{C}^{0,\alpha}(\Omega, \mathbb{R})$ and $\lambda \in \mathbb{R}^+$. We say that $u \in \mathscr{C}^{0,\alpha}(\Omega, \mathbb{R})$ is symmetric under G if $g \cdot u = u$ for all $g \in G$.

From the work of E. N. Dancer [6] it is now standard to introduce the notion of infinitesimal symmetry breaking:

Definition 3. Let Γ be a branch of symmetric solutions of (P_0). A point $(u, \lambda) \in \Gamma$ is an infinitesimal symmetry breaking point if $dM_u(u, \lambda)$, the linearized operator, admits a non-trivial kernel which contains non-symmetric elements. And we will say that this point is a symmetry breaking point if there exists non-symmetric solutions of (P_0) in every neighbourhood of (u, λ) in $\mathscr{C}_0^{2,\alpha}(\Omega, \mathbb{R})$.

Let us note that, from the paper of E. N. Dancer, we know that infinitesimal symmetry breaking is necessary for symmetry breaking to occur.

We can now state the main results of this paper.

3. Main Results

Let us define $A(R) = \{x \in \mathbb{R}^n / R < |x| < 1\}$ for all $R \in (0, 1)$ and $n \geqslant 3$, we consider the problem (3):

Find $u \in \mathscr{C}_0^{2,\alpha}(A(R), \mathbb{R})$ such that

$$\begin{cases} -\Delta u(x) = \lambda e^{u(x)} & \text{on } A(R) \\ u(x) = 0 & \text{on } \partial A(R) \\ u(x) > 0 & \text{on } A(R). \end{cases} \tag{3}$$

The aim of our work is to give a description of the set

$$\mathscr{S}(R) = \{(u, \lambda) \in \mathscr{C}_0^{2,\alpha}(A(R), \mathbb{R}) \times \mathbb{R}^+ / u \text{ is a solution of } (3)\}.$$

We have the following theorem:

THEOREM 3. *For all $R \in (0, 1)$, the set $\mathscr{S}(R)$ contains at least a continuous branch of regular solutions which are symmetric under the group $O(n)$. This branch joins the trivial solution $(0, 0)$ to the point $(\omega, 0)$, where $\omega(x)$ is defined to be equal to $+\infty$ on all $A(R)$.*

An immediate corollary is:

COROLLARY 1. *For all $R \in (0, 1)$ there exists a $\lambda^*(R) > 0$ such that (3) has at least one solution for $\lambda = \lambda^*$ and at least two solutions for $\lambda < \lambda^*$.*

Radial solutions of (3) enjoy the following properties:

PROPOSITION 1. *If u is a radial solution of (3), then the maximum of u is achieved in the area $(1 + R^{2-n})/2(n-2) < |x| < (1 + R)/2$, moreover, along the branch of radial solutions obtained in the previous theorem, the quantity $\lambda \int_{A(R)} e^{(u(x))}\, dx$ goes from 0, at the point $(u, \lambda) = (0, 0)$ to $+\infty$, at the point $(u, \lambda) = (\omega, 0)$.*

Concerning the symmetry breaking problem we can prove:

THEOREM 4. *Along the branch of radial solutions obtained in Theorem 3, there are infinitely many distinct infinitesimal symmetry breaking points.*

For a dense subset of values of R we can be more precise and state:

THEOREM 5. *There is a subset $\Sigma \subset (0, 1)$ of measure equal to 1 such that, for all $R \in \Sigma$, the set $\mathscr{S}(R)$ contains at least a $\mathscr{C}^\infty$ branch of regular solutions, joining the trivial solution $(0, 0)$ to the point $(\omega, 0)$ as before, but, along this branch, there are infinitely many distinct symmetry breaking* *points. Moreover, this branch does not bifurcate in the space of radial solutions* $\mathscr{C}^{2,\alpha}_{0,\mathrm{rad}}(A(R), \mathbb{R}) \times \mathbb{R}^+$.

A more precise description of the symmetry breaking set is given by the two theorems that follow. By a result of C. Popiech [17] we obtain Theorem 6, but before let us recall some definitions:

DEFINITION 4. A cell $k \subset \mathscr{C}^{2,\alpha}_0(A(R), \mathbb{R}) \times \mathbb{R}^+$ of dimension m is a subset topologically equivalent to a m-ball, whose closure is homeomorph to a closed m-ball of $\mathbb{R}^m$.

A cell complex $(K, \mathscr{K})$ is a subset $K \subset \mathscr{C}^{2,\alpha}_0(A(R), \mathbb{R}) \times \mathbb{R}^+$ together with a set of cells $\mathscr{K}$, satisfying:

(i) K is a disjoint union of all the cells in $\mathscr{K}$.

(ii) For every $k \in \mathscr{K}$, $\partial k \equiv \bar{k} - k$ is the union of finitely many cells of $\mathscr{K}$.

(iii) For all $(u, \lambda) \in \mathscr{C}^{2,\alpha}_0(A(R), \mathbb{R}) \times \mathbb{R}^+$, there exists a neighborhood of (u, λ) which meets only a finite number of cells of $\mathscr{K}$.

THEOREM 6 [17]. *Let* $(u_0, \lambda_0) \in \mathscr{C}^{2,\alpha}_0(A(R), \mathbb{R}) \times \mathbb{R}^+$ *be a solution of* (3), *in a closed neighborhood of* (u_0, λ_0) *small enough, the set of solutions of* (3) *forms a cell complex.*

So, when symmetry breaking occurs, we are sure that a branch of non-radial solutions bifurcates from the branch of radial ones.

Concerning the set of non-radial solutions, near a symmetry breaking point, very few results are known, some partial results are given in Section 9.

As in [22], we can prove that:

THEOREM 7. *Suppose that* $(u_0, \lambda_0) \in \Gamma$ *is a symmetry breaking point* (*where* Γ *is the branch of symmetric solutions defined in Theorem* 3) *and that there are two subgroups* $G1$ *and* $G2$ *of* $O(n)$ *satisfying*:

(i) *The group generated by* $G1$ *and* $G2$ *is* $O(n)$.

(ii) *In the kernel of the linearized operator at* (u_0, λ_0), *there are some non-symmetric elements invariant by* $G1$ *and others invariant by* $G2$.

Then there are at least two bifurcating branches respectively symmetric under $G1$ *and* $G2$.

The fact that in (3) the non-linear term is e^u does not play any role in the proofs of the results, so we are going to work now with a more general non-linearity.

Problem (3) will now have the form

$$\begin{cases} -\Delta u(x) = \lambda f(u(x)) & \text{on } \Omega \\ u(x) = 0 & \text{on } \partial\Omega \\ u(x) > 0 & \text{on } \Omega. \end{cases} \tag{4}$$

For which we assume that the following hypotheses hold:

(H1) f is real analytic.

(H2) There exists $\alpha > 0$ such that $f(x) > \alpha \sup(x, 1)$ for all $x \in \mathbb{R}^+$.

(H3) $\lim_{x \to +\infty} (1/x^2) \int_0^x f(t)\, dt = +\infty$ and $f'(x) \geq 0$ for all $x \in \mathbb{R}^+$.

Hypothesis (H3) can be weakened, in order to allow $f'(x)$ to take negative values:

(H3′) $\lim_{x \to +\infty} (1/x^2) \int_0^x f(t)\, dt = +\infty$ and if we define $G(x) = \sup_{u \in [0,x]} f(u)$ and $H(x) = \inf_{u \in [x, +\infty)} f(u)$, we ask that $\liminf_{x \to +\infty} (H(x)/G(x)) > 0$.

We can also replace (H3) by the following hypothesis:

(H3″) $\lim_{x \to +\infty} (f(x)/x) = +\infty$.

The proof that (H3′) or (H3″) are sufficient in order to derive all our results, is quite technical, so we postpone it to Appendix C.

In Section 7, in order to prove the existence of infinitely many symmetry breaking points, we will need an hypothesis stronger than (H3):

(H4) $\lim_{x \to +\infty} xf'(x)/f(x) = +\infty$.

If we only suppose that $\lim_{x \to +\infty} xf'(x)/f(x)$ is "large enough," it will provide us a finite number of symmetry breaking points, but not infinitely many, as stated in Theorems 4 and 5.

The fact that all the solutions of (4) are positive because $\lambda \geq 0$, is not true any more, because we have not assumed that $f(x) \geq 0$ for all $x \in \mathbb{R}$, nevertheless we can prove the proposition:

PROPOSITION 2. *Let C be a connected component of $\mathcal{S}(R) \setminus \{(0, 0)\}$ and assume that there is a point $(u, \lambda) \in C$ such that $u(x) > 0$ on all $A(R)$, then for all point $(v, \lambda) \in C$ we have $v(x) > 0$ on all $A(R)$.*

When $f(u) = e^u$, we have more information about the structure of the sets $\mathcal{P}(\lambda)$, we can be more precise on the number of solutions of (3). The following corollary is just a consequence of our study:

COROLLARY 2. *Assume that $f(u) = e^u$, that $n \in \{3, ..., 9\}$ and that $\lambda = 2(n-2)$, then the number of radial solutions of (3) tends to infinity when the radius of the annulus tends to 0.*

4. General Framework

In Section 5 we use some results of [7] as well as a priori estimates to obtain some information about the location and the value of the maximum of solutions of (4).

Using this, we prove in Section 6, by a shooting method as well as a topological argument, that, for every $R \in (0, 1)$, there exists a branch of radial solutions joining $(0, 0)$ to $(\omega, 0)$ and we study the behaviour of $\lambda \int_{A(R)} f(u(x))\, dx$ along this branch, this ends the proof of Theorem 3. We also prove here that it is not possible for this branch to bifurcate in the space of radial solutions, at least for all R in a dense subset of $(0, 1)$.

Section 7 is devoted to the study of the eigenvalues of the linearized operator near a radial solution obtained in the last section.

We deal with the problem of the parametrization of the branch of radial solutions in Section 8.

In Section 9, we use the Conley index as in [22] to show that symmetry breaking occurs and finish the proof of our results.

Next, Section 10 is concerned with a generalisation of our results to a larger class of problems.

Finally, we give in Section 11 some conjectures.

After we completed this work, we learned that S. S. Lin had obtained similar results in [12], but his proof is quite different.

5. A Priori Estimates

We look for solutions of

$$\begin{cases} -\Delta u(x) = \lambda f(u(x)) & \text{on } A(R) \\ u(x) = 0 & \text{on } \partial A(R) \\ u(x) > 0 & \text{on } A(R). \end{cases} \tag{5}$$

Certainly we can first look for radial solutions, so (5) becomes:
Find u from $[R, 1]$ into $\mathbb{R}^+$ satisfying

$$\begin{cases} \dfrac{d^2u}{dr^2}(r) + \dfrac{n-1}{r}\dfrac{du}{dr}(r) = -\lambda f(u(r)) & \text{on } (R, 1) \\ u(r) > 0 & \text{on } (R, 1) \\ u(R) = 0 & \\ u(1) = 0. & \end{cases} \tag{6}$$

Now if we perform the following change of variables $t = ((n-2)\, r^{n-2})^{-1}$ (recall that we have assumed that $n \geqslant 3$) and if we set $\rho(t) = ((n-2)\, t)^{-k}$, where $k = 2(n-1)(n-2)$, we obtain an equivalent formulation of (6):

Find u from $[t_0, t_1]$ into $\mathbb{R}^+$ satisfying

$$\begin{cases} \dfrac{d^2u}{dt^2}(t) = -\lambda \rho(t) f(u(t)) & \text{on } (t_0, t_1) \\ u(t_0) > 0 & \text{on } (t_0, t_1) \\ u(t_0) = 0 & \\ u(t_1) = 0, & \end{cases} \tag{7}$$

where $t_0 = (n-2)^{-1}$ and $t_1 = (n-2)^{-1} R^{-(n-2)}$.

In the sequel, we are going to work with the three formulations of (5).

As we have already seen in the first section, all regular solutions of (4) on a ball must be radial. Unfortunately it is no longer true on an annulus, nevertheless, applying the theory of B. Gidas *et al.*, we can still say something about the solutions of (4) on an annulus. First, if we apply [7, Theorem 2] to a solution u of (5), we see that $(\partial u/\partial r)(x) < 0$ in the area where $(1+R)/2 \leqslant |x| \leqslant 1$. Next, if we consider u as a solution of (7), and note that $\rho(t)$ is strictly decreasing, we can apply Theorem 1′ of [7] and obtain $(du/dt)(t) < 0$ in the area where $(t_0 + t_1)/2 \leqslant t \leqslant t_1$.

We summarize all this in a lemma:

LEMMA 2. *Every regular solution of* (7) *satisfies*

(i) *for all* $x \in A(R)$, $(1+R)/2 \leqslant |x| \leqslant 1$ *we have* $(\partial u/\partial r)(r) < 0$;

if in addition, u is radial, then it also satisfies

(ii) *for all* $x \in A(R)$, $R \leqslant |x| \leqslant (1 + R^{2-n})/2(n-2)$ *we have* $(\partial u/\partial r)(r) > 0$.

When we consider u, radial solution of (5), as a solution of (7), using (H2) we easily see that u, as a function of t, is concave and admits a unique maximum at some point $\tau \in (t_0, t_1)$.

We have assumed in (H3) that $f'(x) \geqslant 0$ for all $x \geqslant 0$, taking the derivative of (7), a straightforward calculus shows that

$$\frac{d^3u}{dt^3}(t) \geqslant 0$$

on (τ, t_1), hence du/dt is convex and strictly decreasing on (τ, t_1).

Let us define $b = -(du/dt)(t_1) > 0$, we can write

$$\forall t \in (\tau, t_1), \; -b \leqslant \frac{du}{dt} \leqslant -b\left(\frac{t-\tau}{t_1-\tau}\right).$$

Integrating this inequality between τ and t_1, we obtain the estimate of the next lemma:

LEMMA 3. *Let u be a solution of (7), if we define $b = -(du/dt)(t_1) > 0$ then we have the estimate of $u_m \equiv \max_{[t_0, t_1]} u(t)$,*

$$u_m \leqslant b(t_1 - \tau).$$

If in addition we assume that $f'(x) \geqslant 0$ for all $x \geqslant 0$ (i.e., (H3)), we have

$$b(t_1 - \tau)/2 \leqslant u_m.$$

Many other estimates can be obtained, we will denote by $F(x) = \int_0^x f(t)\,dt$ a primitive of $f(x)$, and if we take the derivative of $\frac{1}{2}(du/dt)^2(t) + \lambda\rho(t)\,F(u(t))$ we obtain

$$\frac{d}{dt}\left(\frac{1}{2}\left(\frac{du}{dt}\right)^2(t) + \lambda\rho(t)\,F(u(t))\right) = \frac{d\rho}{dt}\,\lambda F(u(t)) < 0.$$

In the same way

$$\frac{d}{dt}\left(\frac{1}{2\rho(t)}\left(\frac{du}{dt}\right)^2(t) + \lambda F(u(t))\right) = -\frac{d\rho}{dt}\frac{1}{2(\rho(t))^2}\left(\frac{du}{dt}\right)^2(t) > 0.$$

Integrating those two inequalities from t_0 to τ and then from τ to t_1 we obtain the next lemma:

LEMMA 4. *Let u be a solution of (7), if we define as before $b = -(du/dt)(t_1) > 0$ and $a = (du/dt)(t_0) > 0$ then we have*

$$b^2 \leqslant 2\lambda\rho(\tau)\,F(u_m) \leqslant a^2$$

and

$$\frac{a^2}{\rho(t_0)} \leqslant 2\lambda F(u_m) \leqslant \frac{b^2}{\rho(t_1)}.$$

In particular we note that

$$b^2 \leqslant a^2 \leqslant \frac{\rho(t_0)}{\rho(t_1)}\,b^2.$$

If we use the bounds on u_m that we had obtained in Lemma 3 we derive two important estimates:

By (H2) we know that $F(x)$ is strictly increasing therefore

$$\frac{a^2}{\rho(t_0)} \leqslant 2\lambda F(u_m) \leqslant 2\lambda F(b(t_1 - \tau)) \leqslant 2\lambda F(a(t_1 - \tau)), \tag{8}$$

and in the same way

$$2\lambda F\left(\frac{b}{2}(t_1-\tau)\right)\leqslant 2\lambda F(u_m)\leqslant\frac{b^2}{\rho(t_1)}. \tag{9}$$

Here is our last estimate for solutions of (7):
Let u be a solution of (7), as a consequence of (H2), we find that

$$\frac{d^2u}{dt^2}(t)=-\lambda\rho(t)f(u(t))\leqslant-\lambda\rho(t)\alpha.$$

Integrating this inequality we find the estimate

$$u(t_1)\leqslant a(t_1-t_0)-\lambda\alpha\int_{t_0}^{t_1}(t_1-s)\,\rho(s)\,ds.$$

By definition $u(t_1)=0$, we therefore deduce the necessary condition

$$a(t_1-t_0)\geqslant\lambda\alpha\int_{t_0}^{t_1}(t_1-s)\,\rho(s)\,ds.$$

Using the fact that $\rho(t)$ is decreasing, we can simplify this inequality to obtain

$$\frac{a}{\lambda}\geqslant\frac{\alpha}{2}\rho(t_1)(t_1-t_0)\geqslant 0. \tag{10}$$

We define $\mathscr{S}(\lambda_0)$ to be the set of values of the modulus of the normal derivatives on the boundary, for all the radial (regular and singular) solutions of (4), with $\Omega=B^n$ and $\lambda=\lambda_0$. We want to show that:

LEMMA 5. *$\mathscr{S}(\lambda)$ is empty for λ large enough.*

If u is a radial solution (regular or singular) of (4) on a ball, it satisfies

$$\begin{cases}\dfrac{d^2u}{dr^2}(r)+\dfrac{n-1}{r}\dfrac{du}{dr}(r)=-\lambda f(u(r)) & \text{on } (0,1)\\ u(r)>0 & \text{on } (0,1).\\ u(1)=0. & \end{cases}$$

Performing the change of variables $t=((n-2)\,r^{n-2})^{-1}$, we see that u is a solution of

$$\begin{cases}\dfrac{d^2u}{dt^2}(t)=-\lambda\rho(t)f(u(t)) & \text{on } (t_0,+\infty)\\ u(t)>0 & \text{on } (t_0,+\infty).\\ u(t_0)=0. & \end{cases}$$

Note that u, as a function of t, is positive on all $(t_0, \tilde{t}]$ and that, by (H2), we have

$$\frac{d^2u}{dt^2} + \lambda\rho(\tilde{t})\,\alpha u(t) \leqslant 0.$$

Moreover, the equation $d^2v/dt^2 + \omega^2 v(t) = 0$ has a positive solution on $[t_0, \tilde{t}]$ with $v(t_0) = v(\tilde{t}) = 0$, if $\omega = \pi/(\tilde{t} - t_0)$.

Using Strum's theorem (see Appendix B), we obtain the estimate

$$\lambda \leqslant \pi^2/(\rho(\tilde{t})\,\alpha(\tilde{t} - t_0)^2).$$

So we have proved that $\mathscr{P}(\lambda)$ is empty for λ large enough.

6. The shooting Method and the Existence of Solutions

Here, we restrict ourselves to the study of the function $u(t, a, \lambda)$, solution of

$$\begin{cases} \dfrac{d^2u}{dt^2}(t) = -\lambda\rho(t)\,f(u(t)) \\ u(t_0) = 0 \\ \dfrac{du}{dt}(t_0) = a > 0. \end{cases} \tag{11}$$

From the last section we know that positive solutions of (11) are concave, so the solutions are bounded from above on all $[t_0, t]$, where they are defined. The consequence of this is that positive solutions of (11) are defined for all $t > t_0$.

By (H1), f is analytical, so we have the lemma:

Lemma 6. *The unique solution, $u(t, a, \lambda)$ is analytical in the three variables $(t, a, \lambda) \in \mathbb{R}^+/\{0\} \times \mathbb{R}^+/\{0\} \times \mathbb{R}^+/\{0\}$, over his set of definition.*

The proof is standard and uses a fixed point theorem. For the convenience of the reader we shortly recall it here and prove that the lemma holds for $t - t_0$ small enough, the general case can be proved in the same way.

Integrating (11) we derive

$$u(t) = a(t - t_0) - \lambda \int_{t_0}^{t} (t - s)\,\rho(s)\,f(u(s))\,ds.$$

There is a one-to-one correspondence between the solutions of the last equation and the solution of (11). We consider the space

$$\mathscr{E} = \{u \in \mathscr{C}^1(V_{t_0, a_0, \lambda_0}, \mathbb{C}) / |u - a_0(t - t_0)|_{\mathscr{C}^1} < \varepsilon,$$
$$u(t_0, a, \lambda) = 0, u(t, a, \lambda) \text{ is analytical}\},$$

(where V_{t_0, a_0, λ_0} is a neighborhood of (t_0, a_0, λ_0) in $\mathbb{C}^3$). Let us define the operator T on $\mathscr{E}$ by

$$T(u)(t) \equiv a(t - t_0) - \lambda \int_{t_0}^{t} (t - s)\, \rho(s) f(u(s))\, ds.$$

It is easy to see that T sends $\mathscr{E}$ into $\mathscr{E}$, and is a contraction if the neighborhood is chosen small enough. We may apply a fixed point theorem and derive the existence of a unique analytical solution of (11) in $\mathscr{E}$, this proves the lemma.

Now, we define the "time map" as follows:

Let $\lambda \in \mathbb{R}^+$ and $a \in \mathbb{R}^+$ we define

$$t_1(a, \lambda) \equiv \sup\{t \geqslant t_0 / u(s, a, \lambda) \geqslant 0 \quad \text{for all} \quad s \in [t_0, t]\} \in \mathbb{R}^+ \cup \{+\infty\}.$$

For $a \neq 0$, whether, $\{t \geqslant t_0 / u(s, a, \lambda) \geqslant 0$ for all $s \in [t_0, t]\} = [t_0, +\infty)$ or $\{t \geqslant t_0 / u(s, a, \lambda) \geqslant 0$ for all $s \in [t_0, t]\}$ is equal to some $[t_0, t_1]$ for some point $t_1 > t_0$ and in this case, $t_1(a, \lambda)$ is uniquely characterized by the fact that it is the first value $t > t_0$ such that $u(t, a, \lambda) = 0$. If, for some values of $a > 0$ and $\lambda > 0$, $t_1(a, \lambda) < +\infty$, then the unique solution of (11) is a solution of (7), with $t_1 = t_1(a, \lambda)$, so there is a unique point τ where $(\partial u / \partial t)(\tau) = 0$ and in this point $u(\tau) > 0$ (note that $a > 0$ and $\lambda > 0$), we can conclude that $(\partial u / \partial t)(t_1(a, \lambda), a, \lambda) \neq 0$. Applying the implicit function theorem of the Appendix A and the result of the last lemma, we find:

LEMMA 7. *The map $t_1(a, \lambda)$ is analytical on his set of definition.*

In the next part of this section we want to study the behavior of $t_1(a, \lambda)$.

The change of variable we have performed is $t = ((n-2)\, r^{n-2})^{-1}$ so we compute

$$r^{n-1} \frac{du}{dr}(r) = -\frac{du}{dt}(t).$$

In particular $(du/dr)(1) = -(du/dt)(t_0)$.

Let us recall that if $t_1(a, \lambda) = +\infty$, this means that u, the corresponding solution of (6), is defined on all $(0, 1]$, so u is a solution of (4) on a ball minus the origin; this solution can be extended by continuity if it is bounded, otherwise it is a radial singular solution of (4) on a ball.

We have just proved that for all $a_0 \in \mathscr{P}(\lambda_0)$, $t_1(a_0, \lambda_0) = +\infty$. By continuity we obtain $\lim_{(a,\lambda) \to (a_0,\lambda_0)} t_1(a, \lambda) = +\infty$.

6.1. *Study of $t_1(a, \lambda)$*

We now want to investigate what is going on when λ (or a) tends to $+\infty$, or tends to 0.

6.1.1. *Behaviour of $t_1(a, \lambda)$ when λ goes to $+\infty$*. We are going to prove the result:

$\lim_{\lambda \to +\infty} t_1(a, \lambda) = t_0$ uniformly in $a \in \mathbb{R}^+$.

Suppose that it is not true and that there exists $\varepsilon > 0$ and a sequence $(a_n, \lambda_n)_{n \in N}$ such that $\lim_{n \to +\infty} \lambda_n = +\infty$ and $t_1(a_n, \lambda_n) \geqslant t_0 + \varepsilon$. We claim that $t_1(a, \lambda)$ is continuous, for λ large enough. The proof of this fact is as follows:

As we have already seen, the points of $\mathbb{R}^+/\{0\} \times \mathbb{R}^+$ for which $t_1(a, \lambda) = +\infty$ are exactly the points (a, λ) such that $a \in \mathscr{P}(\lambda)$. But, by Lemma 5, $\mathscr{P}(\lambda)$ is empty for λ large enough, $t_1(a, \lambda)$ is finite, so is continuous by Lemma 7. This proves our claim.

As $\lim_{n \to +\infty} \lambda_n = +\infty$, we can assume that for $\lambda = \lambda_n$, $t_1(a, \lambda)$ is a continuous function of a. In addition, it is very easy to see that $t_1(0, \lambda) = t_0$ (cf. Section 6.1.4 for a proof) and, by hypothesis, $t_1(a_n, \lambda_n) \geqslant t_0 + \varepsilon$. So there is a $\tilde{a}_n$ such that $t_1(\tilde{a}_n, \lambda_n) = t_0 + \varepsilon$. We deduce from all this that Eq. (7) has a solution for $\lambda = \lambda_n$ and $t_1 = t_0 + \varepsilon$. So Eq. (5), with $R = ((n-2)t_1)^{1/(2-n)}$, has a regular solution for all $\lambda = \lambda_n$, which is not possible because, by (H2) and the remark following Lemma 1, the set of λ for which (5) has a solution is bounded. This is contradiction, hence we have proved:

LEMMA 8. *$\lim_{\lambda \to +\infty} t_1(a, \lambda) = t_0$ uniformly in $a \in \mathbb{R}^+$.*

6.1.2. *Behavior of $t_1(a, \lambda)$ when λ goes to 0*. We want to show that $\lim_{\lambda \to 0} t_1(a, \lambda) = +\infty$ uniformly in $a \in [\underline{a}, \bar{a}]$. For each (a, λ) we consider the associated solution of (7), with $t_1 = t_1(a, \lambda)$ and $(\partial u/\partial t)(t_0) = a$. In order to obtain the result, we use the *a priori* estimate that we have established in (8) for solutions of (7). We observe as usual that, since (H2) holds, F is increasing, so we can compute

$$\frac{a^2}{\rho(t_0)} \leqslant 2\lambda F(a(t_1 - \tau)) \leqslant 2\lambda F(a(t_1 - t_0)).$$

When λ goes to 0, a staying in $[\underline{a}, \bar{a}]$, the right hand side of the last inequality must stay bounded from below and the result follows easily, we can state:

LEMMA 9. *$\lim_{\lambda \to 0} t_1(a, \lambda) = +\infty$ uniformly in $a \in [\underline{a}, \bar{a}]$.*

6.1.3. *Behavior of* $t_1(a, \lambda)$ *when* a *goes to* $+\infty$. Here, we want to prove the following:

Let $\lambda_0>0$ be fixed, then $\lim_{a\to+\infty} t_1(a,\lambda)=t_0$ uniformly in $\lambda\in[\lambda_0, +\infty)$.

We use here the *a priori* estimate we have derived in the last section.

Suppose the contrary, then there is $\varepsilon>0$ and a sequence $((a_n, \lambda_n))_{n\in N}$, $(a_n)_{n\in N}$ tending to $+\infty$ such that

$$t_0+\varepsilon<t_1(a_n, \lambda_n).$$

As a consequence of Lemma 8, $\lim_{\lambda\to+\infty} t_1(a_n, \lambda)=t_0$, hence, for n large enough, we derive the existence of $\tilde{\lambda}_n\geqslant\lambda_0$ satisfying $t_1(a_n, \tilde{\lambda}_n)=t_0+\varepsilon$.

Then we consider the corresponding solutions of (7) and we see that, by Lemma 4, the sequence $(b_n)_{n\in N}$ also tends to $+\infty$. Now we use Lemma 2 to derive

$$t_1(a_n, \tilde{\lambda}_n)-\tau\geqslant(t_1(a_n, \tilde{\lambda}_n)-t_0)/2=\varepsilon/2.$$

As F is increasing and as ρ is decreasing, we can write, using (9) and Lemma 3

$$b_n^2\geqslant 2\lambda_0\rho(t_0+\varepsilon)\, F\left(b_n\frac{\varepsilon}{4}\right). \tag{12}$$

Let n go to $+\infty$, (H3) provides a contradiction.

We have just proved the lemma:

LEMMA 10. $\lim_{a\to+\infty} t_1(a,\lambda)=t_0$ *uniformly in* $\lambda\in[\lambda_0, +\infty)$.

6.1.4. *Behavior of* $t_1(a, \lambda)$ *when* a *goes to* 0. The case where $a=0$ is very easy to handle because it is obvious to see that for all $\lambda_0>0$, as $(\partial^2u/\partial t^2)(t)<0$ (here we use (H2)), we have $t_1(0,\lambda)=t_0$ and, always by continuity, we obtain:

LEMMA 11. $\lim_{(a,\lambda)\to(0,\lambda_0)} t_1(a,\lambda)=t_0$.

6.2. *The Existence of the Branch of Solutions*

Now that the study of the time map is complete, we can derive the existence of one connected branch of solutions for (5). The proof of this fact is based on a theorem of P. H. Rabinowitz [18], modified for our purpose.

THEOREM 8. [18]. *Let* Θ *be a continuous function defined on* $\mathbb{R}^+\times\mathbb{R}^+\setminus\{(0,0)\}$, *let* $\Lambda=(\underline{\lambda}, \bar{\lambda})$ *and* $\mathscr{S}=\{(a,\lambda)\in\mathbb{R}^+\times\mathbb{R}^+\setminus\{(0,0)\}/\Theta(a,\lambda)=0\}$.

We make the following assumptions:

(i) *First, for all* $a \in (\underline{a}, \bar{a})$, $\Theta(a, \underline{\lambda}) \neq 0$ *and* $\Theta(a, \bar{\lambda}) \neq 0$.

(ii) *Second,* $d(\Theta(a, \cdot), \Lambda, 0) \neq 0$ *for some* $a \in (\underline{a}, \bar{a})$ (*where d is the topological degree*).

Then, there is one connected component of $\mathscr{Z}$ *joining a point of* $(\underline{a}, \bar{a}) \times \{\underline{\lambda}\}$ *to a point of* $(\underline{a}, \bar{a}) \times \{\bar{\lambda}\}$.

Now the argument goes as follows:

First step, fix $\bar{a} > 0$ large enough and $\tilde{t}_1 > t_0$, so we know that $t_1(\bar{a}, \lambda)$ is analytical in λ (on his set of definition), is continuous from $\mathbb{R}^+ \times \mathbb{R}^+ \setminus \{(0, 0)\}$ into $\mathbb{R} \cup \{+\infty\}$ and has the behavior given by Lemmas 8 and 9. We deduce from this that the set of values of λ for which $t_1(\bar{a}, \lambda) = \tilde{t}_1$ is finite; we set $\{\lambda_1^0, ..., \lambda_s^0\} \equiv \{\lambda \in \mathbb{R}^+ / t_1(\bar{a}, \lambda) = \tilde{t}_1\}$.

Then we define the continuous map on $\mathbb{R}^+ \times \mathbb{R}^+ \setminus \{(0, 0)\}$

$$\Theta(a, \lambda) = \frac{t_1(a, \lambda)}{1 + t_1(a, \lambda)} - \frac{\tilde{t}_1}{1 + \tilde{t}_1}.$$

And we easily obtain $d(\Theta(a, \cdot), \Lambda, 0) = 1$ if $\Lambda = (\underline{\lambda}, \bar{\lambda})$ is such that it contains all λ_i^0, $i \in \{1, ..., s\}$.

Applying the result of Lemma 8 we can choose $\bar{\lambda}$ large enough to satisfy:

For all a and $\lambda \geqslant \bar{\lambda}$, $\Theta(a, \lambda) \neq 0$.

Now, by the result of Lemma 9, we can choose $\underline{\lambda}_n$ small enough such that the hypotheses of Theorem 8 hold for $\underline{a} = 1/n$ and $\underline{\lambda} = \underline{\lambda}_n$.

The conclusion of Theorem 8 tells us that there is a connected component of zeros of Θ joining a point of $\{\bar{a}\} \times \Lambda_n$ to a point of $\{1/n\} \times \Lambda_n$, where $\Lambda_n = (\underline{\lambda}_n, \bar{\lambda})$. As this result is true for every n and as the connected component must pass by some $\{\bar{a}\} \times \lambda_i^0$, which belongs to a finite set, thus we can find a $i_0 \in \{1, ..., s\}$ such that the connected component of the zeros of Θ passes through $\{\bar{a}\} \times \lambda_{i_0}^0$ for infinitely many n. Finally, applying (10), we see that the point where this connected component crosses $\{1/n\} \times \Lambda_n$ must satisfy

$$\frac{a}{\lambda} \geqslant \frac{\alpha}{2} \rho(\tilde{t}_1)(\tilde{t}_1 - t_0) \geqslant 0.$$

with $a = 1/n$. So the connected component converges to $(0, 0)$ in the (a, λ) plane. This closes the first step of the proof.

Second step, we have just derived that there was a connected component of $\mathscr{Z}$ joining the point $(0, 0)$ to the point $(\bar{a}, \lambda_{i_0}^0)$. But the result is also true for $2\bar{a}$, namely, there is a connected component of $\mathscr{Z}$ joining the point $(0, 0)$ to the point $(2\bar{a}, \lambda_{i_1}^0)$. And we can go on like this for $3\bar{a}$, $4\bar{a}$, etc. All these connected components must pass through a point of $\{\{\bar{a}\}\times\lambda_i^0/i\in\{1,...,s\}\}$.

Certainly, there is a point in $\{\{\bar{a}\}\times\lambda_i^0/i\in\{1,...,s\}\}$ whose connected component in $\mathscr{Y}$ joins the point $(0,0)$ to a point of $\{\{n\bar{a}\}\times\mathbb{R}^+\}$ for infinitely many n.

Taking the above-mentioned connected component C of $\mathscr{Y}$ passing through this point, gives us the existence of an unbounded connected component in $\mathscr{Y}$ containing $(0,0)$. Using Lemma 10, we see that all unbounded branches of C have $\lambda=0$ as an asymptote.

In C, we can obviously find a continuous branch joining $(0,0)$ to $(\infty,0)$, we call C_0 this branch and we parametrize it by θ: $C_0=\{\{(a(\theta),\lambda(\theta)),\theta\in\mathbb{R}^+\}$.

By means of (11) and doing backward the change of variables, this gives us a continuous branch of solutions $\Gamma=\{(u(\theta),\lambda(\theta))/\theta\in\mathbb{R}^+\}$, for (5) with the appropriate $R=((n-2)\,\tilde{t}_1)^{1/(2-n)}$.

The only point that still remains is the study of the behavior of this branch of solutions when θ goes to $+\infty$. As we have noted before, all unbounded branches of C have $\lambda=0$ as asymptote, so when θ goes to $+\infty$, $(a(\theta),\lambda(\theta))$ goes to $(+\infty,0)$. Using Lemma 4, we see that for the corresponding solutions $h(\theta)=-(du/dt)(\tilde{t}_1,a(\theta),\lambda(\theta))$ goes to $+\infty$. Putting this information back in Lemma 3 we obtain

$$u_m(\theta)\geqslant h(\theta)(\tilde{t}_1-t_0)/4.$$

Hence $u_m(\theta)$ goes to $+\infty$ when θ does. We can now conclude saying that u is concave on $[t_0,\tilde{t}_1]$ so $u(\theta)$ converges to ω when θ goes to $+\infty$ uniformly over $[t_0+\varepsilon,\tilde{t}_1-\varepsilon]$, for all $\varepsilon>0$ (ω being defined in Theorem 3). This completes the proof of the first part of Theorem 3. We can also prove the proposition:

PROPOSITION 3. *Let Γ be the branch of solutions obtained before, then we have*

$$\lim_{\theta\to+\infty}\lambda(\theta)\int_{A(R)}f(u(\theta,x))\,dx=+\infty.$$

The proof of this proposition is obtained directly from Eq. (5)

$$\lambda(\theta)\int_{A(R)}f(u(\theta,x))\,dx=-\int_{A(R)}\Delta u(\theta,x)\,dx=-\int_{\partial A(R)}\frac{\partial u}{\partial \nu}(\theta,x)\,dx$$

$$=|S^{n-1}|\left(R^{n-1}\frac{\partial u}{\partial r}(R)-\frac{\partial u}{\partial r}(1)\right).$$

The last part of this equality can be written as

$$R^{n-1}\frac{\partial u}{\partial r}(R)-\frac{\partial u}{\partial r}(1)=b(\theta)+a(\theta).$$

This, together with Lemma 4, proves the proposition.

In order to obtain smooth branches of solutions, we must show that $t_1(a,\lambda)$ has no critical point on the level set $t_1^{-1}(\tilde{t}_1)\equiv\{(a,\lambda)\in\mathbb{R}^+\times\mathbb{R}^+/t_1(a,\lambda)=\tilde{t}_1\}$. Unfortunately we are not able to prove it directly, therefore we must use Sard's theorem [19], which states that the measure of the critical values of $t_1(a,\lambda)$ is null. We can then find $T\subset[t_0,+\infty)$ such that the measure of $[t_0,+\infty)\setminus T$ is null and for all $t_1\in T$, the set $t_1^{-1}(\tilde{t}_1)$ contains no critical point of $t_1(a,\lambda)$. It means that we can locally parametrize $t_1^{-1}(\tilde{t}_1)$ by a or λ. Note also that the connected component of $\mathscr{Y}$ obtained previously cannot bifurcate, neither can the corresponding branch of solutions Γ if $R\in\Sigma\equiv\{r\in(0,1)/(n-2)^{-1}r^{-(n-2)}\in T\}$. The fact that we do not have bifurcation in the point $(0,0)$ has been established in [5]. The linearized operator near $u=0$ and $\lambda=0$ is equal to $Lw=-\Delta u$, which is invertible from $\mathscr{C}_0^{2,\alpha}(\Omega,\mathbb{R})$ into $\mathscr{C}^{0,\alpha}(\Omega,\mathbb{R})$, thus applying the implicit function theorem, we prove the existence of a unique curve of regular solutions of (5), near $(0,0)$. The first part of Theorem 5 is therefore proved.

Remark. We give here a short proof of Corollary 2.

In Corollary 2, we assume that $n\in\{3,...,9\}$ and that $f(u)=e^u$. As we have seen in the first section, the set of points in $\mathbb{R}^+\times\{\mathbb{R}^+\setminus\{0\}\}$ where $t(a,\lambda)=+\infty$ is a spiral centered at the point $(2,2(n-2))$, Corollary 2 is then a consequence of the fact that $t_1(a,\lambda)$ is continuous from $\mathbb{R}^+\times\mathbb{R}^+\setminus\{(0,0)\}$ into $\mathbb{R}\cup\{+\infty\}$.

7. Study of the Linearized Operator

In this section, we assume that u belongs to the continuous branch Γ obtained in the previous section.

The linearized operator, near a solution of (5), is given by

$$L(\theta)\,w(x)=-\Delta\omega(x)-\lambda(\theta)\,f'(u(\theta,(x)))\,w(x),$$

where $f'(u)$ denotes the derivative of $f(u)$. (Recall that, λ and u depend on θ, the parametrization of Γ.)

We want to study the spectrum of this operator. In order to do this, we work with the modified operator $\tilde{L}(\theta)\,w(x)=|x|^2\,L(\theta)\,w(x)$ which is self-adjoint and uniformly elliptic on the space $L^2(A(R),|x|^{-2}\,dx)$. Next, we denote by φ_k, the eigenfunctions of the Laplacian on S^{n-1}, the unit sphere in $\mathbb{R}^n$ and by λ_k the corresponding eigenvalues

$$-\Delta_{S^{n-1}}\varphi_k(x) = \lambda_k \varphi_k(x) \qquad \text{on } S^{n-1}.$$

Let ψ be an eigenfunction of $\tilde{L}(\theta)$ associated to the eigenvalue μ; i.e., $\tilde{L}(\theta)\,\psi(x) = \mu\psi(x)$.

If we compute

$$\langle \tilde{L}(\theta)\psi, \varphi_k \rangle_{L^2(S^{n-1})} = \mu \langle \psi, \varphi_k \rangle_{L^2(S^{n-1})},$$

we obtain

$$-\frac{d^2 w_k}{dr^2}(r) - \frac{n-1}{r}\frac{dw_k}{dr}(r) + \frac{\lambda_k}{r^2} w_k(r) - \lambda(\theta) f'(u(\theta, r))\, w_k(r) = \frac{\mu}{r^2} w_k(r), \quad (13)$$

where $w_k(r) = \langle \psi, \varphi_k \rangle_{L^2(S^{n-1})}$.

Conversely, if w satisfies (12), then $\psi(x) = w(|x|)\,\varphi_k(x/|x|)$ is an eigenfunction of $\tilde{L}(\theta)$.

So, the study of the eigenvalues of $\tilde{L}(\theta)$ is reduced to the study of the eigenvalues of $\tilde{l}(\theta)$

$$\tilde{l}(\theta)\, w(r) = r^2 \left(-\frac{d^2 w}{dr^2}(r) - \frac{n-1}{r}\frac{dw}{dr}(r) - \lambda(\theta) f'(u(\theta, r))\, w(r) \right)$$

in the space $L^2([R, 1], r^{-2}\, dr)$.

Let us call, $\{\alpha_i(\theta)\}_{i \in N}$ the increasing sequence of the eigenvalues of $\tilde{l}(\theta)$ associated to the eigenfunctions $\{w_i(\theta)\}_{i \in N}$. By the previous calculus we see that all eigenvalues of $\tilde{L}(\theta)$ are of the form $\alpha_i(\theta) + \lambda_k$ and the associated eigenvector is $w_i(\theta, |x|)\,\varphi_k(x/|x|)$.

As Γ is a continuous branch of solutions, the operator $\tilde{l}(\theta)$ and its first eigenvalue depend continuously on θ.

Note that, as $\lambda_0 = 0$, $\alpha_0(\theta)$ is characterized by

$$\alpha_0(\theta) = \inf_{w \in H^1_{0,\mathrm{rad}}(A(R))} \frac{\int_{A(R)} |\nabla w|^2(x)\, dx - \lambda(\theta) \int_{A(R)} f'(u(\theta, x))\, w^2(x)\, dx}{\int_{A(R)} w^2(x)\, |x|^{-2}\, dx}. \quad (14)$$

Here we use (H4) to obtain information about the behavior of $\alpha_0(\theta)$ when θ goes to $+\infty$.

PROPOSITION 4. *Let Γ be the branch of solutions obtained before and assume that* (H4) *holds, then*

$$\lim_{\theta \to +\infty} \alpha_0(\theta) = -\infty.$$

We fix some $w \in \mathscr{C}_0^1((R, 1))$ such that $w(x) > 0$ on $(R, 1)$. In order to prove the proposition, it is sufficient to show that

$$\lim_{\theta \to +\infty} \lambda(\theta) \int_{A(R)} f'(u(\theta, x))\, w^2(x)\, dx = +\infty.$$

We set $R_1 = (2(n-2)R + (1 + R^{2-n}))/4(n-2)$ and $R_2 = (3 + R)/4$.

By Lemma 2, we know that every solution u of (5) is increasing on $[R, (1 + R^{2-n})/2(n-2)]$, moreover f is increasing too, so we can find a constant $c > 0$ independent of u, such that

$$\lambda(\theta) \int_{R < |x| < R_1} f(u(\theta, x))\, dx \leqslant c\lambda(\theta) \int_{R_1 < |x| < R_2} f(u(\theta, x))\, dx.$$

The same thing can be done with R_2, for some $c' > 0$ we have

$$\lambda(\theta) \int_{R_2 < |x| < 1} f(u(\theta, x))\, dx \leqslant c'\lambda(\theta) \int_{R_1 < |x| < R_2} f(u(\theta, x))\, dx.$$

$K > 0$ being fixed, by (H4), we can find a $N > 0$ such that $xf'(x) > Kf(x)$ for all $x > N$. We have already seen that, when θ goes to $+\infty$, $u(\theta)$ converges uniformly to ω on all $[R + \varepsilon, 1 - \varepsilon]$, so for θ large enough $u(\theta, x) > N$ on all $[R_1, R_2]$.

Hence, for θ large enough, we can write

$$\lambda(\theta) \int_{R_1 < |x| < R_2} f'(u(\theta, x))\, w^2(x)\, dx$$

$$\geqslant K\lambda(\theta) \int_{R_1 < |x| < R_2} f(u(\theta, x))\, w^2(x)/u(\theta, x)\, dx.$$

On $[R_1, R_2]$, $w(r)$ is minored by some constant $\underline{w} > 0$ and $u(\theta, r) > u_m(\theta)$ so we can state

$$\lambda(\theta) \int_{A(R)} f'(u(\theta, x))\, w^2(x)\, dx$$

$$\geqslant K\lambda(\theta) \int_{A(R)} f(u(\theta, x))\, dx\, \underline{w}^2/((1 + c + c')\, u_m(\theta)).$$

From Proposition 3 we know that

$$\lambda(\theta) \int_{A(R)} f(u(\theta, x))\, dx = |S^{n-1}|\, (b(\theta) + a(\theta)).$$

So we find that for θ large enough

$$\lambda(\theta)\int_{A(R)} f'(u(\theta,x))\,w^2(x)\,dx$$
$$\geqslant K\,|S^{n-1}|\,(b(\theta)+a(\theta))\,\underline{w}^2/((1+c+c')\,u_m(\theta)).$$

Now using Lemmas 3 and 4 we obtain

$$\lim_{\theta\to+\infty}\lambda(\theta)\int_{A(R)} f'(u(\theta,x))\,w^2(x)\,dx$$
$$\geqslant 2K\,|S^{n-1}|\,\underline{w}^2/((1+c+c')(t_1-t_0)).$$

This completes the proof of the proposition.

Derivating the equation satisfied by u we easily verify that $v=du/dr$ satisfies

$$\tilde{l}v(r)=-(n-1)\,v(r).$$

Therefore, using the Sturm's theorem (see Appendix B), we see that every eigenfunction of $\tilde{l}$ that changes sign must have its corresponding eigenvalue $\alpha>-(n-1)$. It is standard to see that the only eigenfunction that does not change sign is $w_0(\theta)$, so we conclude that $\alpha_i(\theta)>-(n-1)$ for all $i>0$. We summarize all this in a proposition:

PROPOSITION 5. *The eigenvalues of the linearized operator $\tilde{L}(\theta)$ form a sequence $\mu_{i,k}(\theta)=\alpha_i(\theta)+\lambda_k$ and the associated eigenfunctions are given by $\psi_{i,k}(\theta,x)=w_i(\theta,|x|)\,\varphi_k(x/|x|)$, where $\alpha_i(\theta)$ (resp. $w_i(\theta,r)$) are the eigenvalues (resp. eigenfunctions) of the operator $\tilde{l}(\theta)$ and λ_k (resp. φ_k) are the eigenvalues (resp. eigenfunctions) of $-\Delta_{S^{n-1}}$. Moreover, when θ goes to $+\infty$ we have the following behavior:*

For all $k\geqslant 0$

$$\lim_{\theta\to+\infty}\mu_{0,k}(\theta)=-\infty;$$

and, for all $k\geqslant 0$ and $i\geqslant 1$

$$\mu_{i,k}(\theta)>0.$$

We have the corollary:

COROLLARY 3. *Along the branch of solutions found in Section 6, there are infinitely many distinct infinitesimal symmetry breaking points.*

The points θ for which $\mu_{0,k}(\theta)=0$ for some $k\geqslant 0$ are infinitesimal symmetry breaking points.

Furthermore, the sets $Q(k) = \{\theta \in \mathbb{R}^+ / \mu_{0,k}(\theta) = 0\}$ satisfy $Q(k) \cap Q(h) \neq \emptyset$ if and only if $\lambda_k = \lambda_h$. The set of eigenvalues of the Laplacian on S^{n-1} is infinite, this completes the proof of the corollary.

Finally we consider that $R \in \Sigma$, the fact that the branch of solutions obtained in the previous section is an analytical one, gives us some additional information about the infinitesimal symmetry breaking points. Namely, the points where infinitesimal symmetry breaking occurs are isolated because, the branch of solutions being analytical, $\alpha_0(\theta)$ is also analytical (see [10]).

8. The Parametrization of the Branch of Solutions

In the remainder of the paper we will always assume that $R \in \Sigma$ and that u belongs to the smooth branch of solutions Γ obtained in Section 6.

Up to now we have assumed that the branch of solutions Γ was parametrized by θ without further information. We want to give two examples of parametrization that can be very useful.

We first define the mapping from $\mathscr{C}_0^{2,\alpha}(\Omega, \mathbb{R}) \times \mathbb{R}^+$ into $\mathscr{C}^{0,\alpha}(\Omega, \mathbb{R})$ by

$$M(u, \lambda) = -\Delta u - \lambda f(u). \tag{15}$$

The linearized operator dM_u has been studied in the last section, so, applying the implicit function theorem, we obtain

Proposition 6. *If θ satisfies $\mu_{i,0}(\theta) \neq 0$ for all $i \geq 0$, then the set of radial solutions of (7) can be locally parametrized by λ.*

This parametrization is of particular interest because of the following lemma:

Lemma 12. *The equation $(d/ds)u = -M(u, \lambda)$ is a gradient system, i.e., there is a mapping $\mathscr{M}$ such that $M(u, \lambda) = d\mathscr{M}_u(u, \lambda)$ for λ fixed.*

To prove this lemma we need to find $\mathscr{M}$. Just define

$$\mathscr{M}(u, \lambda, \theta) = \frac{1}{2} \int_{A(R)} |\nabla u|^2 (x)\, dx - \lambda \int_{A(R)} F(u(x))\, dx$$

and verify that it provides a good choice.

Now consider the mapping from $\mathscr{C}_0^{2,\alpha}(\Omega, \mathbb{R}) \times \mathbb{R}^+ \times \mathbb{R}$ into $\mathscr{C}^{0,\alpha}(\Omega, \mathbb{R}) \times \mathbb{R}$ defined by

$$\tilde{M}(u, \lambda; \tilde{\theta}) = \left(-\Delta u - \lambda f(u), -\int_{A(R)} F(u(x))\, dx + \frac{\tilde{\theta}}{\lambda} \right). \tag{16}$$

This mapping has been introduced by T. Suzuki in [24]. We want to study the linearized operator $d\tilde{M}_{(u,\lambda)}$ to show that it is invertible in the space of radial functions when $\mu_{i,0}(\theta) = 0$ for some $i \geqslant 0$ (and if $R \in \Sigma$). Suppose for the moment that this is true, the zeros of $\tilde{M}$ are just the solutions of (7), so if we apply the implicit function theorem, we show that the branch of solutions obtained previously can be parametrized by $\tilde{\theta} = \lambda \int_{A(R)} F(u(x))\, dx$ when it cannot be parametrized by λ.

As the spectrum of $d\tilde{M}_{(u,\lambda)}$ is composed of eigenvalues (see Appendix D), we are just left with the study of the kernel of $d\tilde{M}_{(u,\lambda)}$. If (w, v) belongs to this kernel, w is radial, we can write

$$\begin{cases} \Delta w(x) + \lambda f'(u(x))\, w(x) + v f(u(x)) = 0 & \text{on } A(R) \\ w(x) = 0 & \text{on } \partial A(R) \\ \displaystyle\int_{A(R)} f(u(x))\, w(x)\, dx + \tilde{\theta} v/\lambda^2 = 0, & \end{cases} \tag{17}$$

where $(u, \lambda; \tilde{\theta})$ satisfies $\tilde{M}(u, \lambda; \tilde{\theta}) = 0$.

By definition of $\tilde{M}$, we have

$$\int_{A(R)} F(u(x))\, dx = \frac{\tilde{\theta}}{\lambda}.$$

So we obtain

$$\lambda \int_{A(R)} f(u(x))\, w(x)\, dx + v \int_{A(R)} F(u(x))\, dx = 0.$$

All the radial solutions (v, v) of the partial differential equation

$$-\Delta v(x) = \lambda f'(u(x))\, v(x) + v f(u(x)) \tag{18}$$

with the boundary condition $v(1) = 0$, form a vector space of dimension two, and we already know two independent solutions; namely, if we call $\tilde{u}(r, a, \lambda) = u((n-2)^{-1} r^{-(n-2)}, a, \lambda)$, where u is the solution of (7); we derive that $((\partial\tilde{u}/\partial a)(r), 0)$ and $((\partial\tilde{u}/\partial\lambda)(r), 1)$ form a basis of the space of solutions of (18). This is true because one can verify that

$$\frac{\partial\tilde{u}}{\partial a}(1) = 0,$$

$$\frac{\partial\tilde{u}}{\partial\lambda}(1) = 0,$$

$$\frac{\partial^2\tilde{u}}{\partial a\, \partial r}(1) = -1$$

$$\frac{\partial^2\tilde{u}}{\partial\lambda\, \partial r}(1) = 0.$$

Moreover, if we assume that $\mu_{i,0}(\theta)=0$ for some $i \geqslant 0$, it is easy to see that $(\partial\tilde{u}/\partial a)(r)$ is the eigenfunction associated to this eigenvalue, so we have

$$\frac{\partial\tilde{u}}{\partial a}(R)=0.$$

Let us distinguish two cases, first if $v \neq 0$, w is a linear combination of $(\partial\tilde{u}/\partial a)(r)$ and $(\partial\tilde{u}/\partial\lambda)(r)$.

w and $(\partial\tilde{u}/\partial a)(r)$ being constant on $\partial A(R)$, so must be $(\partial\tilde{u}/\partial\lambda)(r)$. Therefore, in this case,

$$\frac{\partial\tilde{u}}{\partial\lambda}(R)=0.$$

Now we turn to the second case, if $v=0$ we can derive the same equality as follows:

If (v_1, ν_1) and (v_2, ν_2) are two radial solutions of (18), it is well-known that the quantity $Q(r)=r^{n-1}((dv_1/dr)(r)\,v_2(r)-(dv_2/dr)(r)\,v_1(r))$ satisfies

$$\frac{dQ}{dr}(r)=(\nu_2 v_1(r)-\nu_1 v_2(r))\,r^{n-1}f(u(r)).$$

Taking $(v_1, \nu_1)=(w, 0)$, $(v_2, \nu_2)=((\partial\tilde{u}/\partial\lambda)(r), 1)$ and using the fact that

$$\lambda\int_{A(R)} f(u(x))\,w(x)\,dx+v\int_{A(R)} F(u(x))\,dx=0,$$

as we have assumed that $v=0$, we obtain

$$\lambda\int_{A(R)} f(u(x))\,w(x)\,dx=0,$$

and we see that, necessarily, $v_2(R)=0$.

A consequence of this is

$$\frac{\partial\tilde{u}}{\partial a}(R)=0,$$

$$\frac{\partial\tilde{u}}{\partial\lambda}(R)=0,$$

independently of the value of v.

Going back to the variable t we obtain

$$\frac{\partial u}{\partial a}(t_1) = 0,$$

$$\frac{\partial u}{\partial \lambda}(t_1) = 0.$$

But, by definition of $t_1(a, \lambda)$, we know that $u(t_1(a, \lambda), a, \lambda) = 0$ identically.

Differentiating with respect to a, we obtain

$$\frac{\partial u}{\partial t}(t_1(a, \lambda), a, \lambda)\frac{\partial t_1}{\partial a}(a, \lambda) + \frac{\partial u}{\partial a}(t_1(a, \lambda), a, \lambda) = 0.$$

So we conclude by $(\partial t_1/\partial a)(a, \lambda) = 0$, because $(\partial u/\partial t)(t_1(a, \lambda), a, \lambda) \neq 0$ as we have already seen before.

The same work can be done with λ and we derive $(\partial t_1/\partial \lambda)(a, \lambda) = 0$.

Thus (a, λ) is a critical point of t_1, which cannot be because we have assumed that $R \in \Sigma$. So necessarily $w \equiv 0$. The kernel of $dM_{(u,\lambda)}$ is thus reduced to the null vector. This ends the proof of the proposition:

PROPOSITION 7. *If we take $R \in \Sigma$, the branch of solutions can locally be parametrized by $\tilde{\theta} = \lambda \int_{A(R)} F(u(x))\,dx$ when it cannot be parametrized by λ.*

We have just finished the study of all radial eigenfunctions of $d\tilde{M}_{(u,\lambda)}$, it remains to study the non-radial ones. It is easy to find that they are in one-to-one correspondence with the non-radial eigenfunctions of Section 7, namely, if $\psi_{i,k}$ is an eigenfunction of Proposition 5, associated to the eigenvalue $\mu_{i,k}$, then $(\psi_{i,k}, 0)$ is an eigenfunction of $d\tilde{M}_{(u,\lambda)}$ associated to the eigenvalue $\mu_{i,k}$. Therefore Proposition 5 holds in the sense:

PROPOSITION 8. *Suppose that $\mu_{i,0}(\theta) = 0$ for some $i \geqslant 0$ then, the eigenvalues of $d\tilde{M}_{(u,\lambda)}$ corresponding to radial eigenfunctions cannot be equal to 0, if $R \in \Sigma$; moreover the eigenvalues associated with non-radial eigenfunctions are exactly the $\mu_{i,k}$, with the corresponding eigenfunctions $(\psi_{i,k}, 0)$, for all $k \geqslant 1$ and $i \geqslant 0$.*

This parametrization is of particular interest because of the following lemma which is equivalent to Lemma 12:

LEMMA 13. *The equation $(d/ds)(u, \lambda) = -\tilde{M}(u, \lambda, \tilde{\theta})$ is a gradient system, i.e., there is a mapping $\tilde{\mathcal{M}}$ such that $\tilde{M}(u, \lambda, \tilde{\theta}) = d\tilde{\mathcal{M}}_{(u,\lambda)}(u, \lambda, \tilde{\theta})$ for $\tilde{\theta}$ fixed.*

To prove this lemma we need to find $\tilde{\mathcal{H}}$. Just define

$$\tilde{\mathcal{H}}(u, \lambda, \tilde{\theta}) = \frac{1}{2} \int_{A(R)} |\nabla u|^2 (x)\, dx - \lambda \int_{A(R)} F(u(x))\, dx + \tilde{\theta} \log \lambda$$

and verify that it provides a good choice.

9. The Symmetry Breaking

Suppose that we are at a point θ_0 such that $\mu_{0,k}(\theta)$ changes sign at this point, using the results of the last section, we know that the branch of radial solutions of (7) can be parametrized by λ or by $\tilde{\theta}$. For example, let us assume that near θ_0 the branch can be parametrized by $\tilde{\theta}$ (we would obtain the same results assuming that the parameter was λ near θ_0).

By Section 7, we know that this point is an infinitesimal symmetry breaking point and that it is isolated.

In order to show that symmetry breaking really occurs, we are going to use, as in [23] the Conley index (see [4, 20]). Let $\tilde{M}$ be as defined in the last section. We give one more definition:

Definition 5. Then eigenspace $N_{\tilde{\theta}}$ is the closed subvector space spanned by all eigenvectors of $d\tilde{M}_{(u,\lambda)}$ corresponding to all non-positive eigenvalues.

Here is a theorem from [23] which exactly fits our situation:

Theorem 9 [23]. *If, for some* $\tilde{\theta}_1, \tilde{\theta}_2 \in \mathbb{R}^+$, $\dim N_{\tilde{\theta}_1} \neq \dim N_{\tilde{\theta}_2}$ *and assume that* $d\tilde{M}_{(u(\tilde{\theta}), \lambda(\tilde{\theta}))}$ *is non-singular for* $\tilde{\theta} = \tilde{\theta}_1$ *and for* $\tilde{\theta} = \tilde{\theta}_2$, *then there exists some* $\tilde{\theta}$ *in* $(\tilde{\theta}_1, \tilde{\theta}_2)$ *such that* $(u(\tilde{\theta}), \lambda(\tilde{\theta}), \tilde{\theta})$ *is a symmetry breaking point for* $\tilde{M}$, *so it is a symmetry breaking point for* (7).

The proof of the existence of bifurcating solutions is done in [23], it uses a global Liapunov–Schmidt reduction to reduce to the finite dimensional case as well as the Conley index. The fact that the operator must be gradient as defined in Lemma 13, is crucial (see [23]).

We here give the main ideas of the proof.

Let us assume that the problem is a finite dimensional one (this is true once the global Liapunov–Schmidt reduction has been performed). First we replace (u, λ) by $(u - u(\tilde{\theta}), \lambda - \lambda(\tilde{\theta}))$, an we replace $\tilde{M}(u, \lambda, \tilde{\theta})$ by the corresponding operator $\tilde{M}(u + u(\tilde{\theta}), \lambda + \lambda(\tilde{\theta}), \tilde{\theta})$.

The fact that $d\tilde{M}_{(0,0)}$ is non-singular when $\tilde{\theta} = \tilde{\theta}_1$ (or when $\tilde{\theta} = \tilde{\theta}_2$), allows us to compute the Conley index of the isolated invariant set $\{(0,0)\}$ for the flow associated with the operator $\tilde{M}$

$$\frac{d}{dt}(u, \lambda) = \tilde{M}(u + u(\tilde{\theta}), \lambda + \lambda(\tilde{\theta}), \tilde{\theta}).$$

As $\dim N_{\tilde{\theta}_1} \neq \dim N_{\tilde{\theta}_2}$, the Conley index of $\{(0,0)\}$ for $\tilde{\theta} = \tilde{\theta}_1$ is different then the Conley index of $\{(0,0)\}$ for $\tilde{\theta} = \tilde{\theta}_2$. Applying Conley's continuation theorem, if we take some neighborhood V of $(0,0)$, we see that $(0,0)$ cannot be the maximal invariant set in V for all $\tilde{\theta} \in [\tilde{\theta}_1, \tilde{\theta}_2]$.

So there exists a point $\tilde{\theta} \in [\tilde{\theta}_1, \tilde{\theta}_2]$ and a point of $V(u, \lambda) \neq (0,0)$ such that (u, λ) belongs to the maximal invariant set in V.

The α-limit and ω-limit of (u, λ) are distinct rest points because the equation is gradient like, so there exists another rest point in V different from $(0,0)$. Taking a decreasing sequence of neighborhoods V_n, we obtain a sequence $\tilde{\theta}_n \in [\tilde{\theta}_1, \tilde{\theta}_2]$ and a corresponding sequence of rest points $(u_n, \lambda_n) \neq (0,0)$. Taking, if necessary a subsequence, we can assume that $\tilde{\theta}_n$ converges to some point $\tilde{\theta} \in (\tilde{\theta}_1, \tilde{\theta}_2)$ and therefore $(0, 0, \tilde{\theta})$ is a bifurcation point for the equation $\tilde{M}(u + u(\tilde{\theta}), \lambda + \lambda(\tilde{\theta}), \tilde{\theta}) = 0$.

In fact symmetry breaking occurs and not only bifurcation because radial bifurcation cannot occur. This finishes the proof of Theorem 5.

In order to discard some degenerate situation (cf. [17]) we use the work of C. Popiesch to derive Theorem 6.

Now taking into account that Eq. (7) is equivariant with respect to $O(n)$ and defining, for every subgroup G of $O(n)$, $N^G_{\tilde{\theta}}$ to be the subspace of $N_{\tilde{\theta}}$ which is invariant with respect to G, we can derive the more general theorem [23]:

THEOREM 10 [23]. *With the above hypothesis, let G_1 and G_2 be two subgroups of $O(n)$ such that the group generated by G_1 and G_2 is $O(n)$ and* $\dim N^{G_i}_{\tilde{\theta}_1} \neq \dim N^{G_i}_{\tilde{\theta}_2}$ *for $i = 1$ and $i = 2$; then there are at least two distinct symmetry breaking solutions with symmetry at least G_1 and G_2.*

The proof of this theorem, is divided in two steps. First we can perform the same demonstration as in theorem 9 with spaces of functions invariant under a subgroup of $O(n)$, to derive the fact that there are at least two symmetry breaking solutions with symmetry at least G_1 and G_2. Second, we show that these two solutions are distinct because elements belonging to the two sets of symmetry breaking solutions must be invariant under both G_1 and G_2, so under $G_1 G_2 = O(n)$, which is not possible because radial bifurcation does not occur. The existence of subgroups of $O(n)$ satisfying the hypothesis of theorem 10 is proved in [23].

We already have some partial results for some bifurcating points, but it is not so clear for example to know how many distinct manifolds do bifurcate. We concentrate here on one symmetry breaking point θ_0 and suppose that near this point the set of radial solutions is parametrized by $\tilde{\theta}$ (if the parametrization was λ, we would obtain the same results). We will denote by E_k the kernel of the linearized operator at this point, by what we have said in Section 8, E_k is spanned by elements like $(w_0(\theta_0, |x|)\,\varphi_k(x/|x|), 0)$, in particular the dimension of E_k is equal to the dimension of the eigenspace of the operator $-\Delta_{S^{n-1}}$ associated to the eigenvalue λ_k. We already know that our equation is equivariant under the action of $O(n)$ (defined on $\mathscr{C}_0^{2,\alpha}(\Omega, \mathbb{R})$), we can define an action of $O(n)$ over E_k by $g \cdot (w, v) = (g \cdot w, v)$.

Now, it is possible to define for some point x in E_k, the subgroup of all the elements of $O(n)$ which leave x invariant

$$\mathrm{Fix}(x) \in \{g \in O(n)\}/g \cdot x = x\},$$

and, for some subgroup G of $O(n)$ the subspace of E_k spanned by all the vectors under the action of G:

$$\mathrm{Inv}(G) = \{y \in E_k / \forall g \in G g \cdot y = y\}.$$

Suppose that we choose some $x \in E_k$, $x \neq 0$ such that $\mathrm{Inv}(\mathrm{Fix}(x)) = \mathrm{Vect}(x)$, the one dimensional space spanned by x.

If we consider, as in the last section, Eq. (7) over the space of functions invariant with respect to $\mathrm{Fix}(x)$, we can derive the existence of a symmetry breaking branch having at least the symmetry $\mathrm{Fix}(x)$. Assume that there exists some $g \in O(n)$ such that $g \cdot x = \pm x$, we derive that $\mathrm{Fix}(g \cdot x) = g\,\mathrm{Fix}(x)\,g^{-1}$ and that $\mathrm{Inv}(\mathrm{Fix}(g \cdot x)) = g \cdot \mathrm{Inv}(\mathrm{Fix}(x)) = g \cdot \mathrm{Vect}(x)$. As before we obtain the existence of a symmetry breaking branch having at least the symmetry $\mathrm{Fix}(g \cdot x)$.

The two branches obtained are different in a neighborhood of the symmetry breaking point. In fact, if we suppose the contrary, using the result of E. N. Dancer [6], we derive the existence of an element in $E(k) \setminus \{0\}$ invariant under both $\mathrm{Fix}(x)$ and $\mathrm{Fix}(g \cdot x)$, which is not possible because we have assumed that $g \cdot x = \pm x$. From what we have just seen, we can see that the dimension of the symmetry breaking branch is at least equal to the dimension of the manifold $\{g \in O(n)/gx \neq \pm x\}$ plus one.

Finally, we give the proof of Proposition 2.

Suppose that we have two solutions of (7) (u_0, λ_0) and (u_1, λ_1), belonging to the same connected component of $\mathscr{S}(R) \setminus \{(0, 0)\}$, such that $u_0(x) > 0$ on all $A(R)$ and $u_1(x)$ changes sign on $A(R)$. Then one can find a continuous map $\phi(t) = (u(t, x), \lambda(t))$ from $[0, 1]$ into $\mathscr{S}(R) \setminus \{(0, 0)\}$ such that $\phi(0) = (u_0(x), \lambda_0)$ and $\phi(1) = (u_1(x), \lambda_1)$.

If we take the largest value of $t \in [0, 1]$ such that $u(s, x) > 0$ for all $(s, x) \in [0, t) \times A(R)$, we can see that $\lambda(s) > 0$ for all $s < t$, so by continuity we find that $\lambda(t)\geqslant 0$ and $u(t,x)\geqslant 0$ for all $x\in A(R)$. Using (H2) we see that $f(x)\geqslant 0$ on some $[\varepsilon,+\infty)$ so, applying the maximum principle (see [8, Theorem 2.2]) we find that $u(t,x)>0$ on $A(R)$, for $|t'-t|$ small enough, which cannot be.

This completes the proof of the proposition.

10. Generalization

In all we have done we have worked with the equation

$$-\Delta u=\lambda f(u),$$

but is it possible to obtain the same results if we assume that f is also a function of λ.

To be more precise, let us assume that we work with the equation

$$-\Delta u=f(u,\lambda).$$

Then all our results still hold if we change the hypothesis (H1)–(H4) into:

(I1) f is real analytical in both variables λ and u.

(I2) $f(u,0)=0$ for all $u\in\mathbb{R}^+$, moreover there exists a continuous function $\alpha(\lambda)>0$ for $\lambda>0$, satisfying

$$\lim_{\lambda\to+\infty}\alpha(\lambda)=+\infty$$

and

$$f(u,\lambda)>\alpha(\lambda)\sup(u,1)\qquad\text{for all } u\in\mathbb{R}^+ \text{ and all } \lambda>0.$$

(I3) For all $\lambda_0>0$, $\lim_{x\to+\infty}(1/x^2)\int_0^x f(t,\lambda)\,dt=+\infty$ uniformly in $[\lambda_0,+\infty)$ and $f_u(u,\lambda)\geqslant 0$.

And finally (H4) must be changed into:

(I4) There exists $A>0$ such that $\lim_{u\to+\infty}uf_u(u,\lambda)/f(u,\lambda)=+\infty$ uniformly in $[A,+\infty)$.

f_u denotes the derivative of f with respect to u.

We can verify that the new hypothesis allows us to reach the conclusion in all the proofs of our paper. Let us now say some words about the changes that must be done in Section 8.

In the first parametrization, the mapping considered does not change

$$M(u,\lambda)=-\Delta u-f(u,\lambda).$$

And the energy associated is just

$$\mathcal{M}(u, \lambda, \theta) = \frac{1}{2} \int_{A(R)} |\nabla u|^2 (x)\, dx - \int_{A(R)} F(u(x), \lambda)\, dx,$$

where $F(u, \lambda) = \int_0^u f(t, \lambda)\, dt$.

Now, if we turn to the second parametrization, it is interesting to note the changes.

The mapping we have to consider is now

$$\tilde{M}(u, \lambda; \tilde{\theta}) = \left(-\Delta u - f(u, \lambda), -\int_{A(R)} F_\lambda(u(x), \lambda)\, dx + \frac{\tilde{\theta}}{\lambda} \right),$$

where $F_\lambda(u, \lambda) = \int_0^u f_\lambda(t, \lambda)\, dt$ and f_λ is the derivative of f with respect to λ. The associated energy is then

$$\tilde{\mathcal{M}}(u, \lambda, \tilde{\theta}) = \frac{1}{2} \int_{A(R)} |\nabla u|^2 (x)\, dx - \int_{A(R)} F(u(x), \lambda)\, dx + \tilde{\theta} \log(\lambda).$$

So, we see that the new parameter is not $\int_{A(R)} F(u(x), \lambda)\, dx$ but rather $\lambda \int_{A(R)} F_\lambda(u(x), \lambda)\, dx$.

11. Conjectures

We give here some conjectures:

Conjecture 1. $t_1(a, \lambda)$ has no critical point.

This problem is very important because if the conjecture is true we can prove that the set of radial solutions on an annulus or on a ball is connected.

We can ask, what is the behavior of the symmetry breaking branches?

Conjecture 2. The non-radial branches of bifurcation converge to some singular solutions.

Conjecture 3. Is it true that the branch of radial solutions can be parametrized by the parameter $\tilde{\theta}$ introduced in Section 8?

12. Appendix A

We recall here the implicit function theorem we use here [3]:

Theorem 11. *Let X, Y, Z be three (real or complex) Banach spaces and $U \subset X$, $V \subset Y$ two opens; we take a $\mathcal{C}^1$ map F from $U \times V$ into Z such that*

$F(x_0, y_0)=0$ and $dF_x(x_0, y_0)$ has bounded inverse. Then there are $U_1 \subset U$, $V_1 \subset V$ open neighborhoods of x_0 and y_0 and a function f from V_1 into U_1 such that $F(x, y)=0$ in $U_1 \times V_1$ if and only if $x=f(y)$. Moreover, f is as regular as F is.

13. APPENDIX B

THEOREM 12. *Given a function $a(x)>0$ and two functions $b_1(x) \leqslant b_2(x)$, if u_i is a solution of $(d/dx)(a(x)(du_i/dx)(x))+b_i(x)\,u_i(x)=0$, for $i \in \{1, 2\}$, then, between two zeros of u_1, there is a zero of u_2.*

14. APPENDIX C

Here we want to show that if (H3′) holds, all our results are true.

Recall that (H3′) is just the hypothesis:

(H3′) $\lim_{x\to+\infty}(1/x^2)\int_0^x f(t)\,dt = +\infty$ and if we define $G(x) = \sup_{u\in[0,x]} f(u)$ and $H(x) = \inf_{u\in[x,+\infty)} f(u)$, we ask that $\liminf_{x\to+\infty}(H(x)/G(x))>0$.

The assumption "f is increasing" was used in order to derive the second part of Lemma 3, to obtain a lower bound of u_m/a. This lower bound was used in Section 6 once in order to prove Lemma 10 and after that, once in order to show that along a branch of radial solutions, $u(t)$ converges to $+\infty$ uniformly over $[t_0-\varepsilon, t_1-\varepsilon]$ when a goes to $+\infty$ (t_0 and t_1 being fixed).

We are going to show that these two results are still true if we do not assume f to be increasing, like in (H3), but only that $\liminf_{x\to+\infty}(H(x)/G(x))=\beta>0$, like in (H3′).

First, let us prove that if (a, λ) belongs to the set of points for which $t_1(a, \lambda)=t_1$, then the maximum value of u goes to $+\infty$ when a does (t_0 and t_1 being fixed).

Suppose that this is not true, so we can find a point (a, λ) satisfying $t_1(a, \lambda)=t_1$ with a as large as we want, such that the corresponding solutions are bounded independently of a. As the set of λ for which (7) has a solution is bounded, going back to Eq. (7), we see that the second derivative of u is bounded on $[t_0, t_1]$ independently of a. But this is not possible because necessarily $(du/dt)(t_1)<0$, which cannot be if a is large enough.

Then, choosing $\varepsilon>0$ and using the fact that u is concave, we see that $u(t)$ converges to $+\infty$ uniformly over $[t_0-\varepsilon, t_1-\varepsilon]$ when a goes to $+\infty$.

Now we are going to prove that Lemma 10 is still true under assumption

(H3′). Let us recall that we have a sequence $((a^k, \lambda^k))_{k\in N}$ for which $(a^k)_{k\in N}$ tends to $+\infty$, $\lambda^k \geqslant \lambda^* > 0$ and satisfying

$$t_0 + \varepsilon = t_1(a^k, \lambda^k).$$

Then we consider u^k, the corresponding solutions of (5), with the appropriate radius $R = ((n-2)t_1)^{1/(2-n)}$.

As the sequence $(\lambda^k)_{k\in N}$ is bounded, passing to a subsequence if necessary, we can always assume that it converges to some $\lambda > 0$. Therefore, we can prove, as above, that $\{u^k(x)\}_{k\in N}$ converges to $+\infty$ uniformly over $[R+\varepsilon, 1-\varepsilon]$.

We set $R_1 = (2(n-2)R + (1+R^{2-n}))/4(n-2)$ and $R_2 = (3+R)/4$.

By Lemma 2, we know that every solution u of (5) is increasing on $[R, (1+R^{2-n})/2(n-2)]$. We can choose $\varepsilon > 0$ small enough to ensure that $[R_1, R_2] \subset [R+\varepsilon, 1-\varepsilon]$. Then, using (H3′), there is some $x_0 > 0$ such that $H(x)/G(x) > \beta/2$ for all $x > x_0$.

For k large enough, $u^k(r) > x_0$ for all $r \in [R+\varepsilon, 1-\varepsilon]$. So we can find some constant $c > 0$, depending on β and R, such that

$$\lambda^k \int_{R<|x|<R_1} f(u^k(x))\,dx \leqslant c\left(\frac{2}{\beta}\right)\lambda^k \int_{R_1<|x|<R_2} f(u^k(x))\,dx.$$

The same thing can be done with R_2; for some $c' > 0$ and k large enough we have

$$\lambda^k \int_{R_2<|x|<1} f(u^k(x))\,dx \leqslant c'\left(\frac{2}{\beta}\right)\lambda^k \int_{R_1<|x|<R_2} f(u^k(x))\,dx.$$

Hence, for k large enough, we can write

$$\lambda^k \int_{A(R)} f(u^k(x))\,dx \leqslant \left(1 + (c+c')\left(\frac{2}{\beta}\right)\right)\lambda^k \int_{R_1<|x|<R_2} f(u^k(x))\,dx.$$

Moreover, as in Lemma 1, we can obtain

$$\lambda_1 \int_{A(R)} \varphi_1(x)\,u^k(x)\,dx = \lambda^k \int_{A(R)} \varphi_1(x)\,f(u^k(x))\,dx.$$

If we define $u_m^k = \sup_{r\in A(R)} u(r)$, we can write

$$u_m^k \lambda_1 \int_{A(R)} \varphi_1(x)\,dx \geqslant \lambda_1 \int_{A(R)} \varphi_1(x)\,u^k(x)\,dx = \lambda^k \int_{A(R)} \varphi_1(x)\,f(u^k(x))\,dx.$$

On $[R_1, R_2]$, $\varphi_1(x)$ is minored by some constant $\eta > 0$, so we find that, for k large enough,

$$u_m^k \lambda_1 \int_{A(R)} \varphi_1(x) \geqslant \eta \lambda^k \int_{A(R)} f(u^k(x))\, dx \beta/(\beta + 2(c + c')).$$

Finally, from Proposition 3 we know that

$$\lambda^k \int_{A(R)} f(u^k(x))\, dx = |S^{n-1}|\, (b^k + a^k).$$

So we derive

$$u_m^k \geqslant C(b^k + a^k)$$

for some constant $C > 0$. The remainder of the proof is unchanged so we omit it here.

We can also replace (H3) by the following hypothesis:

(H3″) $\lim_{x \to +\infty} (f(x)/x) = +\infty$.

The proof of Lemma 10 is then analogues to the previous one, we only outline it here.

With the same notations one can prove that

$$\lambda_1 \int_{R_1 < |x| < R_2} \varphi_1(x)\, u^k(x)\, dx \geqslant c \lambda_1 \int_{A(R)} \varphi_1(x)\, u^k(x)\, dx,$$

for some constant $c > 0$.

Then, given some $q > \lambda_1/(c\lambda)$, as u^k converges to $+\infty$ uniformly over $[R + \varepsilon, 1 - \varepsilon]$, one can see that for k large enough, $f(u^k(x)) > q u^k(x)$ for all $x \in [R_1, R_2]$.

So we can derive

$$\lambda_1 \int_{R_1 < |x| < R_2} \varphi_1(x)\, u^k(x)\, dx \geqslant c \lambda^k \int_{A(R)} \varphi_1(x) f(u^k(x))\, dx$$

$$\geqslant c q \lambda^k \int_{R_1 < |x| < R_2} \varphi_1(x)\, u^k(x)\, dx.$$

Therefore

$$\lambda_1 \geqslant c q \lambda^k.$$

As λ^k goes to $\lambda > 0$ when k tends to $+\infty$, we obtain a contradiction. This ends the proof of Lemma 10 if (H3″) is assumed to hold.

15. Appendix D

We want to study the spectrum of the operator $d\tilde{M}_{(u,\lambda)}$ defined from $(H_0^1(\Omega)\cap H^2(\Omega))\times\mathbb{R}$ into $L^2(\Omega)\times\mathbb{R}$ by

$$d\tilde{M}_{(u,\lambda)}(w,v)=\Big(-\Delta w(x)-\lambda f'(u(x))\,w(x)-vf(u(x)),$$
$$-\int_\Omega f(u(x))\,w(x)\,dx-\frac{v}{\lambda}\int_\Omega F(u(x))\,dx\Big). \tag{19}$$

First let us show that there exists a number $\alpha\in\mathbb{R}$ suitably chosen such that the operator $d\tilde{M}_{(u,\lambda)}+\alpha I$ is invertible. As the spectrum of the operator $-\Delta\cdot-\lambda f'(u(x))\cdot$ is composed of eigenvalues, we can assume that $-\Delta\cdot-\lambda f'(u(x))\cdot+\alpha_0\cdot$ is invertible. Let us call L this operator. Now, we take $h\in L^2(\Omega)$ and $\gamma\in\mathbb{R}$, we want to show that the system

$$\begin{cases} Lw(x)-vf(u(x))+\alpha w(x)=h(x) & \text{on } \Omega\\ w(x)=0 & \text{on } \partial\Omega\\ -\int_\Omega f(u(x))\,w(x)\,dx-\frac{v}{\lambda}\int_\Omega F(u(x))\,dx+(\alpha+\alpha_0)v=\gamma \end{cases} \tag{20}$$

has a unique solution in $(H_0^1(\Omega)\cap H^2(\Omega))\times\mathbb{R}$.

The operator L being invertible, so is $L\cdot+\alpha\cdot$, for α small. So we can compute $w=(L+\alpha I)^{-1}(h)+v(L+\alpha I)^{-1}(f(u))$.

Putting w back into the second equation of (20), gives us

$$-\int_\Omega f(u(x))(L+\alpha I)^{-1}(h)(x)\,dx-v\int_\Omega f(u(x))(L+\alpha I)^{-1}(f(u))(x)\,dx$$
$$-\frac{v}{\lambda}\int_\Omega F(u(x))\,dx+(\alpha+\alpha_0)v=\gamma. \tag{21}$$

Suppose that the constant

$$\int_\Omega f(u(x))\,L^{-1}(f(u))(x)\,dx+\frac{1}{\lambda}\int_\Omega F(u(x))\,dx-\alpha_0\neq 0.$$

Then we can take $\alpha=0$ in (20) and we can compute v using the last equation, this gives us the existence as well as the uniqueness of the solution (w,v) of the system (20). So the operator $d\tilde{M}_{(u,\lambda)}+\alpha_0 I$ is invertible.

Next we suppose that

$$\int_\Omega f(u(x))\,L^{-1}(f(u))(x)\,dx+\frac{1}{\lambda}\int_\Omega F(u(x))\,dx-\alpha_0=0.$$

We expand, in Eq. (21), the operator $(L+\alpha I)^{-1}$ for α small to see that the coefficient of v is not zero for α small enough. More precisely, in Eq. (21), the coefficient of v is

$$-\int_{\Omega} f(u(x))(L+\alpha I)^{-1}(f(u))(x)\,dx-\frac{1}{\lambda}\int_{\Omega} F(u(x))\,dx+\alpha_0+\alpha$$

$$=\alpha\left(\int_{\Omega}(L^{-1}(f(u))(x))^2\,dx+1\right)+o(\alpha).$$

So, for α small enough, Eq. (21) has a unique solution, so, the system (20), has also a unique solution.

Finally, we note that $(d\tilde{M}+\alpha I)^{-1}$ is a self adjoint operator for the scalar product on $L^2(\Omega)\times\mathbb{R}$ defined by

$$((u,v);(v,\mu))=\int_{\Omega} u(x)\,v(x)\,dx+v\mu.$$

Moreover, it is easy to see that $(d\tilde{M}_{(u,\lambda)}+\alpha I)^{-1}$ is a compact operator from $L^2(\Omega)\times\mathbb{R}$ into itself. Schauder's theory allows us to conclude that the spectrum of $d\tilde{M}(u,\lambda)$ is only composed of eigenvalues.

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