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Asymptotic Stability in Singular Perturbation Problems. II: Problems Having Matched Asymptotic Expansion Solutions*

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The stability of systems of ordinary differential equations of the form

$$dx/dt = f(t, x, y, \epsilon), \qquad \epsilon\, dy/dt = g(t, x, y, \epsilon),$$

where ϵ is a real parameter near zero, is studied. It is shown that if the reduced problem

$$dx/dt = f(t, x, y, 0), \qquad 0 = g(t, x, y, 0),$$

is stable, and certain other conditions which ensure that the method of matched asymptotic expansions can be used to construct solutions are satisfied, then the full problem is asymptotically stable as $t \to \infty$, and a domain of stability is determined which is independent of ϵ. Moreover, under certain additional conditions, it is shown that the solutions of the perturbed problem have limits as $t \to \infty$. In this case, it is shown how these limits can be calculated directly from the equations

$$f(\infty, x, y, \epsilon) = 0 \qquad g(\infty, x, y, \epsilon) = 0$$

as expansions in powers of ϵ.

Many investigations have been made into the stability of singular perturbation problems of the form

$$(P_\epsilon) \qquad dx/dt = f(t, x, y, \epsilon), \qquad \epsilon\, dy/dt = g(t, x, y, \epsilon),$$

where ϵ is a positive real parameter near zero and x and y are m- and n-vectors, respectively. Usually, the reduced problem

$$(P_0) \qquad dx/dt = f(t, x, y, 0) \qquad 0 = g(t, x, y, 0)$$

is assumed to have an asymptotically stable solution for $t \geqslant 0$, and then conditions are placed on f and g which ensure the full problem is stable

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in some sense for $\epsilon > 0$. In particular, conditions were given in [1] which ensure that for initial data near those of the stable solution of (P_0) and for ϵ near zero, the problem (P_ϵ) has a unique solution existing for $t \geqslant 0$ and converging to the stable solution of (P_0) as $\epsilon \to 0$. In [2] the behavior of solutions of (P_ϵ) as $t \to \infty$ was studied under the conditions given in [1]. However, these results are difficult to apply since they involve quite technical stability assumptions. On the other hand, some results have been obtained under conditions less general than those needed here or in [1, 2]; e.g., this problem was considered in [3] where f and g were not allowed to depend explicitly on ϵ, and in [7] where the full problem was studied only for initial data near those of the reduced problem.

In this paper, general, yet directly applicable, results about the stability of (P_ϵ) are obtained. This is done by restricting attention to systems whose solutions can be constructed by the method of matched asymptotic expansions. The results needed here about this method of analysis are summarized in Theorem 1.

The main results of this paper are given in Theorem 2. Roughly, these show that if the problem (P_0) is stable, then for small $\epsilon > 0$, the full problem (P_ϵ) is also stable. Moreover, the domain of stability for (P_ϵ) is essentially determined by that for the problem (P_0) and another zero-order auxiliary problem. This means that a set is found such that any two solutions of (P_ϵ) beginning in that set approach each other as $t \to \infty$. Thus, the initial data are restricted only to a set which is determined by zero-order problems rather than being required to be "small" as in [7]. This explicit determination of a domain of stability for the problem results from our construction of approximate solutions of (P_ϵ). Furthermore, this stability result is combined with the matched asymptotic expansion solution to give a straightforward method for approximating the state of the system in the distant future.

Finally, the case where f, g, and the solution of (P_0), $(x = x_0(t), y = y_0(t))$, all have finite limits as $t \to \infty$ is studied. It is shown that all solutions of (P_ϵ) beginning in the domain of stability have finite limits as $t \to \infty$. Moreover, these limits can be determined by solving the equations

$$f(\infty, x, y, \epsilon) = 0 \qquad g(\infty, x, y, \epsilon) = 0$$

for (x, y) near $(x_0(\infty), y_0(\infty))$ and ϵ near zero. An expansion for this solution in powers of ϵ can be constructed by the implicit function theorem. This fact is established by a study of the matched asymptotic expansion solution as $t \to \infty$.

The novelty of these results lies in the determination of a domain of stability and in the use of matched asymptotic expansions to compute approximations to the long time state of the system.

Asymptotic Stability of (P_ϵ)

We will suppose the reduced problem (P_0) has a solution, $x = x_0(t)$, $y = y_0(t)$, existing for $0 \leqslant t < \infty$. The functions f and g are assumed to have continuous and bounded derivatives with respect to t, ϵ, and the components of x and y up to order $N + 2$ in some tube about (x_0, y_0) and for ϵ near zero. Moreover, we suppose there is a smooth function $\phi(t, x)$ which defines the branch of $g(t, x, y, 0) = 0$ on which (x_0, y_0) lies; that is, $y_0(t) = \phi(t, x_0(t))$ for $t \geqslant 0$ and

$$g(t, x, \phi(t, x), 0) = 0$$

for $0 \leqslant t < \infty$, $|x - x_0(t)| \leqslant \Delta$ (Δ is some fixed positive number).

There are two stability conditions which we place on the problem. The first deals with the stability of the reduced problem.

SI. The linear system

$$dz/dt = A(t)z, \tag{1}$$

where $A(t) = (f_x - f_y g_y^{-1} g_x)(t, x_0(t), y_0(t), 0)$, is exponentially asymptotically stable. That is, if $\Phi(t)$ is the fundamental matrix for this system defined by

$$d\Phi/dt = A(t)\Phi, \qquad \Phi(0) = \text{identity},$$

then there are positive constants K and α such that

$$|\Phi(t)\,\Phi^{-1}(s)| \leqslant K \exp[-\alpha(t - s)]$$

for $0 \leqslant s \leqslant t < \infty$. Here the matrix norm is any convenient one, such as the Euclidean norm. The notation f_x, etc., is used here to denote the Jacobian matrix $(\partial f_i/\partial x_j)$, etc.

We observe that SI has often been replaced in other investigations by the equivalent condition that the linear system (1) be uniformly asymptotically stable (see [5]).

The second stability assumption guarantees that the solutions of (P_ϵ) have limits as $\epsilon \to 0$:

SII. The eigenvalues of the Jacobian matrix $g_y(t, x_0(t), y_0(t), 0)$ all have real parts less than some fixed negative number for all $0 \leqslant t < \infty$.

Condition SI guarantees that the solution $x = x_0(t)$ of

$$dx/dt = f(t, x, \phi(t, x), 0) \tag{2}$$

is asymptotically stable. Let its domain of attraction be denoted by D.

Thus, if $\xi \in D$, there is a unique solution of (2) having ξ as its initial value. Moreover, this solution approaches $x_0(t)$ as $t \to \infty$.

We restrict, if needed, D so that the eigenvalues of the matrix

$$g_y(0, \xi, \phi(0, \xi), 0)$$

all have negative real part for $\xi \in D$. Next, we consider the "fast time" system which is obtained from (P_ϵ) by setting $t = \epsilon\tau$ and letting $\epsilon \to 0$:

$$dY/d\tau = g(0, \xi, \phi(0, \xi) + Y, 0). \tag{3}$$

Condition SII and the restriction on D ensure that for each $\xi \in D$, the zero solution of this system is asymptotically stable. Let its domain of attraction be denoted by E_ξ. We write this in such a way that if $Y(0) + \phi(0, \xi) \in E_\xi$, then the solution $Y(\tau)$ of (3) determined by the initial condition $Y(0)$ exists for $\tau \geqslant 0$ and approaches zero as $\tau \to \infty$.

Now, let $\xi(\epsilon) \in R^m$, $\eta(\epsilon) \in R^n$, be smooth functions of ϵ at $\epsilon = 0$ such that $\xi_0 = \xi(0) \in D$ and $\eta(0) \in E_{\xi_0}$. Then with SI and SII, the conditions of Theorem 2 [4] are satisfied, and we have

THEOREM 1. *If the conditions listed above are satisfied, then for each small* $\epsilon > 0$, *the initial value problem*

$$dx/dt = f(t, x, y, \epsilon), \qquad x(0) = \xi(\epsilon),$$

$$\epsilon\, dy/dt = g(t, x, y, \epsilon), \qquad y(0) = \eta(\epsilon),$$

has a unique solution $(x(t, \epsilon), y(t, \epsilon))$ *for* $0 \leqslant t < \infty$. *Moreover, it can be written as*

$$x = x^*(t, \epsilon) + X(t/\epsilon, \epsilon), \qquad y = y^*(t, \epsilon) + Y(t/\epsilon, \epsilon),$$

where (X, Y) *satisfies* $X(\infty, \epsilon) = 0$, $Y(\infty, \epsilon) = 0$, *and the functions* x^* *and* y^* *are smooth functions of* t *and* ϵ *at* $\epsilon = 0$ *with*

$$\begin{aligned} x^*(t, \epsilon) &= \sum_{r=0}^{N} x_r^*(t)\, \epsilon^r + O(\epsilon^{N+1}), \\ y^*(t, \epsilon) &= \sum_{r=0}^{N} y_r^*(t)\, \epsilon^r + O(\epsilon^{N+1}), \end{aligned} \tag{4}$$

where $O(\cdot)$ *holds uniformly for* $0 \leqslant t < \infty$ *as* $\epsilon \to 0$. *Finally, the functions* x_r^* *and* y_r^* *are determined successively by solving the differential equations*

$$\begin{gathered} dx_0^*/dt = f(t, x_0^*, y_0^*, 0), \qquad x_0^*(0) = \xi(0), \\ y_0^* = \phi(t, x_0^*(t)), \end{gathered} \tag{5}$$

and for $r = 1, \ldots, N$,

$$dx_r^*/dt = f_x^*(t)\, x_r^* + f_y^*(t)\, y_r^* + p_r(t),$$

$$dy_{r-1}^*/dt = g_x^*(t)\, x_r^* + g_y^*(t)\, y_r^* + q_r(t),$$

where to shorten notation we have let $f_x^*(t)$, *etc., denote* $f_x(t, x_0^*(t), y_0^*(t), 0)$, *etc. The functions* $p_r(t)$ *and* $q_r(t)$ *only depend on* $t, x_0^*, y_0^*, \ldots, x_{r-1}^*, y_{r-1}^*$.

These last equations can be simplified by eliminating y_r^*:

$$dx_r^*/dt = A(t)\, x_r^* + R_r(t), \tag{6}$$

where A is given in SI and

$$R_r(t) = p_r(t) + f_y^*(t)\, g_y^{*-1}(t)[(dy_{r-1}^*/dt) - q_r(t)].$$

The initial conditions for the Eq. (6) are given by formulas in [4] which will not be needed here. The function $(x^*(t, \epsilon), y^*(t, \epsilon))$ is called the outer solution of the problem. The function $(X(t/\epsilon, \epsilon), Y(t/\epsilon, \epsilon))$ is called the boundary layer solution (or correction) for the problem, and it can be shown to approach zero exponentially as $t/\epsilon \to \infty$ at a rate independent of ϵ.

The following theorem is the principal result of this paper. It shows that with conditions SI and SII, the problem (P_ϵ) is asymptotically stable and that the domain of stability for this system is determined exclusively by the auxiliary problems (2) and (3).

THEOREM 2. *Let conditions* SI *and* SII *be satisfied. If* $(x(t, \epsilon), y(t, \epsilon))$ *and* $(\tilde{x}(t, \epsilon), \tilde{y}(t, \epsilon))$ *are solutions of* (P_ϵ) *such that* $(x(0, \epsilon), y(0, \epsilon))$ *and* $(\tilde{x}(0, \epsilon), \tilde{y}(0, \epsilon))$ *define functions* $(\xi(\epsilon), \eta(\epsilon))$ *and* $(\tilde{\xi}(\epsilon), \tilde{\eta}(\epsilon))$, *respectively, which are* (i) *smooth functions of* ϵ *at* $\epsilon = 0$, *and* (ii) *satisfy* $\xi_0 \equiv \xi(0)$, $\tilde{\xi}_0 \equiv \tilde{\xi}(0) \in D$ *and* $\eta(0) \in E_{\xi_0}$, $\tilde{\eta}(0) \in E_{\tilde{\xi}_0}$, *then for small* $\epsilon > 0$, *these solutions exist on the whole interval* $0 \leqslant t < \infty$, *and*

$$\lim_{t\to\infty} [(x(t, \epsilon), y(t, \epsilon)) - (\tilde{x}(t, \epsilon), \tilde{y}(t, \epsilon))] = 0.$$

The proof of Theorem 2 is given at the end of the paper. However, we note here that the difference $(x, y) - (\tilde{x}, \tilde{y})$ will actually be shown to approach zero at an exponential rate determined by α in SI.

We now combine this result with the method of matched asymptotic expansions solutions to approximate the "long-time" state of the system. Let us construct an outer solution of (P_ϵ), $(\tilde{x}(t, \epsilon), \tilde{y}(t, \epsilon))$, choosing the data to make the computations easy. First, we want $\tilde{x}(t, 0) = x_0(t)$ and $\tilde{y}(t, 0) = y_0(t)$. To construct the first-order approximation, we take $\tilde{x}_1(0) = 0$ and, therefore, are forced to take $\tilde{y}_1(0) = g_y^{-1}(0)[y_0'(0) - g_\epsilon(0)]$. Thus, for the choice of initial data $\tilde{x}(0, \epsilon) = x_0(0) + O(\epsilon^2), \tilde{y}(0, \epsilon) = y_0(0) + \epsilon\tilde{y}_1(0) + O(\epsilon^2)$, Theorem 1 can be applied to prove the existence of a unique solution of (1) for $t \geqslant 0$ and small $\epsilon > 0$. Moreover, this solution satisfies

$$\tilde{x}(t, \epsilon) = x_0(t) + \epsilon\tilde{x}_1(t) + O(\epsilon^2),$$

$$\tilde{y}(t, \epsilon) = y_0(t) + \epsilon\tilde{y}_1(t) + O(\epsilon^2),$$

where $O(\cdot)$ holds uniformly for $t \geqslant 0$.

According to Theorem 2, any solution of (1) which lies in $\tilde{D} = D \times \bigcup_{\xi \in D} E_\xi$ for $\epsilon = 0$, approaches $(\tilde{x}(t, \epsilon), \tilde{y}(t, \epsilon))$ as $t \to \infty$ at an exponential rate δ independent of ϵ. Thus, the formula for $(\tilde{x}, \tilde{y})$ gives an approximation of this solution up to order $O(\epsilon^2)$ which is valid uniformly on the interval $2 \mid \log \epsilon \mid (\delta)^{-1} \leqslant t < \infty$. Thus, $(\tilde{x}, \tilde{y})$ gives an approximation for large t to any solution of (1) beginning in $\tilde{D}$. Obviously, the approximation can be improved by defining higher-order terms in the expansion of $(\tilde{x}, \tilde{y})$.

This approach is developed further in the next section under some additional assumptions which ensure that the steady state problem exists.

The Steady-State Case

In certain cases, it is possible to actually develop an asymptotic expansion valid as $\epsilon \to 0$ for the limit

$$\lim_{t \to \infty} (x(t, \epsilon), y(t, \epsilon)).$$

To illustrate this, we make the following additional assumptions. First, we assume the equations make sense in the limit $t = \infty$.

HI. The solution $(x_0(t), y_0(t))$ of (P_0) has a finite limit $(x_0(\infty), y_0(\infty))$ as $t \to \infty$, and the functions f, g, ϕ, and their derivatives with respect to t, ϵ, and the components of x and y to order $N + 2$, when evaluated near $x = x_0(t)$, $y = y_0(t)$, $\epsilon = 0$, have finite limits as $t \to \infty$.

Next, we suppose instead of SI that we have

SI′. $A(t) = (f_x - f_y g_y^{-1} g_x)(t, x_0(t), y_0(t), 0)$ approaches a stable matrix $A(\infty)$ as $t \to \infty$; i.e., $A(\infty)$ is an $m \times m$-matrix whose eigenvalues satisfy $\operatorname{Re} \lambda < 0$.

It is shown below that condition SI′ implies that SI is satisfied. It can also be shown that HI and SI imply that SI′ is satisfied. With these new conditions, we have the following result.

THEOREM 3. *Let conditions* HI, SI′, *and* SII *be satisfied. If* $(\xi(\epsilon), \eta(\epsilon))$ *are smooth functions of* ϵ *with* $\xi(0) \in D$ *and* $\eta(0) \in E_{\xi_0}$, *then for small* $\epsilon > 0$, *the problem* (P_ϵ) *has a unique solution* $(x(t, \epsilon), y(t, \epsilon))$ *for* $0 \leqslant t < \infty$ *which satisfies*

$$x(0, \epsilon) = \xi(\epsilon), \qquad y(0, \epsilon) = \eta(\epsilon).$$

Moreover,

$$\lim_{t\to\infty} (x(t, \epsilon), y(t, \epsilon)) = (x_0(\infty), y_0(\infty)) + \sum_{r=1}^{N} (x_r(\infty), y_r(\infty))\, \epsilon^r + O(\epsilon^{N+1}), \tag{8}$$

where

$$x_r(\infty) = -A^{-1}(\infty)\, R_r(\infty)$$

and

$$y_r(\infty) = -g_y^{-1}(\infty)[g_x(\infty)\, x_r(\infty) + q_r(\infty)],$$

and the functions R_r *and* q_r *are defined as in* (6).

Note that the expansion (8) is the same as would result if the implicit function theorem were applied directly to the steady-state equations discussed in the introduction.

Theorem 3 is easy to derive. From Theorem 1, we have

$$x(t, \epsilon) = x^*(t, \epsilon) + X(t/\epsilon, \epsilon),$$

$$y(t, \epsilon) = y^*(t, \epsilon) + Y(t/\epsilon, \epsilon).$$

Since the functions (X, Y) vanish in the limit $t = \infty$, they make no contribution to the result. Moreover, we also have that

$$x^*(t, \epsilon) = x_0^*(t) + \sum_{r=1}^{N} x_r^*(t)\, \epsilon^r + O(\epsilon^{N+1}),$$

$$y^*(t, \epsilon) = y_0^*(t) + \sum_{r=1}^{N} y_r^*(t)\, \epsilon^r + O(\epsilon^{N+1}),$$

where $O(\cdot)$ holds uniformly for $0 \leqslant t < \infty$. Assumption HI ensures that $(x_0^*(\infty), y_0^*(\infty))$ exists, and since $x_0^*(0) \in D$, $(x_0^*(\infty), y_0^*(\infty)) = (x_0(\infty), y_0(\infty))$. We will show in the following lemma that $(x_r^*(\infty), y_r^*(\infty))$ also exists. Then a reference to (6) shows that $x_r^*(\infty)$ satisfies

$$A(\infty)\, x_r^*(\infty) + R_r(\infty) = 0.$$

With this the proof of Theorem 3 is complete.

We now prove the lemma used in the proof of Theorem 3.

LEMMA. *Suppose* $z \in R^k$ *satisfies*

$$dz/dt = B(t)z + b(t), \qquad z(0) = z_0,$$

where B is a continuous, real $k \times k$*-matrix such that* (i) $\lim_{t\to\infty} B(t) = B(\infty)$ *exists and is a stable matrix, and the vector b is a k-vector such that* (ii) $\lim_{t\to\infty} b(t) = b(\infty)$ *exists. Then for any* $z_0 \in R^k$, $\lim_{t\to\infty} z(t) = -B^{-1}(\infty)\, b(\infty)$.

Proof. We first show that SI′ implies SI. Let $\Psi(t)$ denote the fundamental matrix defined by

$$d\Psi/dt = B(t)\Psi, \qquad \Psi(0) = \text{identity}. \tag{9}$$

Therefore, $\Psi(t)$ satisfies

$$d\Psi/dt = B(\infty)\Psi + [B(t) - B(\infty)]\Psi.$$

Thus, Corollary II in [8, p. 70] can be applied to show there are positive constants K_1, α_1 such that

$$|\Psi(t)| \leqslant K_1 e^{-\alpha_1 t}$$

for $0 \leqslant t < \infty$.

Next, it follows from the formula

$$z(t) = \Psi(t)\, z_0 + \int_0^t \Psi(t)\, \Psi^{-1}(s)\, b(s)\, ds$$

that for any z_0, $z(t)$ is bounded uniformly for $0 \leqslant t < \infty$.

By adding and subtracting the appropriate quantities in the equation for z, and again applying the variation of constants formula to the result, we obtain the formula

$$z(t) = \exp[B(\infty)t]\, z_0 + \int_0^t \exp[B(\infty)(t-s)]\, b(\infty)\, ds$$
$$+ \int_0^t \exp[B(\infty)(t-s)]\{b(s) - b(\infty) + (B(s) - B(\infty))\, z(s)\}\, ds. \tag{10}$$

The integrand in the last integral is of the form $\exp[B(\infty)(t-s)]\, h(s)$ where $h(s) \to 0$ as $s \to \infty$. With this, it is easily shown that

$$\lim_{t\to\infty} \int_0^t \exp[B(\infty)(t-s)]\, h(s)\, ds = 0.$$

Finally, we have

$$\int_0^t \exp[B(\infty)(t-s)]\, ds\, b(\infty) = -B^{-1}(\infty)\{I - \exp[B(\infty)t]\}\, b(\infty).$$

Putting all of this information back into (10) and passing to the limit $t = \infty$, we have

$$z(\infty) = -B^{-1}(\infty)\, b(\infty),$$

which is the desired result.

Proof of Theorem 2

The differences $\tilde{x} - x$, $\tilde{y} - y$ satisfy the equations

$$\begin{aligned} d(\tilde{x} - x)/dt &= f_x(t, \epsilon)(\tilde{x} - x) + f_y(t, \epsilon)(\tilde{y} - y) + F, \\ \epsilon d(\tilde{y} - y)/dt &= g_x(t, \epsilon)(\tilde{x} - x) + g_y(t, \epsilon)(\tilde{y} - y) + G, \end{aligned} \tag{11}$$

where $f_x(t, \epsilon) = f_x(t, x(t, \epsilon), y(t, \epsilon), \epsilon)$, etc., and the functions F and G depend on t, $\tilde{x} - x$, $\tilde{y} - y$, and ϵ. If we set

$$z = \begin{pmatrix} \tilde{x} - x \\ \tilde{y} - y \end{pmatrix},$$

then it follows from the form of F and G that

$$F, G = O(|z|)\,|z| \tag{12}$$

uniformly for $0 \leqslant t < \infty$, as $|z| \to 0$.

Next, we use the fact that the method of matched expansions shows $x(t, \epsilon)$, $y(t, \epsilon)$ can be written in the form

$$x(t, \epsilon) = \hat{x}_0(t) + O(\epsilon), \qquad y(t, \epsilon) = \hat{y}_0(t) + O(\epsilon),$$

where $O(\epsilon)$ holds uniformly on any set of the form $0 < s \leqslant t < \infty$. Because of this,

$$f_x(t, \epsilon) = f_x(t) + O(\epsilon) + O(|x_0(t) - \hat{x}_0(t)|), \text{ etc.,} \tag{13}$$

in (11) where $f_x(t)$, etc., are defined in SI. It was shown in [3] that if $\Phi_0(t, s, \epsilon)$ is the fundamental matrix defined by

$$\frac{d\Phi_0}{dt} = \begin{pmatrix} f_x(t) & f_y(t) \\ \dfrac{g_x(t)}{\epsilon} & \dfrac{g_y(t)}{\epsilon} \end{pmatrix} \Phi_0, \qquad \Phi_0(s, s, \epsilon) = \text{identity}, \tag{14}$$

then for sufficiently small $\epsilon > 0$, there are positive constants C', μ' (independent of ϵ) such that

$$|\Phi_0(t, s, \epsilon)| \leqslant C' e^{-\mu'(t-s)} \qquad \text{for} \quad 0 \leqslant s \leqslant t < \infty.$$

It follows from (13), (14) and the argument in [3] that if $\Phi(t, s, \epsilon)$ is the fundamental matrix defined by

$$\frac{d\Phi}{dt} = \begin{pmatrix} f_x(t, \epsilon) & f_y(t, \epsilon) \\ \dfrac{g_x(t, \epsilon)}{\epsilon} & \dfrac{g_y(t, \epsilon)}{\epsilon} \end{pmatrix} \Phi, \qquad \Phi(s, s, \epsilon) = \text{identity},$$

then for sufficiently small $\epsilon > 0$, there are positive constants C and μ such that

$$|\Phi(t, s, \epsilon)| \leqslant C e^{-\mu(t-s)} \qquad \text{for} \quad 0 \leqslant s \leqslant t < \infty. \tag{15}$$

Using Φ, we obtain from (11) that for any $0 \leqslant s \leqslant t < \infty$, z satisfies

$$z(t) = \Phi(t, s, \epsilon)\, z(s) + \int_s^t \Phi(t, s', \epsilon) \begin{pmatrix} F \\ G/\epsilon \end{pmatrix} (s')\, ds'.$$

Therefore, according to (12) and (15), there is a constant C_1 such that

$$|z(t)| \leqslant C e^{-\mu(t-s)} |z(s)| + \left(\frac{C_1 C}{\epsilon}\right) \int_s^t e^{-\mu(t-s')} |z(s')|^2\, ds'.$$

If for $s \leqslant s' \leqslant t$ we know that

$$|z(s')| \leqslant \epsilon\mu/2C_1 C, \tag{16}$$

then

$$|z(t)| \leqslant C\, |z(s)|\, e^{-\mu(t-s)} + \left(\frac{\mu}{2}\right) \int_s^t e^{-\mu(t-s')} |z(s')|\, ds'$$

and so, by Gronwall's inequality [6, p. 32],

$$|z(t)| \leqslant C\, |z(s)| \exp[-(\mu/2)(t-s)]. \tag{17}$$

Therefore, (17) guarantees that (16) is satisfied provided

$$|z(s)| \leqslant \epsilon\mu/2C_1 C^2. \tag{18}$$

We will now show that for any small $\epsilon > 0$, it is possible to choose s so that (18) is satisfied. This will complete the proof.

Now, we use Theorem 1 again to show

$$z(t, \epsilon) = \begin{pmatrix} (\tilde{x}_0 - \hat{x}_0)(t) + \epsilon(\tilde{x}_1 - \hat{x}_1)(t) \\ (\tilde{y}_0 - \hat{y}_0)(t) + \epsilon(\tilde{y}_1 - \hat{y}_1)(t) \end{pmatrix} + O(\epsilon^2). \tag{19}$$

Let us choose ϵ so small that the term $O(\epsilon^2)$ here satisfies

$$O(\epsilon^2) \leqslant \epsilon\mu/6C_1C^2.$$

Next, since $\hat{x}_0(0)$, $\tilde{x}_0(0) \in D$, we have that

$$\Delta(t) = |\tilde{x}_0(t) - \hat{x}_0(t)| + |\tilde{y}_0(t) - \hat{y}_0(t)| \to 0 \text{ as } t \to \infty.$$

Therefore, we choose s_1 so that

$$\Delta(t) \leqslant \epsilon\mu/6C_1C^2$$

for $t \geqslant s_1$. Finally, by an argument like that in the lemma, we have that

$$\Delta_1(t) = |\tilde{x}_1(t) - \hat{x}_1(t)| + |\tilde{y}_1(t) - \hat{y}_1(t)| \to 0$$

as $t \to \infty$. We therefore choose $s_2 \geqslant s_1$ so that

$$\Delta_1(t) \leqslant \epsilon\mu/6C_1C^2$$

for $t \geqslant s_2$. It follows that if $s \geqslant s_2$, (18) is satisfied and therefore

$$|z(t)| \leqslant C\,|z(s_2)|\,e^{-(\mu/2)(t-s_1)}.$$

The exponent μ here can be shown to be arbitrarily near 2α for small ϵ.

This completes the proof of Theorem 2.

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