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Discrete Mathematics 306 (2006) 2131 – 2133
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

Note

A note on maximal progression-free sets

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Received 29 November 2004; received in revised form 4 April 2006; accepted 12 April 2006
Available online 12 June 2006

Abstract

Erdős et al [Greedy algorithm, arithmetic progressions, subset sums and divisibility, Discrete Math. 200 (1999) 119–135.] asked whether there exists a maximal set of positive integers containing no three-term arithmetic progression and such that the difference of its adjacent elements approaches infinity. This note answers the question affirmatively by presenting such a set in which the difference of adjacent elements is strictly increasing. The construction generalizes to arithmetic progressions of any finite length.

Keywords: Progression-free sets

1. Introduction

Let $\mathbb{N}$ denote the set of positive integers. A subset of $\mathbb{N}$ is said to be ℓ-*free* if it contains no ℓ-term arithmetic progression ($\ell \geqslant 3$). We also say *progression-free* instead of ℓ-free if the value of ℓ is irrelevant. An ℓ-*free* set is called *maximal* if it is not properly contained in any other ℓ-*free* set. Maximal ℓ-*free* sets are clearly infinite. In the sequel, the notation $\{a_1, a_2, \ldots\}$ with $a_k \in \mathbb{N}$ assumes that $a_1 < a_2 < \cdots$. The following question was asked in [1]:

Does there exist a maximal 3-*free set* $\{a_1, a_2, \ldots, a_n, \ldots\} \subseteq \mathbb{N}$ with the property that $\lim_{n\to\infty}(a_{n+1} - a_n) = \infty$?

The authors noted that the answer was almost certainly affirmative, which is confirmed in this note. We construct a maximal 3-free subset of $\mathbb{N}$ such that the difference of its adjacent elements increases strictly, and hence approaches infinity. Moreover, the rate of growth of this difference can be made arbitrarily fast. A generalization of the construction extends the result to progressions of any finite length. The examples obtained show that, essentially speaking, maximality is unrelated to traditional notions of largeness for progression-free sets, such as asymptotic density and sum of the reciprocals of their elements.

2. The construction

Let $A \subseteq \mathbb{N}$ be an ℓ-free set ($\ell \geqslant 3$). We say that a positive integer b is a *hole* of A if b is not in A and adding b to A does not create any ℓ-term arithmetic progression. Thus, a progression-free set is maximal if it has no holes. A finite progression-free set is perhaps closest to the notion of maximality if its holes are all greater than its largest element.

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0012-365X/$ - see front matter
doi:10.1016/j.disc.2006.04.008

However (surprisingly at first glance), the constructions below involve finite progression-free sets with minimum holes less than their second largest elements. Starting with the case $\ell=3$, we call *feasible* each finite set $A=\{a_1,\ldots,a_n\}\subseteq\mathbb{N}$ such that $n\geqslant 3$ and:

- A is 3-free;
- A has a hole which is less than a_{n-1};
- $a_2-a_1<a_3-a_2<\cdots<a_{n-1}-a_{n-2}<a_n-a_{n-1}$;
- $a_{n-1}\geqslant 2a_{n-2}+3$ and $a_n\geqslant 2a_{n-1}+3$.

An example is the set $\{a, 2a+3, 4a+9\}$, where $a\in\mathbb{N}$ is arbitrary.

Let A be a feasible set with minimum hole b (which is less than a_{n-1}). Define

$$a_{n+1}=2a_n-b,\quad a_{n+2}=2a_{n+1}+3,\ \text{ and }\ a_{n+3}=2a_{n+2}+3.$$

We claim that the set $A'=\{a_1,\ldots,a_n,a_{n+1},a_{n+2},a_{n+3}\}\subseteq\mathbb{N}$ is also feasible and its minimum hole is greater than b.

Clearly, A' satisfies the last two conditions from the definition of a feasible set; note that $a_n-a_{n-1}<a_{n+1}-a_n$ is equivalent to $b<a_{n-1}$. Suppose that A' contains a 3-term progression P; clearly one of the newly added elements a_{n+1},a_{n+2},a_{n+3} belongs to P. If P has two terms among $a_1,\ldots,a_{n-1}$, its difference d is less than a_{n-1}. Hence, the third term a_k satisfies $a_k\leqslant a_{n-1}+d<2a_{n-1}<a_n$, which is impossible. So at most one term of P is in $\{a_1,\ldots,a_{n-1}\}$. On the other hand, a_{n+2} and a_{n+3} are evidently too large to be in P, implying that $P=\{a_i,a_n,a_{n+1}\}$ for some $i<n$. In view of $a_{n+1}=2a_n-b$ we infer that $a_i=b$. But $b\notin A$, which proves that A' is 3-free .

Let us now show that $c=a_{n+2}-2=2a_{n+1}+1$ is a hole of A'. It is clear that $c\notin A'$. Suppose that adding c to A' creates a 3-term progression P; then P must contain c. Like above, a_{n+3} is too large to be in P. Similarly, c is too large to form a progression with some two numbers among $a_1,\ldots,a_n,a_{n+1}$. And if a_{n+2} is in P, then P has difference 2, which is easily rejected. Therefore, c is a hole of A' which is less than its second largest element a_{n+2}.

Finally, the minimum hole of A' is strictly greater than b, because the inclusion of a_{n+1} eliminates b as a hole: $\{b,a_n,a_{n+1}\}$ is a 3-term progression.

Thus, by adding three new elements, every feasible set can be expanded to a feasible set whose minimum hole is greater than the one of the original set. With this in mind, we carry out an inductive construction starting from an arbitrary feasible set of size 3. At each step the current feasible set is updated as explained above. The procedure gives rise to an infinite set A. By the properties of feasible sets, A is 3-free and the difference of its adjacent elements is strictly increasing. Moreover, A is maximal because a hole at any finite stage is destined to be eliminated by the definition of some later term. In conclusion:

> There exists a maximal 3-free set $\{a_1,a_2,\ldots,a_n,\ldots\}\subseteq\mathbb{N}$ with the property that $a_n-a_{n-1}<a_{n+1}-a_n$ for all $n>1$. In particular, $\lim_{n\to\infty}(a_{n+1}-a_n)=\infty$.

The construction readily generalizes to all $\ell>3$. Let $A=\{a_1,a_2,\ldots,a_n\}\subseteq\mathbb{N}, n\geqslant\ell$, be an ℓ-free set with the properties: A has a hole less than a_{n-1}; the difference of adjacent elements is nondecreasing and a_n-a_{n-1} is greater than the previous differences; $a_{n-1}\geqslant 2a_{n-2}+3$ and $a_n\geqslant 2a_{n-1}+3$. An example of such a set is, for instance, $\{a_1,a_2,\ldots,a_\ell\}$ with any $a_1\in\mathbb{N}$ and $a_i=2a_{i-1}+3$ for $2\leqslant i\leqslant\ell$. Let b be the minimum hole of A. Define $\ell-2$ new elements $a_{n+1},\ldots,a_{n+\ell-2}$ so that $\{b,a_n,a_{n+1},\ldots,a_{n+\ell-2}\}$ is an ℓ-term progression. This eliminates the current minimum hole b. Then define two more elements by $a_{n+\ell-1}=2a_{n+\ell-2}+3$ and $a_{n+\ell}=2a_{n+\ell-1}+3$. An argument similar to the above shows that the new set is ℓ-free and has a hole which is less than $a_{n+\ell-1}$. The difference of adjacent elements is nondecreasing, with $a_{n+\ell}-a_{n+\ell-1}$ strictly greater than the previous differences. Finally, by definition each of the last two elements is at least twice the previous one plus 3. So one may apply an inductive construction again, obtaining a maximal ℓ-free set $\{a_1,a_2,\ldots,a_n,\ldots\}$ such that the difference $a_{n+1}-a_n$ approaches infinity. More exactly:

> For each $\ell>3$ there exists a maximal ℓ-free set $\{a_1,a_2,\ldots,a_n,\ldots\}\subseteq\mathbb{N}$ such that $a_n-a_{n-1}\leqslant a_{n+1}-a_n$ for all $n>1$, with strict inequality for infinitely many n. In particular, $\lim_{n\to\infty}(a_{n+1}-a_n)=\infty$.

3. Comments

When constructing maximal progression-free sets inductively, at each finite stage one may try avoiding holes between elements already defined. In one way or another, such constructions would be based on the greedy algorithm. But the examples known seem to indicate that in greedily generated progression-free sets, the difference of adjacent elements is unlikely to approach infinity (on the contrary, perhaps this difference drops down to 1 infinitely often). From this viewpoint, the idea to allow appropriately chosen "small" holes is essential for our simple construction. In the case $\ell=3$, the choice of a_{n+1} is indispensable to this idea, while there is a great deal of flexibility in how a_{n+2} and a_{n+3} are defined. In particular, they can be chosen so that the difference of adjacent elements grows arbitrarily fast, hence producing maximal 3-free sets as thin as desired. Furthermore (in the case $\ell = 3$ again), the set $\{a_1, a_2, \ldots, a_n, \ldots\}$ obtained in the process of construction satisfies $a_i > 3a_{i-2}$ for all $i \geqslant 3$, regardless of the initial feasible set $\{a_1, a_2, a_3\}$. It follows that $a_{2n+1} > 3^n a_1$ and $a_{2n+2} > 3^n a_2$ for all $n \geqslant 1$. Therefore, $\sum_{n=1}^{\infty} 1/a_n < \frac{3}{2}\,(1/a_1 + 1/a_2) < 3/a_1$. By choosing a_1 large enough, one can obtain maximal 3-free sets with the sum of the reciprocals of their elements arbitrarily small. Analogous conclusions hold for maximal ℓ-free sets as well. Such observations confirm the intuitively obvious fact that maximality of progression-free sets implies neither high asymptotic density nor large sum of the reciprocals of their elements.

References

[1] P. Erdős, V. Lev, G. Rauzy, C. Sándor, A. Sárközy, Greedy algorithm, arithmetic progressions, subset sums and divisibility, Discrete Math. 200 (1999) 119–135.