Existence of positive solutions for 2n-th-order singular sublinear boundary value problems

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Abstract

This paper investigates the existence of positive solutions for 2n-th-order (n > 1) singular sublinear boundary value problems. A necessary and sufficient condition for the existence of C^{2n-2}[0, 1] as well as C^{2n-1}[0, 1] positive solutions is given by constructing lower and upper solutions and with the maximal theorem. Our nonlinearity f(t, x_1, x_2, ..., x_n) may be singular at x_i = 0, i = 1, 2, ..., n, t = 0 and/or t = 1.

Keywords: Singular boundary value problem; Positive solution; Lower and upper solution; Maximum principle; 2n-th-order

1. Introduction

The singular ordinary differential equations arise in the fields of gas dynamics, Newtonian fluid mechanics, the theory of boundary layer and so on, the theory of singular boundary value problems has become an important area of investigation in recent years (see [1–5] and references therein). Consider the singular boundary value problems of 2n-th-order ordinary differential equation
\(( -1)^{n} x^{(2n)} (t) = f(t, x(t), -x''(t), \ldots, (-1)^{i} x^{(2i)}(t), \ldots, (-1)^{n-1} x^{(2n-2)}(t)),\)

\(t \in (0, 1),\)

\(x^{(2i)}(0) = x^{(2i)}(1) = 0, \quad i = 0, 1, 2, \ldots, n - 1,\)

where \(n > 1\) is an integer, and \(f\) satisfies the following hypothesis:

\((H)\) \(f \in C((0, 1) \times (0, \infty)^{n}, [0, \infty]),\) and there exist constants \(\lambda_{i}, \mu_{i} (-\infty < \lambda_{i} \leq 0 \leq \mu_{i},\)

\(i = 1, 2, \ldots, n, \mu_{n} < 1, \sum_{i=1}^{n} \mu_{i} < 1\) such that for \(t \in (0, 1), x_{i} \in (0, \infty),\)

\[c^{\lambda_{i}} f(t, x_{1}, x_{2}, \ldots, x_{i}, \ldots, x_{n}) \leq f(t, x_{1}, x_{2}, \ldots, c x_{i}, \ldots, x_{n}) \leq c^{\mu_{i}} f(t, x_{1}, x_{2}, \ldots, x_{i}, \ldots, x_{n}), \quad \text{if} 0 < c \leq 1, \ i = 1, 2, \ldots, n.\]

**Remark 1.** (1.3) implies

\[c^{\lambda_{i}} f(t, x_{1}, x_{2}, \ldots, x_{i}, \ldots, x_{n}) \leq f(t, x_{1}, x_{2}, \ldots, c x_{i}, \ldots, x_{n}) \leq c^{\mu_{i}} f(t, x_{1}, x_{2}, \ldots, x_{i}, \ldots, x_{n}), \quad \text{if} c \geq 1, \ i = 1, 2, \ldots, n.\]

Typical functions that satisfy the above sub-linear hypothesis are those taking the form \(f(t, x_{1}, x_{2}, \ldots, x_{n}) = \sum_{i=1}^{m} p_{i}(t) x_{1}^{\ell_{1i}} x_{2}^{\ell_{2i}} \ldots x_{n}^{\ell_{ni}};\) here \(p_{j}(t) \in C(0, 1), p_{j}(t) > 0\) on \((0, 1), \ell_{jk} \in \mathbb{R}, \ell_{jn} < 1, j = 1, 2, \ldots, m, k = 1, 2, \ldots, n, \sum_{i=1}^{n} \sum_{k=1}^{m} \ell_{ik} < 1.\)

By singularity we mean that the function \(f(t, x_{1}, x_{2}, \ldots, x_{i}, \ldots, x_{n})\) in (1.1) is allowed to be unbounded at \(x_{i}, i = 1, 2, \ldots, n, t = 0\) and/or \(t = 1.\) A function \(x(t) \in C^{2n-2}[0, 1] \cap C^{2n}(0, 1)\) is called a \(C^{2n-2}[0, 1]\) (positive) solution of (1.1) and (1.2) if it satisfies (1.1) and (1.2) \((-1)^{i} x(t) > 0, \ i = 0, 1, 2, \ldots, n - 1\) for \(t \in (0, 1).\) A \(C^{2n-2}[0, 1]\) (positive) solution of (1.1) and (1.2) is called a \(C^{2n-1}[0, 1]\) (positive) solution if \(x^{(2n-1)}(0^{+})\) and \(x^{(2n-1)}(1^{-})\) both exist \((-1)^{i} x(t) > 0, \ i = 0, 1, 2, \ldots, n - 1\) for \(t \in (0, 1).\)

For the special case \(n = 2,\) the function \(f(n, 1) \times R \times R, R)\) in (1.1), \(i.e. f\) is continuous, problem (1.1) and (1.2) is nonsingular, the existence and uniqueness of solutions of (1.1) and (1.2) have been studied by papers [6–20]. A sufficient condition for the existence of solutions of the singular problem (1.1) and (1.2) was given by D. O’Regan in [21] with a topological transversal theorem. The existence of one or more positive solutions of singular boundary value problems (1.1) and (1.2) and has been studied by several authors [22–25]. However, all of these authors consider an equation of the form

\[x^{(4)}(t) = f(t, x(t))\]

with a diverse kind of boundary conditions, \(f(t, x)\) in (1.5) cannot be singular at \(x = 0.\)

For the general case \(n \geq 2,\) the function \(f \in C([0, 1] \times R^{n}, R)\) in (1.1), \(i.e. f\) is continuous, problem (1.1) and (1.2) is nonsingular, the existence of one or more solutions of (1.1) and (1.2) have been studied by papers [26–28]. The function \(f \in C([0, 1] \times R^{+}, R^{+})\) in (1.1), \(i.e. f\) is singular at \(t = 0\) or \(t = 1,\) but \(f\) is continuous at \(x_{i}, \ i = 1, 2, \ldots, n,\) paper [29] has investigated super-linear singular boundary value problem (1.1) and (1.2) and obtained some necessary and sufficient conditions for the existence of \(C^{2n-2}[0, 1]\) as well as \(C^{2n-1}[0, 1]\) positive solutions by means of the fixed point theorems on cones.
In this paper, we shall study the existence of positive solutions for 2nth-order singular sub-linear boundary value problems (1.1) and (1.2). A necessary and sufficient condition for the existence of $C^{2n-2}[0, 1]$ as well as $C^{2n-1}[0, 1]$ positive solutions is given by constructing lower and upper solutions and with the maximal theorem. Our nonlinearity $f(t, x_1, x_2, \ldots, x_n)$ may be singular at $x_i = 0, i = 1, 2, \ldots, n, t = 0$ and/or $t = 1$.

2. Several lemmas

To prove the main result, we need the following maximum principle. Suppose that $0 \leq a < b$, and

$$F = \{ x \in C^{2n-2}[a, b] \cap C^{2n}(a, b), (-1)^j x^{(2j)}(a) \geq 0, (-1)^j x^{(2j)}(b) \geq 0, i = 0, 1, 2, \ldots, n - 1 \}.$$

**Lemma 2.1** (Maximum principle). If $x \in F$ such that $(-1)^i x^{(2i)}(t) \geq 0, t \in (a, b)$, then

$$(-1)^i x^{(2i)}(t) \geq 0, t \in [a, b], i = 0, 1, 2, \ldots, n - 1.$$

**Proof.** Set

$$(-1)^i x^{(2i)}(a) = x_a^{(i)}, (-1)^i x^{(2i)}(b) = x_b^{(i)}, i = 0, 1, 2, \ldots, n - 1,$$

then

$$x_a^{(i)} \geq 0, x_b^{(i)} \geq 0, i = 0, 1, 2, \ldots, n - 1,$$

$$(-1)^i x^{(2i)}(t) = \left(\frac{t-a}{b-a} x_a^{(i)} + \frac{b-t}{b-a} x_b^{(i)} + \int_a^b G(t, s) \left[(-1)^i x^{(2i+2)}(s)\right] ds\right),$$

$$t \in [a, b], i = 0, 1, 2, \ldots, n - 1,$$

where

$$G(t, s) = \begin{cases} \frac{(b-t)(s-a)}{b-a}, & s < t, \\ \frac{(b-s)(t-a)}{b-a}, & t \leq s. \end{cases}$$

By means of condition $(-1)^i x^{(2i)}(t) \geq 0, t \in (a, b)$, (2.2), (2.3) and induction, we can obtain

$$(-1)^i x^{(2i)}(t) \geq 0, t \in [a, b], i = 0, 1, 2, \ldots, n - 1.$$

The proof is complete. $\square$

**Definition 2.1.** Suppose $\alpha \in C^{2n-2}[0, 1] \cap C^{2n}(0, 1)$, if $\alpha$ satisfies

$$(-1)^n \alpha^{(2n)}(t) \leq f(t, \alpha(t), -\alpha''(t), \ldots, (-1)^j \alpha^{(2j)}(t), \ldots, (-1)^{n-1} \alpha^{(2n-2)}(t)),$$

$$t \in (0, 1),$$

$$(-1)^j \alpha^{(2j)}(0) \leq 0, (-1)^j \alpha^{(2j)}(1) \leq 0, i = 0, 1, 2, \ldots, n - 1.$$

Then $\alpha$ is called a lower solution of the singular boundary value problem (1.1) and (1.2).
Definition 2.2. Suppose $\beta \in C^{2n-2}[0, 1] \cap C^2(0, 1)$, if $\beta$ satisfies
\[
(-1)^n \beta^{(2n)}(t) \geq f(t, \beta(t), -\beta'', \ldots, (-1)^1 \beta^{(2)}(t), \ldots, (-1)^{n-1} \beta^{(2n-2)}(t)),
\]
$t \in (0, 1)$,
\[
(-1)^1 \beta^{(2)}(0) \geq 0, \quad (-1)^1 \beta^{(2)}(1) \geq 0, \quad i = 0, 1, 2, \ldots, n - 1.
\]
Then $\beta$ is called a upper solution of the singular boundary value problem (1.1) and (1.2).

Lemma 2.2. Assume that there exist lower and upper solutions of (1.1) and (1.2), respectively $\alpha(t)$ and $\beta(t)$, such that $\alpha(t), \beta(t) \in C^{2n-2}[0, 1] \cap C^2(0, 1)$, $0 < (-1)^1 \alpha^{(2)}(t) \leq (-1)^1 \beta^{(2)}(t)$, for $t \in (0, 1)$, $(-1)^j \alpha^{(2)}(j) = (-1)^j \beta^{(2)}(j) = 0$, $j = 0, 1, 2, \ldots, n - 1$. Then problem (1.1) and (1.2) has at least one $C^{2n-2}[0, 1]$ positive solution $x(t)$ such that $(-1)^j \alpha^{(2)}(t) \leq (-1)^j x^{(2)}(t) \leq (-1)^j \beta^{(2)}(t)$, for $t \in [0, 1], i = 0, 1, 2, \ldots, n - 1$. If, in addition, there exists $F(t) \in L^1[0, 1]$ such that
\[
|f(t, x(t), -x''(t), \ldots, (-1)^1 x^{(2)}(t), \ldots, (-1)^{n-1} x^{(2n-2)}(t))| \leq F(t),
\]
for $(-1)^1 \alpha^{(2)}(t) \leq (-1)^1 x^{(2)}(t) \leq (-1)^1 \beta^{(2)}(t)$,
t $\in (0, 1)$, $i = 0, 1, 2, \ldots, n - 1$, (2.5)
then the solution $x(t)$ of (1.1) and (1.2) is a $C^{2n-1}[0, 1]$ positive solution.

Proof. First of all, we define a partial ordering in $C^{2n-2}[0, 1] \cap C^2(0, 1)$ by $x \leq y$ if and only if
\[
(-1)^1 x^{(2)}(t) \leq (-1)^1 y^{(2)}(t), \quad t \in [0, 1], \quad i = 0, 1, 2, \ldots, n - 1.
\] (2.6)
Then, we shall define an auxiliary function. $\forall (t, x) \in C^{2n-2}[0, 1] \cap C^2(0, 1)$,
\[
g(t, x) = \begin{cases} 
    f(t, x(t), -x''(t), \ldots, (-1)^1 x^{(2)}(t), \ldots, (-1)^{n-1} x^{(2n-2)}(t)), \\
    \text{if } x \not\leq x, \\
    f(t, x(t), -x''(t), \ldots, (-1)^1 x^{(2)}(t), \ldots, (-1)^{n-1} x^{(2n-2)}(t)), \\
    \text{if } x \leq x \leq \beta, \\
    f(t, \beta(t), -\beta''(t), \ldots, (-1)^1 \beta^{(2)}(t), \ldots, (-1)^{n-1} \beta^{(2n-2)}(t)), \\
    \text{if } x \not\geq \beta.
\end{cases}
\] (2.7)
By the condition (H), we have $g : (0, 1) \times R \rightarrow [0, +\infty)$ is continuous.

Let $[a_m], [b_m]$ be sequences satisfying $0 < \cdots < a_{m+1} < a_m < \cdots < a_1 < 1/2 < b_1 < \cdots < b_m < b_{m+1} < \cdots < 1$, $a_m \rightarrow 0$ and $b_m \rightarrow 1$ as $m \rightarrow \infty$, and let $\{r_{i0}^{(m)}\}, \{r_{i1}^{(m)}\}$ be sequences satisfying
\[
(-1)^1 \alpha^{(2)}(a_m) \leq r_{i0}^{(m)} \leq (-1)^1 \beta^{(2)}(a_m),
\]
\[
(-1)^1 \alpha^{(2)}(b_m) \leq r_{i1}^{(m)} \leq (-1)^1 \beta^{(2)}(b_m),
\]
$m = 1, 2, \ldots, i = 0, 1, 2, \ldots, n - 1$. (2.8)
For each $m$, consider the nonsingular problem
\[
\begin{cases}
    (-1)^n x^{(2n)}(t) = g(t, x), & t \in [a_m, b_m], \\
    (-1)^1 x^{(2)}(a_m) = r_{i0}^{(m)}, \quad (-1)^1 x^{(2)}(b_m) = r_{i1}^{(m)}, & i = 0, 1, 2, \ldots, n - 1.
\end{cases}
\] (2.9)
Obviously, the problem (2.9) is equivalent to the integral equation

\[ x(t) = A_m x(t) = R_0(t) + \int_{a_m}^{b_m} G_m(t, \xi_1) R_1(\xi_1) d\xi_1 \]

\[ + \int_{a_m}^{b_m} G_m(t, \xi_1) d\xi_1 \int_{a_m}^{b_m} G_m(\xi_1, \xi_2) R_2(\xi_2) d\xi_2 + \cdots \]

\[ + \int_{a_m}^{b_m} G_m(t, \xi_1) d\xi_1 \int_{a_m}^{b_m} \cdots \int_{a_m}^{b_m} G_m(\xi_{n-2}, \xi_{n-1}) R_{n-1}(\xi_{n-1}) d\xi_{n-1} \]

\[ + \int_{a_m}^{b_m} G_m(t, \xi_1) d\xi_1 \int_{a_m}^{b_m} \cdots \int_{a_m}^{b_m} G_m(\xi_{n-1}, \xi_n) g(\xi_n, x(\xi_n)) d\xi_n, \]

\( t \in [a_m, b_m], \) (2.10)

where

\[ G_m(t, s) = \begin{cases} (b_m - t)(s - a_m), & s < t, \\ (b_m - s)(t - a_m), & t \leq s, \end{cases} \]

\[ R_i(t) = \frac{t - a_m}{b_m - a_m} r_{i1} + \frac{b_m - t}{b_m - a_m} r_{i0}, \quad i = 0, 1, 2, \ldots, n - 1. \] (2.11)

It is easy to verify that \( A_m : X_m \to X_m = C^{2n-2}[a_m, b_m] \) is completely continuous and \( A_m(X_m) \) is a bounded set. Moreover, \( x \in C^{2n-2}[a_m, b_m] \) is a solution of (2.9) if and only if \( A_m x = x \). Using the Schauder’s fixed point theorem, we assert that \( A_m \) has at least one fixed point \( x_m \in C^{2n}[a_m, b_m] \).

We claim that

\[ \alpha \leq x_m \leq \beta, \]

that is

\[ (-1)^i \alpha^{(2i)}(t) \leq (-1)^j x_m^{(2i)}(t) \leq (-1)^j \beta^{(2i)}(t), \]

\( t \in [a_m, b_m], \quad i = 0, 1, 2, \ldots, n - 1, \) (2.13)

and hence \( x_m(t) \in C^{2n}[a_m, b_m] \) and satisfies

\[ (-1)^n x^{(2n)}(t) = f(t, x(t), -x''(t), \ldots, (-1)^i x^{(2i)}(t), \ldots, (-1)^{n-1} x^{(2n-2)}(t)), \]

\( t \in [a_m, b_m]. \) (2.14)

Indeed, suppose by contradiction that \( x_m \not\leq \beta \). By the definition of \( g \),

\[ g(t, x_m(t)) = f(t, \beta(t), -\beta''(t), \ldots, (-1)^i \beta^{(2i)}(t), \ldots, (-1)^{n-1} \beta^{(2n-2)}(t)), \]

\( t \in [a_m, b_m]. \)

and therefore
On the other hand, since $\beta$ is an upper solution of (1.1) and (1.2), we also have

\[(−1)^n \beta^{(2n)}(t) \geq f(t, \beta(t), −\beta''(t), \ldots, (−1)^i \beta^{(2i)}(t), \ldots, (−1)^{n−1} \beta^{(2n−2)}(t)),\]

\[t \in [a_m, b_m].\]  

(2.15)

Then set

\[z(t) = \beta(t) − x_m(t), \quad t \in [a_m, b_m].\]

By (2.8), (2.9), (2.15) and (2.16), we obtain \((−1)^n z^{(2n)}(t) \geq 0, t \in (a_m, b_m),\)

\[z \in C^{2n−2}[a_m, b_m] \cap C^2(a_m, b_m),\]

\[(−1)^i z^{(2i)}(a_m) \geq 0, \quad (−1)^i z^{(2i)}(b_m) \geq 0, \quad i = 0, 1, 2, \ldots, n − 1.\]

By Lemma 2.1, we can conclude that

\[(−1)^i x^{(2i)}(t) \geq 0, \quad t \in [a_m, b_m], \quad i = 0, 1, 2, \ldots, n − 1,\]

a contradiction with the assumption $x_m \not\equiv \beta$. Therefore $x_m \not\equiv \beta$ is impossible.

Similarly, we can show that $\alpha \leq x_m$. So, we have shown that (2.13) holds.

Using the method of [3] and the Theorem 3.2 in [30], we can obtain that there is a $C^{2n−2}[0, 1]$ positive solution $x(t)$ of (1.1) such that $\alpha \leq x \leq \beta$ and a subsequence of \[\{(−1)^i x_0^{(2i)}(t)\}, \quad i = 0, 1, 2, \ldots, n − 1,\] converges to $y(t) = (−1)^i x^{(2i)}(t)$, $i = 0, 1, 2, \ldots, n − 1$, on any compact subintervals of (0, 1).

In addition, if (2.5) holds, then $|x^{(2n)}(t)| \leq F(t)$, and hence $x^{(2n)}(t)$ is absolutely integrable on [0, 1]. This implies $x(t) \in C^{2n−1}[0, 1]$, so $x(t)$ is a $C^{2n−1}[0, 1]$ positive solution of the problem (1.1) and (1.2). The proof is complete. \(\square\)

**Lemma 2.3.** Suppose that (H) holds. Let $x(t)$ be a $C^{2n−1}[0, 1]$ positive solution of (1.1) and (1.2). Then there are constants $l_1^{(i)}$ and $l_2^{(i)}$, $0 < l_1^{(i)} < l_2^{(i)}$, such that

\[l_1^{(i)} t(1 − t) \leq (−1)^i x^{(2i)}(t) \leq l_2^{(i)} t(1 − t),\]

\[t \in [0, 1], \quad i = 0, 1, 2, \ldots, n − 1.\]  

(2.17)

**Proof.** Assume that $x(t)$ is a $C^{2n−1}[0, 1]$ positive solution of (1.1) and (1.2). Then \((−1)^{n−1} x^{(2n−1)}(0) > 0\) and \((−1)^{n−1} x^{(2n−1)}(1) < 0\), \((−1)^{n−1} x^{(2n−2)}(t) > 0\) for $t \in (0, 1)$.

By integration of (1.1), we have

\[
\int_0^1 f(t, x(t), −x''(t), \ldots, (−1)^i x^{(2i)}(t), \ldots, (−1)^{n−1} x^{(2n−2)}(t)) \, dt
\]

\[= (−1)^{n−1} x^{(2n−1)}(0) − (−1)^{n−1} x^{(2n−1)}(1) < \infty.\]  

(2.18)

Let $d_1$ be a constant sufficiently small satisfying \((−1)^{n−1} x^{(2n−2)}(1/2) − d_1/2 \geq 0\), and let $y(t) = (−1)^{n−1} x^{(2n−2)}(t) − d_1 t$, $t \in [0, 1/2]$, and denote
Then
\[
\begin{cases}
-y''(t) = F(t, x(t)), & \quad t \in (0, 1), \\
y(0) = 0, & \quad y(1/2) = (-1)^{n-1}x^{(2n-2)}(1/2) - d_1 1/2 \geq 0.
\end{cases}
\]

By the maximum principle, we have \(y(t) \geq 0\), for \(t \in [0, 1/2]\). Therefore,
\[
(-1)^{n-1}x^{(2n-2)}(t) \geq d_1 t, \quad t \in [0, 1/2].
\]

On the other hand, let \(d_2\) be a constant sufficiently large such that \(d_2 1/2 - ((-1)^{n-1} \times x^{(2n-2)}(1/2)) = r_0\), \(r_0 \geq \int_0^{1/2} F(s, x(s)) \, ds\), \(r_0 \geq 2 \int_0^{1/2} (1/2 - s) F(s, x(s)) \, ds\). And let
\[
y(t) = d_2 t - ((-1)^{n-1}x^{(2n-2)}(t)).
\]

Then
\[
\begin{cases}
-y''(t) = -F(t, x(t)), & \quad t \in (0, 1/2), \\
y(0) = 0, & \quad y(1/2) = d_2 1/2 - ((-1)^{n-1}x^{(2n-2)}(1/2)) = r_0 > 0.
\end{cases}
\]

(2.20) has a unique solution \(y(t)\) satisfying
\[
y(t) = \frac{t}{1/2} r_0 - \frac{1}{1/2} \int_0^{t} (1/2 - s) F(s, x(s)) \, ds
\]
\[
- \frac{1}{1/2} \int_0^{t} (1/2 - s) F(s, x(s)) \, ds
\]
\[
\geq 2t \left[ \frac{r_0}{2} - \int_0^{1/2} F(s, x(s)) \, ds \right]
\]
\[
+ 2t \left[ \frac{r_0}{2} - \int_0^{1/2} (1/2) F(s, x(s)) \, ds \right] \geq 0, \quad t \in [0, 1/2].
\]

Hence,
\[
(-1)^{n-1}x^{(2n-2)}(t) \leq d_2 t, \quad t \in [0, 1/2].
\]

Similarly, we can verify that there exist two numbers \(d_3\) and \(d_4\) satisfying
\[
d_3 (1 - t) \leq ((-1)^{n-1}x^{(2n-2)}(t)) \leq d_4 (1 - t), \quad t \in [1/2, 1].
\]

Let
\[
I_1^{(n-1)} = \min\{d_1, d_3\}, \quad I_2^{(n-1)} = \max\{2d_2, 2d_4\}.
\]

Then, from (2.19), (2.21) and (2.22) imply that
\[
I_1^{(n-1)} t (1 - t) \leq (-1)^{n-1}x^{(2n-2)}(t) \leq I_2^{(n-1)} t (1 - t), \quad t \in [0, 1].
\]
By direct computation, we have
\[
\int_0^1 H(t,s)s(1-s)ds = \frac{1}{12} t(1-t)(1+t(1-t)),
\] (2.24)
and
\[
(-1)^i x^{(2i)}(t) = \int_0^1 H(t,s)[(-1)^{i+1} x^{(2i+2)}(s)]ds,
\] (2.25)
where
\[
H(t,s) = \begin{cases} 
(1-t)s, & s < t, \\
(1-s)t, & t \leq s,
\end{cases}
\] (2.26)
(23)–(25) imply that
\[
\frac{1}{12} I_1^{(i+1)} t(1-t) \leq (-1)^i x^{(2i)}(t) \leq \frac{1}{6} I_2^{(i+1)} t(1-t),
\] (2.27)
\[
0 < I_1^{(i)} < I_2^{(i)}, \ i = 0, 1, 2, \ldots, n-2.
\] (2.28)
(2.27) and (2.28) imply that (2.17) holds. The proof of Lemma 2.3 is complete. \(\square\)

3. The main result

For convenience, we set
\[
F_1(t) = f(t, t(1-t), t(1-t), \ldots, t(1-t), 1),
\]
\[
F_2(t) = f(t, t(1-t), t(1-t), \ldots, t(1-t), t(1-t)).
\]
The main results of this paper are the following two theorems.

**Theorem 3.1.** Suppose (H) holds, then a necessary and sufficient condition for problem (1.1) and (1.2) to have \(C^{2n-2}\) positive solutions is that the following integral conditions hold:
\[
0 < \int_0^1 t(1-t) F_1(t) dt < \infty,
\] (3.1)
\[
\lim_{t \to 0^+} t \int_0^1 (1-s)F_1(s) \, ds = 0,
\]
(3.2)

\[
\lim_{t \to 1^-} \int_0^t s F_1(s) \, ds = 0.
\]
(3.3)

**Theorem 3.2.** Suppose \((H)\) holds, then a necessary and sufficient condition for problem (1.1) and (1.2) to have \(C^{2n-1}[0, 1]\) positive solutions is that the following integral conditions hold:

\[
0 < \int_0^1 F_2(t) \, dt < \infty.
\]
(3.4)

**Proof of Theorem 3.1.** Necessity. Let \(x(t) \in C^{2n-2}[0, 1] \cap C^2(0, 1)\) be a positive solution of (1.1) and (1.2). Then \((-1)^j x^{(2j)}(t) > 0, t \in (0, 1), j = 1, 2, \ldots, n-1\). By the proof of Lemma 2.3, there are constants \(I_1^{(i)}\) and \(I_2^{(i)}\) such that

\[
0 < I_1^{(i)} < I_2^{(i)}, \quad i = 0, 1, 2, \ldots, n-2,
\]
(3.5)

\[
I_1^{(i)} t(1-t) \leq (-1)^j x^{(2j)}(t) \leq I_2^{(i)} t(1-t),
\]
(3.6)

Let \(c_0\) be a constant such that \(c_0 I_2^{(n-2)} \leq 1, 1/c_0 \geq 1, c_0 (-1)^{(n-1)} x^{(2n-2)}(t) \leq 1\). Then from (1.3)–(1.6) and (3.6), we have

\[
f(t, x(t), -x''(t), \ldots, (-1)^{n-1} x^{(2n-2)}(t))
\]

\[
\geq (1/c_0)^{\lambda_1} f\left(t, \frac{c_0 x(t)}{t(1-t)} t(1-t), -x''(t), \ldots, (-1)^{n-1} x^{(2n-2)}(t)\right)
\]

\[
\geq c_0^{-\lambda_1} \left(\frac{x(t)}{t(1-t)}\right)^{\mu_1} f(t, t(1-t), -x''(t), \ldots, (-1)^{n-1} x^{(2n-2)}(t))
\]

\[
\geq c_0^{-\lambda_1} \left[\frac{f(t, t(1-t), -x''(t), \ldots, (-1)^{n-1} x^{(2n-2)}(t))}{\mu_1}\right]^{\mu_1} \times f(t, t(1-t), \ldots, t(1-t)), \quad t \in (0, 1).
\]
(3.7)

According to (1.1), we have

\[
F_1(t) \leq a_0 \left[(-1)^{n-1} x^{(2n-2)}(t)\right]^{-\mu_1} \left[(-1)^{n} x^{(2n)}(t)\right], \quad t \in (0, 1),
\]
(3.8)
where
\[
a_0 = \left[ \left( \prod_{i=1}^{n} c_0^{(\mu_i - \lambda_i)} \right) \left( \prod_{i=1}^{n-1} I(i-1)[i] \right) \right]^{-1}.
\]

From (1.2), there exist \( t_0 \in (0, 1) \) such that
\[
\begin{align*}
%0 \leq n-1 \sum_{i=1}^{n} c_0^{(\mu_i - \lambda_i)} (i-1)[i] &< 0, \quad t \in (t_0, 1), \\
%0 \leq n-1 \sum_{i=1}^{n} c_0^{(\mu_i - \lambda_i)} (i-1)[i] &> 0, \quad t \in (0, t_0).
\end{align*}
\]

And let \( u(t) = (-1)^{n-1} x^{(2n-2)}(t) \). For \( t \in (0, t_0) \), by integration of (3.8), we obtain
\[
\begin{align*}
\int_0^n F_1(s) s ds &\leq -a_0 u'(s) u^{-\mu_n}(s) \frac{\int_0^n a_0 u^{-\mu_n-1}(s) u'(s)^2 ds}{1 - \mu_n} \\
&\leq a_0 u^{-\mu_n}(t) u'(t), \quad t \in (0, t_0).
\end{align*}
\]

Integrating (3.10), we have
\[
\begin{align*}
\int_0^n \int_0^s F_1(s) ds dt &\leq a_0 \frac{u^{-\mu_n}(t_0)}{1 - \mu_n} < \infty,
\end{align*}
\]
so
\[
0 < \int_0^{t_0} s F_1(s) ds < \infty.
\]

Similarly, by integration of (3.9), we obtain
\[
0 < \int_0^{1} (1 - s) F_1(s) ds < \infty.
\]

(3.11) and (3.12) imply that (3.1) holds.

For \( t \in (0, t_0) \), by integration of (3.10), we have
\[
\begin{align*}
\int_0^n \int_0^s F_1(\tau) d\tau ds &\leq a_0 \frac{(-1)^{n-1} x^{(2n-2)}(t)}{1 - \mu_n}.
\end{align*}
\]

therefore,
\[
\begin{align*}
t &\int_0^t F_1(\tau) d\tau \leq a_0 \frac{(-1)^{n-1} x^{(2n-2)}(t)}{1 - \mu_n},
\end{align*}
\]

Letting \( t \to 0 \) in (3.13) and noting \( x^{(2n-2)}(0) = 0 \), we have \( \lim_{t \to 0^+} t \int_0^t F_1(s) ds = 0 \).

These imply that (3.2) holds. Similarly, we can prove (3.3).
Sufficiency. Suppose that (3.1)–(3.3) hold. Let
\[
h(t) = \int_0^1 H(t, \xi_1) d\xi_1 \int_0^1 H(\xi_1, \xi_2) d\xi_2 \ldots \int_0^1 H(\xi_{n-1}, \xi_n) F_1(\xi_n) d\xi_n,
\]
where \(H(t, s)\) is given by (2.26). Then \(h(t) \in C^{2n-2}[0, 1] \cap C^2(0, 1)\), and there exist constants \(0 < \ell^{(i)}_1 < \ell^{(i)}_2\) such that
\[
\ell^{(i)}_1 t(1 - t) \leq (-1)^i h^{(2i)}(t) \leq \ell^{(i)}_2 t(1 - t), \quad i = 0, 1, 2, \ldots, n - 2,
\]
(3.15)
holds, where
\[
\ell^{(n-2)}_1 = \min \left\{ 1, \frac{1}{12n-2} \ell^{(n-2)}_1 \right\}, \quad \ell^{(n-2)}_2 = \max \left\{ 1, \frac{1}{6n-2} \ell^{(n-2)}_1 \right\}, \quad i = 0, 1, 2, \ldots, n - 2.
\]
(3.16)
Choose a constant \(m \geq 2\) such that \(m(\mu_n - \lambda_n) > 1\), and let
\[
q(t) = \int_0^1 H(t, s) F_1(s) ds \leq \int_0^1 s(1 - s) F_1(s) ds < \infty,
\]
(3.19)
\[
Q(t) = \left[q(t)\right]^{1/(m(\mu_n - \lambda_n))}.
\]
(3.20)
Then \(q(t), Q(t) \in C[0, 1] \cap C^2(0, 1)\) satisfying \(q(t) > 0, Q(t) > 0, t \in (0, 1)\), and
\[-q''(t) = (-1)^n h^{(2n)}(t) = F_1(t), \quad -Q''(t) \geq 0, \quad \text{for } t \in (0, 1),
\]
an and from (3.1)–(3.3), we have \(q(i) = Q(i) = 0\), for \(i = 0, 1,\)
\[(1-t)^{1-1/m}(1-1/m)^{-1}\left(\int_{0}^{t} s F_1(s) \, ds\right)^{1-1/m}\]
\[\leq (1-1/m)^{-1}\left(\int_{0}^{1} (1-s) s F_1(s) \, ds\right)^{1-1/m} < \infty. \quad (3.21)\]

Similarly, we have
\[t \int_{t}^{1} (1-s) Q^{-(-\mu_n-\lambda_n)}(s) F_1(s) \, ds\]
\[\leq (1-1/m)^{-1}\left(\int_{0}^{1} (1-s) s F_1(s) \, ds\right)^{1-1/m} < \infty. \quad (3.22)\]

Let \(c_1 > 0\) such that \((1/c_1) Q(t) \leq 1, \ c_1 \geq 1.\) From (1.3) and (1.4), we have
\[Q^{-\mu_n} f(t, t(1-t), \ldots, t(1-t)), \ P\]
\[\leq Q^{-\mu_n} (Q/c_1)^{\lambda_n} f(t, t(1-t), \ldots, t(1-t), c_1)\]
\[\leq Q^{-\mu_n} (Q/c_1)^{\lambda_n} c_1^{\mu_n} f(t, t(1-t), \ldots, t(1-t), 1)\]
\[= c_1^{\mu_n-\lambda_n} Q^{\lambda_n-\mu_n} f(t, t(1-t), \ldots, t(1-t), 1). \quad (3.23)\]

Let
\[h_1(t) = \int_{0}^{1} H(t, \xi_1) d\xi_1 \int_{0}^{1} H(\xi_1, \xi_2) d\xi_2 \ldots \int_{0}^{1} H(\xi_{n-1}, s) [s(1-s)]^{\mu_n} F_1(s) \, ds, \quad t \in [0, 1],\]
\[h_2(t) = \int_{0}^{1} H(t, \xi_1) d\xi_1 \int_{0}^{1} H(\xi_1, \xi_2) d\xi_2 \ldots \times \int_{0}^{1} H(\xi_{n-1}, s) Q^{-\mu_n}(s) f(s, s(1-s), \ldots, s(1-s), Q(s)) \, ds\]
\[+ \int_{0}^{1} H(t, \xi_1) d\xi_1 \int_{0}^{1} H(\xi_1, \xi_2) d\xi_2 \ldots \int_{0}^{1} H(\xi_{n-2}, s) Q(s) \, ds, \quad t \in [0, 1].\]

Thus, (3.20)–(3.23) imply that
\[0 \leq (-1)^i h_1^{(2i)}(t) < \infty, \quad 0 \leq (-1)^i h_2^{(2i)}(t) < \infty,\]
for \(t \in [0, 1], \ i = 0, 1, 2, \ldots, n-1.\)
One can check that \( h_j \in C^{2n-2}[0, 1] \cap C^2(0, 1), h_j^{(2i)}(0) = h_j^{(2i)}(1) = 0, i = 0, 1, 2, \ldots, n - 1, \ j = 1, 2, \) and

\[
L_1 t(1 - t) \leq (-1)^{n-1} h_1^{(2n-2)}(t) \leq L_1, \\
Q(t) \leq (-1)^{n-1} h_2^{(2n-2)}(t) \leq L_2, \quad t \in [0, 1], \\
(-1)^n h_2^{(2n)}(t) = (t(1-t))^{\mu_n} F_1(t), \quad t \in (0, 1), \\
(-1)^n h_2^{(2n)}(t) \geq Q^{-\mu_n} f(t, t(1-t), \ldots, t(1-t), Q(t)), \quad t \in (0, 1), \\
a_1^{(i)} t(1-t) \leq (-1)^2 h_1^{(2i)}(t) \leq a_2^{(i)} t(1-t), \\
t \in [0, 1], \ i = 0, 1, 2, \ldots, n - 2, \\
b_1^{(i)} t(1-t) \leq (-1)^2 h_2^{(2i)}(t) \leq b_2^{(i)} t(1-t), \\
t \in [0, 1], \ i = 0, 1, 2, \ldots, n - 2.
\]

Here,

\[
L_1 = \int_0^1 \left[ s(1-s) \right]^{1+\mu_n} F_1(s) \, ds, \\
L_2 = \max_{t \in [0, 1]} (-1)^{n-1} h_2^{(2n-2)}(t), \\
a_1^{(n-2)} = \min \left\{ 1, \int_0^1 \xi(1-\xi) \int_0^1 H(\xi, s)(s(1-s))^{\mu_n} F_1(s) \, ds \, d\xi \right\}, \\
a_2^{(n-2)} = \max \left\{ 1, \int_0^1 \int_0^1 H(\xi, s)(s(1-s))^{\mu_n} F_1(s) \, ds \, d\xi \right\}, \\
a_1^{(i)} = \frac{1}{12n^{2-i}} a_1^{(n-2)}, \quad a_2^{(i)} = \frac{1}{6n^{2-i}} a_2^{(n-2)}, \quad i = 0, 1, 2, \ldots, n - 2, \\
b_1^{(n-2)} = \min \left\{ 1, \int_0^1 \xi(1-\xi) \int_0^1 H(\xi, s) \right. \\
\left. \times \left( Q^{-\mu_n}(s) f(s, s(1-s), \ldots, s(1-s), Q(s)) \right) \, ds \, d\xi \right\}, \\
b_2^{(n-2)} = \max \left\{ 1, \int_0^1 \int_0^1 H(\xi, s) \\
\times Q^{-\mu_n}(s) f(s, s(1-s), \ldots, s(1-s), Q(s)) \, ds + Q(\xi) \right. \right. \right. \\
\left. \left. \left. \left. \int_0^1 H(\xi, s) \right) \, d\xi \right\}, \\
b_1^{(i)} = \frac{1}{12n^{2-i}} b_1^{(n-2)}, \quad b_2^{(i)} = \frac{1}{6n^{2-i}} b_2^{(n-2)}, \quad i = 0, 1, 2, \ldots, n - 2.
\]
Let $\alpha(t) = k_1 h_1(t)$, $\beta(t) = k_2 h_2(t)$, $t \in [0, 1]$; here $k_1, k_2$ are constants satisfying $0 < k_1 \leq 1 \leq k_2$ and will be determined later. Suppose $c_2, c_3$ are positive constants such that $c_2 L_1 \leq 1$, $c_2 d_2^{(n-2)} \leq 1$, $1/c_2 \geq 1$, $c_3 \geq 1$, $c_3 b_1^{(0)} \geq 1$. From (1.3), (1.4), we have

\[ f(t, \alpha(t), -\alpha''(t), \ldots, (-1)^i \alpha^{(2i)}(t), \ldots, (-1)^{n-1} \alpha^{(2n-2)}(t)) \]

\[ \geq (1/c_2)^{\mu_n} f(t, \alpha(t), -\alpha''(t), \ldots, (-1)^i \alpha^{(2i)}(t), \ldots, c_2 (-1)^{n-1} \alpha^{(2n-2)}(t)) \]

\[ \geq c_2^{\mu_n - \lambda_n} \left( (-1)^{n-1} \alpha^{(2n-2)}(t) \right)^{\mu_n} \]

\[ \times f(t, \alpha(t), -\alpha''(t), \ldots, (-1)^i \alpha^{(2i)}(t), \ldots, (-1)^{n-2} \alpha^{(2n-4)}(t), 1) \]

\[ \geq c_2^{\mu_n - \lambda_n} (k_1 L_1)^{\mu_n} (t(1-t))^{\mu_n} \]

\[ \times f(t, \alpha(t), -\alpha''(t), \ldots, (-1)^i \alpha^{(2i)}(t), \ldots, (-1)^{n-2} \alpha^{(2n-4)}(t), 1) \]

\[ \geq c_2^{\mu_n - \lambda_n} (k_1 L_1)^{\mu_n} (t(1-t))^{\mu_n} \left( \prod_{i=1}^{n-1} c_2^{-1} \lambda_i (k_1 a_1^{(i-1)})^{\mu_i} \right) \]

\[ \times f(t, t(1-t), \ldots, t(1-t), 1) \]

\[ = \left( \prod_{i=1}^{n} c_2^{\mu_i - \lambda_i} k_1^{\mu_i} \right) \left( \prod_{i=1}^{n-1} (a_1^{(i-1)})^{\mu_i} \right) L_1^{\mu_n} [t(1-t)]^{\mu_n} F_1(t) \]

\[ \geq k_1 [t(1-t)]^{\mu_n} F_1(t) = (-1)^n \alpha^{(2n)}(t), \quad t \in (0, 1), \quad (3.28) \]

\[ f(t, \beta(t), -\beta''(t), \ldots, (-1)^i \beta^{(2i)}(t), \ldots, (-1)^{n-1} \beta^{(2n-2)}(t)) \]

\[ \leq c_3^{\mu_n - \lambda_n} \left( (-1)^{n-1} \beta^{(2n-2)}(t) / Q(t) \right)^{\mu_n} \]

\[ \times f(t, \beta(t), -\beta''(t), \ldots, (-1)^i \beta^{(2i)}(t), \ldots, (-1)^{n-2} \beta^{(2n-4)}(t), Q(t)) \]

\[ \leq c_3^{\mu_n - \lambda_n} (k_2 L_2)^{\mu_n} Q^{-\mu_n} (t) \]

\[ \times f(t, \beta(t), -\beta''(t), \ldots, (-1)^i \beta^{(2i)}(t), \ldots, (-1)^{n-2} \beta^{(2n-4)}(t), Q(t)) \]

\[ \leq c_3^{\mu_n - \lambda_n} (k_2 L_2)^{\mu_n} \left( \prod_{i=1}^{n-1} c_3^{-1} \lambda_i (k_2 b_2^{(i-1)})^{\mu_i} \right) Q^{-\mu_n} (t) \]

\[ \times f(t, t(1-t), \ldots, t(1-t), Q(t)) \]

\[ = \left( \prod_{i=1}^{n} c_3^{\mu_i - \lambda_i} k_2^{\mu_i} \right) \left( \prod_{i=1}^{n-1} (b_2^{(i-1)})^{\mu_i} \right) L_2^{\mu_n} Q^{-\mu_n} (t) \]

\[ \times f(t, t(1-t), \ldots, t(1-t), Q(t)) \]

\[ \leq k_2 Q^{-\mu_n} (t) f(t, t(1-t), \ldots, t(1-t), Q(t)) \leq (-1)^n \beta^{(2n)}(t), \quad t \in (0, 1). \quad (3.29) \]
By virtue of (1.3), (1.4), we can find a $k_0$ such that
\[ f(t, t(1 - t), \ldots, t(1 - t), Q(t)) \geq k_0 Q^\mu(t) f(t, t(1 - t), \ldots, t(1 - t), 1), \]
and hence, from the definitions of $h_1(t), h_2(t)$, we have, when $k > k_0^{-1}$, $(-1)^j h_1(2) \leq k^{-1} h_2(2)(t)$ for $t \in [0, 1], i = 0, 1, 2, \ldots, n - 1$. Now we choose
\[ k_1 = \min\left\{1, k_0^{-1}, L_1^{\mu_0} \left( \prod_{i=1}^n c_i^{\mu_i - \lambda_i} \right) \left( \prod_{i=1}^{n-1} \left( a_i^{(i-1)} \right)^{\mu_i} \right)^{1/(1 - \sum_{i=1}^n \mu_i)} \right\} \]
and
\[ k_2 = \max\left\{1, k_0^{-1}, L_2^{\mu_0} \left( \prod_{i=1}^n c_i^{\mu_i - \lambda_i} \right) \left( \prod_{i=1}^{n-1} \left( b_i^{(i-1)} \right)^{\mu_i} \right)^{1/(1 - \sum_{i=1}^n \mu_i)} \right\}. \]

Then $\alpha(t), \beta(t) \in C^{2n-2}[0, 1] \cap C^2(0, 1), 0 < (-1)^j \alpha^{(2)}(t) \leq (-1)^j \beta^{(2)}(t)$ for $t \in (0, 1), (-1)^j \alpha^{(2)}(j) = (-1)^j \beta^{(2)}(j) = 0, j = 0, 1, i = 0, 1, 2, \ldots, n - 1$. From (3.28) and (3.29), we obtain that for such choice of $k_1$ and $k_2$, $\alpha(t)$ and $\beta(t)$ are lower and upper solutions of (1.1) and (1.2), respectively.

From the first conclusion of Lemma 2.2, we conclude that the problem (1.1) has at least a $C^{2n-2}[0, 1]$ positive solution $x(t)$ satisfying $(-1)^j \alpha^{(2)}(t) \leq (-1)^j x^{(2)}(t) \leq (-1)^j \beta^{(2)}(t)$ for $t \in [0, 1], i = 0, 1, 2, \ldots, n - 1$. This completes the proof of Theorem 3.1.

**The proof of Theorem 3.2.** Necessity. Suppose that $x(t)$ is a $C^{2n-1}[0, 1]$ positive solution of (1.1) and (1.2). Then both $(-1)^{n-1} x^{(2n-1)}(0) > 0$ and $(-1)^{n-1} x^{(2n-1)}(1) < 0$ exist. By Lemma 2.3 and the proof of Theorem 3.1, there are constants $0 < I_1^{(i)} < I_2^{(i)}$ such that
\[ I_1^{(i)} t(t - 1) \leq (-1)^j x^{(2)}(t) \leq I_2^{(i)} t(t - 1), \]
\[ t \in [0, 1], i = 0, 1, 2, \ldots, n - 1. \] (3.30)

Here,
\[ I_1^{(i)} = \frac{1}{12^{n-1-i}} I_1^{(n-1)}, \quad I_2^{(i)} = \frac{1}{6^{n-1-i}} I_2^{(n-1)}, \quad i = 0, 1, 2, \ldots, n - 2. \] (3.31)

Let $c_4$ be a constant satisfying $c_4 < 1, 1/c_4 \geq 1$ and let $e(t) = t(1 - t)$. Then (1.3), (1.4) and (3.30), (3.31), lead to
\[ f(t, x(t), -x''(t), \ldots, (-1)^{n-1} x^{(2n-2)}) \geq (1/c_4)^{\lambda_1} f(t, c_4 x(t)e(t)/e(t), -x''(t), \ldots, (-1)^{n-1} x^{(2n-2)}) \geq c_4^{\mu_1 - \lambda_1} (x(t)/e(t))^{\mu_1} f(t, e(t), -x''(t), \ldots, (-1)^{n-1} x^{(2n-2)}) \geq c_4^{\mu_1 - \lambda_1} \left( I_1^{(0)} \right)^{\mu_1} f(t, e(t), -x''(t), \ldots, (-1)^{n-1} x^{(2n-2)}) \geq \left( \prod_{i=1}^n c_i^{\mu_i - \lambda_i} \left( I_1^{(i-1)} \right)^{\mu_i} \right) f(t, e(t), e(t), \ldots, e(t)), \quad t \in (0, 1). \]
Consequently,
\[ \int_0^1 f(t, e(t), e(t), \ldots, e(t)) \, dt \leq \left( \prod_{i=1}^{n} c_i^{-\lambda_i} \left[I_1^{(i-1)} \right]^\mu_i \right)^{-1} \int_0^1 f(t, x(t), -x''(t), \ldots, (-1)^{n-1} x^{(2n-2)}(t)) \, dt \]
\[ = \left( \prod_{i=1}^{n} c_i^{-\lambda_i} \left[I_1^{(i-1)} \right]^\mu_i \right)^{-1} \left[ (1)^{n-1} x^{(2n-1)}(0) - (-1)^{n-1} x^{(2n-1)}(1) \right] < \infty. \]

Thus (3.4) holds.

**Sufficiency.** Suppose that (3.4) holds. Let
\[ h(t) = \int_0^1 H(t, \xi_1) \, d\xi_1, \quad t \in [0, 1]; \quad (3.32) \]
Then \( h(t) \in C^{2n-1}[0, 1] \cap C^{2n-2}(0, 1) \) and (3.30), (3.31) hold if \( x(t) \) is replaced by \( h(t) \), and
\[ I_1^{(n-1)} = \int_0^1 e(s) F_2(s) \, ds, \quad I_2^{(n-1)} = \int_0^1 F_2(s) \, ds. \]
Suppose that positive constants \( c_5 \) and \( c_6 \) satisfy \( c_5 I_1^{(n-1)} \leq 1, \ 1/c_5 \geq 1, \ c_6 I_2^{(0)} \geq 1, \ 1/c_6 \leq 1 \). Let \( \alpha(t) = k_1 h(t), \ \beta(t) = k_2 h(t), \ t \in [0, 1] \); here
\[ k_1 = \min \left\{ 1, \left[ \prod_{i=1}^{n} c_i^{-\lambda_i} \left[I_1^{(i-1)} \right]^\mu_i \right]^{1/(1-\sum_{i=1}^{n} \mu_i)} \right\} \]
and
\[ k_2 = \max \left\{ 1, \left[ \prod_{i=1}^{n} c_i^{-\lambda_i} \left[I_2^{(i-1)} \right]^\mu_i \right]^{1/(1-\sum_{i=1}^{n} \mu_i)} \right\}; \]
A similar argument to that we have checked in the sufficiency proof of Theorem 3.1 yields that
\[ f(t, \alpha(t), -\alpha''(t), \ldots, (-1)^{n-1} \alpha^{(2n-2)}(t)) \]
\[ \geq (1/c_5)^{\beta_1} f(t, c_5 \alpha(t), -\alpha''(t), \ldots, (-1)^{n-1} \alpha^{(2n-2)}(t)) \]
\[ \geq c_5^{\beta_1-\lambda_1} \left( \alpha(t)/\alpha(t) \right)^{\mu_1} f(t, e(t), -\alpha''(t), \ldots, (-1)^{n-1} \alpha^{(2n-2)}(t)) \]
\[ \geq c_5^{\beta_1-\lambda_1} (k_1 I_1^{(0)})^{\mu_1} f(t, e(t), -\alpha''(t), \ldots, (-1)^{n-1} \alpha^{(2n-2)}(t)) \]
positive solution \( x(t) \) and (1.2) satisfying \( 0 < \alpha(t), \beta(t) \). From (3.4), we have

\[
\begin{align*}
\prod_{i=1}^{n} c_{5}^{-\lambda_{i}} (k_{1}I_{1}^{2-1}(i)^{-1})^{\mu_{i}} f(t, e(t), e(t), \ldots, e(t)) \\
\geq k_{1} f(t, e(t), e(t), \ldots, e(t)) = (-1)^{n} \alpha^{(2n)}(t), \quad t \in (0, 1), \tag{3.33}
\end{align*}
\]

\[
f(t, \beta(t), -\beta''(t), \ldots, (-1)^{n-1} \beta^{(2n-2)}(t)) \leq \frac{1}{c_{6}} \lambda_{1} f(t, c_{6} \beta(t), -\beta''(t), \ldots, (-1)^{n-1} \beta^{(2n-2)}(t)) \leq \frac{1}{c_{6}} \lambda_{1} \left( \frac{c_{6} x}{k_{1} e(t)} \right)^{\mu_{1}} f(t, e(t), -x''(t), \ldots, (-1)^{n-1} x^{(2n-2)}(t)) \leq \frac{c_{6}}{k_{1}} \lambda_{1} \left( k_{2}I_{2}^{(0)} \right)^{\mu_{1}} f(t, e(t), -x''(t), \ldots, (-1)^{n-1} x^{(2n-2)}(t)) \leq \prod_{i=1}^{n} \left[ \frac{c_{6}}{k_{1}} \right]^{\mu_{1}-\lambda_{i}} \left( k_{2}I_{2}^{(i-1)} \right)^{\mu_{i}} f(t, e(t), e(t), \ldots, e(t)) = F(t),
\end{align*}
\]

\[
t \in (0, 1), \tag{3.35}
\]

So, \( \alpha(t), \beta(t) \in C^{2n-1}[0, 1] \cap C^{2n}(0, 1) \) are, respectively, lower and upper solutions of (1.1) and (1.2) satisfying \( 0 < (-1)^{i} \alpha^{(2i)}(t) \leq (-1)^{j} \beta^{(2j)}(t) \) for \( t \in (0, 1), \ (-1)^{i} \alpha^{(2i)}(j) = (-1)^{j} \beta^{(2j)}(j) \), \( j = 0, 1, i = 0, 1, 2, \ldots, n-1 \). Additionally, when \( t \in (0, 1) \) and \( 0 < (-1)^{i} \alpha^{(2i)}(t) \leq (-1)^{j} \beta^{(2j)}(t) \), we have

\[
f(t, x(t), -x''(t), \ldots, (-1)^{n-1} x^{(2n-2)}(t)) \leq \frac{1}{c_{6}} \lambda_{1} f(t, c_{6} x, e(t), -x''(t), \ldots, (-1)^{n-1} x^{(2n-2)}(t)) \leq \frac{1}{c_{6}} \lambda_{1} \left( \frac{c_{6} x}{k_{1} e(t)} \right)^{\mu_{1}} f(t, e(t), -x''(t), \ldots, (-1)^{n-1} x^{(2n-2)}(t)) \leq \frac{c_{6}}{k_{1}} \lambda_{1} \left( k_{2}I_{2}^{(0)} \right)^{\mu_{1}} f(t, e(t), -x''(t), \ldots, (-1)^{n-1} x^{(2n-2)}(t)) \leq \prod_{i=1}^{n} \left[ \frac{c_{6}}{k_{1}} \right]^{\mu_{1}-\lambda_{i}} \left( k_{2}I_{2}^{(i-1)} \right)^{\mu_{i}} f(t, e(t), e(t), \ldots, e(t)) = F(t),
\]

\[
t \in (0, 1).
\]

From (3.4), we have \( \int_{0}^{1} F(t) \, dt < \infty \). By Lemma 2.2, we assert that problem (1.1) admits a positive solution \( x(t) \in C^{2n-1}[0, 1] \cap C^{2n}(0, 1) \) such that \( (-1)^{i} x^{(2i)}(t) \leq (-1)^{j} x^{(2j)}(t) \) for \( t \in [0, 1], \ i = 0, 1, 2, \ldots, n-1 \). The proof of Theorem 3.2 is complete. □

References


