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Zero cycles of degree one on principal homogeneous spaces ☆

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ABSTRACT

Let k be a field of characteristic different from 2. Let G be an absolutely simple, simply connected or adjoint semisimple algebraic k-group of classical type. We show that if a principal homogeneous space under G over k admits a zero cycle of degree 1 then it has a k-rational point.

Introduction

Let k be a field and let G be an absolutely simple algebraic group defined over k. Let $S(G)$ be the set of homological torsion primes of G defined by Serre [22].

Definition 0.1. We say that a number d is *coprime to* $S(G)$ if none of its prime factors is contained in $S(G)$.

The following question of Serre [22, p. 233] is open in general.

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Q: Let k be a field and let G be an absolutely simple k-group. Let $\{L_i\}_{1 \leqslant i \leqslant m}$ be a set of finite field extensions of k and let the greatest common divisor of the degrees of the extensions $[L_i : k]$ be d. If d is coprime to $S(G)$, does the canonical map

$$H^1(k, G) \to \prod_{i=1}^{m} H^1(L_i, G)$$

have trivial kernel?

The above question has great implications. For instance, a positive answer for exceptional groups would lead to the solution of Serre's Conjecture II [23, Chapter III, §3.1] for these groups, which is still open. Zinovy Reichstein [19, Section 5] has distinguished between *Type* 1 and *Type* 2 problems in Galois Cohomology. The former type can be conveniently handled with current methods while the latter poses greater difficulties. A positive answer to **Q** would reduce the Type 2 problem of finding points on principal homogeneous spaces over a general field to the Type 1 problem of finding points over fields with absolute Galois group a pro-p group.

The first major result in this direction for a general field k is due to Bayer-Fluckiger and Lenstra [3, Section 2] for groups of isometries of algebras with involution. Our approach in this paper is to build on the theorem of Bayer–Lenstra to prove the following:

Theorem 0.2. *Let k be a field of characteristic different from 2. Let $\{L_i\}_{1 \leqslant i \leqslant m}$ be a set of finite field extensions of k and let $\gcd([L_i : k]) = d$. Let G be an absolutely simple algebraic k-group which is not of type E_8 and which is either a simply connected or adjoint classical group or a quasisplit exceptional group. If d is coprime to $S(G)$, then the canonical map*

$$H^1(k, G) \to \prod_{i=1}^{m} H^1(L_i, G)$$

has trivial kernel.

Remark 0.3. Grothendieck [24, Theorem 1.1] proved that X admits a zero cycle whose degree has all its prime factors in $S(G)$. Thus if X admits a zero cycle whose degree is coprime to the homological torsion primes, then X admits a zero cycle of degree one. It suffices then to consider **Q** in the case where X admits a zero cycle of degree one. However, with the same effort as it would take to consider the degree one case and without appealing to the theorem of Grothendieck we give a proof of 0.2.

Notable consequences of this result include the following:

Theorem 0.4. *Let k be a field of characteristic different from 2 and let G be an absolutely simple algebraic k-group which is not of type E_8 and which is either a simply connected or adjoint classical group or a quasisplit exceptional group. Let X be a principal homogeneous space under G over k. If X admits a zero cycle of degree one then X has a rational point.*

This is just the case $d = 1$ of 0.2 and gives a positive answer to a question posed by Serre [23, p. 192] for these groups. We remark that 0.4 does not hold if X is instead a quasi-projective homogeneous variety [7] or a projective homogeneous variety [18].

Theorem 0.2 implies that the Rost invariant $R_G : H^1(k, G) \to H^3(k, \mathbb{Z}/3\mathbb{Z})$ is injective for G an absolutely simple, simply connected group of type A_2. In turn, one recovers a classification of unitary involutions on algebras of degree 3 [15, Theorem 19.6] which had appeared in [13].

Besides the Bayer–Lenstra theorem [3], the Gille–Merkurjev norm principle theorems [11, Théorème II.3.2], [17, Theorem 3.9] are critical for our proof for simply connected classical groups. In the case of adjoint classical groups, we make regular use of Bayer–Lenstra's [3] extension of Scharlau's transfer homomorphism to hermitian forms. Some cases of adjoint groups have been proven by Barquero-Salavert [2]. Triviality of the kernel of the Rost invariant on torsors under a quasisplit, simply connected group of exceptional type not E_8 [9,6] leads to the proof of 0.2 in these cases.

As we wish to exploit Weil's classification of classical semisimple algebraic groups as groups associated to algebras with involution, we begin by recalling relevant notions from the theory of algebras with involution. In Section 2 we recall the classification of simply connected and adjoint absolutely simple groups and list values of $S(G)$ for each group G. In Section 3 we consider the question **Q** for G a simply connected, absolutely simple, classical group and in Section 4 for G an adjoint, absolutely simple, classical group. Section 5 is a discussion of the question **Q** for G quasisplit, simple exceptional of type other than E_8. Finally in Section 6 we show that our main result 0.2 is an easy consequence of our results in the preceding three sections.

1. Algebras with involution

Let A be a central simple algebra over a field K of characteristic different from 2. An *involution* σ on A is an anti-automorphism of period 2. We will often write (A,σ) for a central simple K-algebra A with involution σ. Let k denote the set of elements in K fixed by σ. If $k = K$ we call σ an *involution of the first kind*. An involution of the first kind is called *orthogonal* if it is a form for the transpose over $\bar{k}$ and *symplectic* otherwise. If $k \neq K$ we call σ an *involution of the second kind*. In this case, $[K:k] = 2$. An involution of the second kind will also be referred to as an *involution of unitary type*.

A *similitude* of a central simple algebra with involution (A,σ) is an element $a \in A$ such that $\sigma(a)a$ is in k^*. This element $\sigma(a)a$ is called the *multiplier* of a written $\mu(a)$. Following [15] we denote the group of similitudes of (A,σ) by $GO(A,\sigma)$ if σ is of orthogonal type, $GSp(A,\sigma)$ if σ is of symplectic type and $GU(A,\sigma)$ if σ is of unitary type. Let the quotients of these groups by their centers be denoted by $PGO(A,\sigma)$, $PGSp(A,\sigma)$ and $PGU(A,\sigma)$ respectively, and let them be referred to as the group of *projective similitudes* of (A,σ) in each case. The group of similitudes with multiplier 1 is called the group of *isometries* of (A,σ) and is denoted $O(A,\sigma)$, $Sp(A,\sigma)$ and $U(A,\sigma)$ in the cases σ orthogonal, symplectic and unitary respectively.

Let $SU(A,\sigma)$ be the elements in $U(A,\sigma)$ with trivial reduced norm. For σ an orthogonal involution on a central simple K-algebra A of even degree, let $GO^+(A,\sigma)$ denote the set of elements a in $GO(A,\sigma)$ such that $\mathrm{Nrd}(a) = \mu(a)^{\deg(A)/2}$ and $PGO^+(A,\sigma)$ be the quotient of $GO^+(A,\sigma)$ by its center. Let $GO^-(A,\sigma)$ be the coset of $GO^+(A,\sigma)$ in $GO(A,\sigma)$ consisting of elements a such that $\mathrm{Nrd}(a) = -\mu(a)^{\deg(A)/2}$. We will call elements of $GO^+(A,\sigma)$ *proper similitudes* and those of $GO^-(A,\sigma)$ *improper similitudes*. For (A,σ) an algebra of even degree with orthogonal involution, let $Spin(A,\sigma)$ be the subgroup of the Clifford group consisting of elements g with $g\tilde{\sigma}(g) = 1$ where $\tilde{\sigma}$ is the map on the Clifford group induced by σ. We also recall that for a K-algebra A, $SL_1(A)$ is the kernel of the reduced norm map on $GL_1(A)$ and $PGL_1(A)$ is the quotient of $GL_1(A)$ by its center.

Given a central simple K-algebra A with involution σ and $K^\sigma = k$, a *hermitian form* h on a right A-module V is a map $h : V \times V \to A$ such that for all $v, w \in V$ and $a, b \in A$, $h(va, wb) = \sigma(a)h(v, w)b$, $h(v, w) = \sigma(h(w, v))$ and h is bi-additive. We will also assume that all hermitian forms satisfy a non-degeneracy condition, that is to say, for all $v \in V - \{0\}$ there is a $w \in V - \{0\}$ such that $h(v, w) \neq 0$.

We associate to any hermitian form h over an algebra with involution (A,σ), the adjoint involution τ_h on the space of endomorphisms of V over A. This association gives a bijective correspondence between hermitian forms on V modulo factors in k^* and involutions on $\mathrm{End}_A(V)$ whose restriction to K is σ. Since by Wedderburn's theorem [12, Theorem 2.1.3] we may write A as $\mathrm{End}_D V$ for V a vector space over a division algebra D, we may write any central simple algebra with involution (A,σ) as $(\mathrm{End}_D V, \tau_h)$ where h is a hermitian form over (D,θ) and θ is an involution whose restriction to K is σ.

If a is an algebraic element over k, consider the k-linear map $s : k(a) \to k$ given by $s(1) = 1$ and $s(a^j) = 0$ for all $1 \leqslant j < m$ where $m = [k(a) : k]$. The map s induces a transfer homomorphism s_*

from the Witt group of hermitian forms over $(D_{k(a)}, \theta_{k(a)})$ to the Witt group of hermitian forms over (D, θ). We will refer to this homomorphism as Scharlau's transfer homomorphism. Bayer-Fluckiger and Lenstra have shown [3] that if $[k(a) : k]$ is odd, r^* is the extension of scalars from k to $k(a)$, and h is a hermitian form over (D, θ) then $s_*(r^*(h)) = h$ in $W(D, \theta)$. We may regard $W(D_{k(a)}, \theta_{k(a)})$ as a $W(k(a))$-module. For example, we may write $ar^*(h) \in W(D_{k(a)}, \theta_{k(a)})$ as $\langle a\rangle \otimes r^*(h) \in W(k(a)) \otimes W(D_{k(a)}, \theta_{k(a)})$. Bayer-Fluckiger and Lenstra have shown [3] that $s_*(\langle a\rangle \otimes r^*(h)) = s_*(\langle a\rangle) \otimes h$.

2. Properties of algebraic groups

Let k be a field of characteristic different from 2. A simply connected (respectively adjoint), semisimple algebraic k-group G is a product of groups of the form $R_{E_j/k}(G_j)$ where each E_j is a finite, separable extension of k and each G_j is an absolutely simple, simply connected (respectively adjoint) group [15, Theorem 26.8].

An absolutely simple, simply connected, classical k-group G has one of the following forms [15, 26.A], [4]:

Type $^1A_{n-1}$: $G = SL_1(A)$ for a central simple algebra A of degree n over k.
The unitary case: $G = SU(A, \sigma)$ associated to a central simple algebra A over K of degree n at least 2, with σ a unitary involution on A with $K^\sigma = k$.
The symplectic case: $G = Sp(A, \sigma)$ associated to a central simple algebra A over k of even degree with a symplectic involution σ.
The orthogonal case: $G = Spin(A, \sigma)$ associated to a central simple algebra A over k of degree at least 3, with σ an orthogonal involution on A.

Let k be field of characteristic different from 2. An absolutely simple, adjoint, classical k-group G has one of the following forms: [15, 26.A]:

Type $^1A_{n-1}$: $G = PGL_1(A)$ for a central simple algebra A of degree n over k.
The unitary case: $G = PGU(A, \sigma)$ associated to A is a central simple algebra over a field K of degree n at least 2 and σ is a unitary involution on A with $K^\sigma = k$.
The symplectic case: $G = PGSp(A, \sigma)$ associated to a central simple algebra A over k of even degree and σ a symplectic involution on A.
The orthogonal case: We distinguish between groups of type B and D in this case.
Type B_n: $G = O^+(A, \sigma)$ associated to a central simple algebra A over k of odd degree at least 3 and σ an orthogonal involution on A.
Type D_n: $G = PGO^+(A, \sigma)$ associated to a central simple algebra A over k of even degree at least 4 and σ an orthogonal involution on A.

In the classification of semisimple algebraic groups, exceptional groups are precisely those of types $^{3,6}D_4$, E_6, E_7, E_8, F_4 and G_2 and a group of type E_8, F_4 or G_2 is both simply connected and adjoint.

In [22] Serre defines a set of primes $S(G)$ associated to an absolutely simple k-group G which we will refer to as the *homological torsion primes* of G. $S(G)$ is the set of prime numbers p each of which satisfies one of the following conditions:

(1) p divides the order of the automorphism group of the Dynkin graph of G;
(2) p divides the order of the center of the universal cover of G;
(3) p is a torsion prime of the root system of G.

The values of the homological torsion primes for each of the absolutely simple semisimple groups is shown in Table 1.

We mention that for each absolutely simple group G, the prime factors of the Dynkin index of G are contained in $S(G)$ [10].

Table 1

Group	$S(G)$
type ${}^1A_{n-1}$	prime divisors of n
unitary case	2, prime divisors of n
symplectic case	2
orthogonal case	2
G_2	2
F_4	2, 3
${}^{3,6}D_4$, E_6, E_7	2, 3
E_8	2, 3, 5

3. Absolutely simple simply connected groups of classical type

The main result of this section is the following:

Theorem 3.1. *Let k be a field of characteristic different from 2. Let G be an absolutely simple, simply connected, classical algebraic group over k. Let $\{L_i\}_{1\leqslant i\leqslant m}$ be a set of finite field extensions of k and let the greatest common divisor of the degrees of the extensions $[L_i : k]$ be d. If d is coprime to $S(G)$, then the canonical map*

$$H^1(k, G) \to \prod_{i=1}^{m} H^1(L_i, G)$$

has trivial kernel.

If $G = Sp(A, \sigma)$ for σ a symplectic involution on a central simple algebra over k, then $H^1(k, Sp(A, \sigma))$ classifies rank one hermitian forms over (A, σ). Then for any finite extension of odd degree L over k, triviality of the kernel of the map $H^1(k, Sp(A, \sigma)) \to H^1(L, Sp(A, \sigma))$ is a consequence of the Bayer–Lenstra theorem [3, Theorem 2.1]. We discuss the remaining cases in 3.3, 3.4, 3.7 below.

We will need the following lemma in the rest of this section.

Lemma 3.2. *Let K be a field and let A be a central simple algebra over K of index s. Let* Nrd *be the reduced norm. For every $\alpha \in K^*$, there exists $\beta \in A^*$ such that $\mathrm{Nrd}(\beta) = \alpha^s$.*

Proof. By [12, Proposition 4.5.4], choose a splitting field E for A such that $[E : K] = s$. Since A_E is split, $\mathrm{Nrd} : A_E \to E$ is onto. In particular α is in $\mathrm{Nrd}(A_E)$. Since $N_{E/K}(\mathrm{Nrd}(A_E)) \subset \mathrm{Nrd}(A)$ [5, Corollary 2.3] and $N_{E/K}(\alpha) = \alpha^s$, it follows that α^s is in $\mathrm{Nrd}(A)$. □

Type ${}^1A_{n-1}$

Theorem 3.3. *Let k be a field, A a central simple algebra of degree n over k and $G = SL_1(A)$. Let $\{L_i\}_{1\leqslant i\leqslant m}$ be a set of finite extensions of k let $\gcd([L_i : k]) = d$. If d is coprime to n, then the canonical map*

$$H^1(k, G) \to \prod_{i=1}^{m} H^1(L_i, G)$$

has trivial kernel.

Proof. Consider the short exact sequence

$$1 \longrightarrow SL_1(A) \longrightarrow GL_1(A) \xrightarrow{\mathrm{Nrd}} G_m \longrightarrow 1 \tag{3.3.1}$$

which by Hilbert's Theorem 90 induces the following commutative diagram with exact rows.

$$\begin{array}{ccccccc} A^* & \xrightarrow{\mathrm{Nrd}} & k^* & \xrightarrow{\delta} & H^1(k, SL_1(A)) & \longrightarrow & 1 \\ \downarrow & & \downarrow g & & \downarrow h & & \\ \prod A^*_{L_i} & \xrightarrow{\mathrm{Nrd}} & \prod L_i^* & \xrightarrow{\delta} & \prod H^1(L_i, SL_1(A)) & \longrightarrow & 1 \end{array} \tag{3.3.2}$$

Choose $\lambda \in \ker(h)$. By the exactness of the top row of the diagram, choose $\lambda' \in k^*$ such that $\delta(\lambda') = \lambda$. Fix an index i. Since $\delta(g(\lambda')) = \text{point}$, by exactness of the bottom row choose $(\lambda_i'') \in A^*_{L_i}$ such that $\mathrm{Nrd}(\lambda_i'') = g(\lambda')$. By restriction–corestriction [12, Proposition 4.2.10], $N_{L_i/k}(g(\lambda')) = (\lambda')^{m_i}$ where $m_i = [L_i : k]$. By the norm principle for reduced norms [5, Corollary 2.3], $N_{L_i/k}(\mathrm{Nrd}(A^*_{L_i})) \subset \mathrm{Nrd}(A^*)$. In particular, $(\lambda')^{m_i}$ is in $\mathrm{Nrd}(A^*)$. Since $d = \sum m_i n_i$ for appropriate choice of integers n_i, $(\lambda')^d = \prod((\lambda')^{m_i})^{n_i}$ is in $\mathrm{Nrd}(A^*)$.

Let s be the index of A. Then by 3.2, $(\lambda')^s$ is in $\mathrm{Nrd}(A^*)$. Since s divides n and by assumption d and n are coprime, then d and s are coprime. So choose a and b such that $sa + db = 1$. Then $\lambda' = (\lambda')^{sa}(\lambda')^{db}$ is in $\mathrm{Nrd}(A^*)$ and by exactness of the top row $\lambda = \delta(\lambda')$ is the point in $H^1(k, SL_1(A))$. □

The unitary case

Theorem 3.4. *Let A be a central simple algebra of degree n with center K and σ a unitary involution on A with $K^\sigma = k$. Suppose $\deg_K(A) \geqslant 2$. Let $G = SU(A, \sigma)$. Let $\{L\}_{1 \leqslant i \leqslant m}$ be a set of finite field extensions of k with $\gcd([L_i : k]) = d$. If d is odd and coprime to n, then the canonical map*

$$H^1(k, G) \to \prod_{i=1}^{m} H^1(L_i, G)$$

has trivial kernel.

Proof. Consider the short exact sequence

$$1 \longrightarrow SU(A, \sigma) \longrightarrow U(A, \sigma) \xrightarrow{\mathrm{Nrd}} R^1_{K/k}G_m \longrightarrow 1 \tag{3.4.1}$$

which induces the following commutative diagram in Galois Cohomology with exact rows.

$$\begin{array}{ccccccc} U(A,\sigma)(k) & \xrightarrow{\mathrm{Nrd}} & R^1_{K/k}G_m(k) & \xrightarrow{\delta} & H^1(k, SU(A,\sigma)) & \xrightarrow{j} & H^1(k, U(A,\sigma)) \\ \downarrow & & \downarrow f & & \downarrow g & & \downarrow h \\ \prod U(A,\sigma)(L_i) & \xrightarrow{\mathrm{Nrd}} & \prod R^1_{K/k}G_m(L_i) & \xrightarrow{\delta} & \prod H^1(L_i, SU(A,\sigma)) & \xrightarrow{j} & \prod H^1(L_i, U(A,\sigma)) \end{array} \tag{3.4.2}$$

Choose $\lambda \in \ker(g)$. By assumption, there is an index i such that $[L_i : k]$ is odd. Fix that index i and let $L_i = L$. By the Bayer–Lenstra theorem [3, Theorem 2.1], $H^1(k, U(A, \sigma)) \to H^1(L, U(A, \sigma))$ has trivial kernel. In particular, h has trivial kernel and λ is in $\ker(j)$. So choose $\lambda' \in R^1_{K/k}G_m(k)$ such that $\delta(\lambda') = \lambda$. Since $\delta(f(\lambda')) = \text{point}$, exactness of the bottom row of the diagram gives $(\lambda_i'') \in \prod U(A, \sigma)(L_i)$ such that $\mathrm{Nrd}(\lambda_i'') = f(\lambda')$. Applying $N_{L_i/k}$ to both sides of this equality we find $N_{L_i/k}(\mathrm{Nrd}(\lambda_i'')) = N_{L_i/k}(f(\lambda'))$. Since $U(A, \sigma)$ is a rational group, [17, Theorem 3.9] gives that for each i, $N_{L_i/k}(\mathrm{Nrd}(\lambda_i''))$ is in the image of $\mathrm{Nrd} : U(A, \sigma)(k) \to R^1_{K/k}G_m(k)$. By restriction–corestriction

[12, Proposition 4.2.10], for each i, $N_{L_i/k}(f(\lambda')) = (\lambda')^{m_i}$ for $m_i = [L_i : k]$. So for each i, $(\lambda')^{m_i}$ is in the image of $\mathrm{Nrd} : U(A,\sigma)(k) \to R^1_{K/k}G_m(k)$. Since $(\lambda')^d = \prod((\lambda')^{m_i})^{n_i}$ for appropriate choice of integers n_i, then $(\lambda')^d$ is in the image of $\mathrm{Nrd} : U(A,\sigma)(k) \to R^1_{K/k}G_m(k)$.

By Classical Hilbert 90, which is a consequence of [15, Theorem 29.2], write $\lambda' = \mu^{-1}\bar{\mu}$ for $\mu \in K^*$ and $\bar{\mu}$ the image of μ under the nontrivial automorphism of K over k. Let s be the index of A and write $(\lambda')^s = (\mu^s)^{-1}\bar{\mu}^s$. By 3.2, $\mu^s = \mathrm{Nrd}(a)$ for some $a \in A^*$. Thus $(\lambda')^s = \mathrm{Nrd}(a^{-1}\sigma(a))$ and by Merkurjev's theorem [17, Proposition 6.1] $(\lambda')^s$ is in the image of $\mathrm{Nrd} : U(A,\sigma)(k) \to R^1_{K/k}G_m(k)$.

Certainly, s divides n and since by assumption d is coprime to n, then d is coprime to s. In particular, there exist $v, w \in \mathbb{Z}$ such that $dv + sw = 1$. Therefore $\lambda' = (\lambda')^{dv}(\lambda')^{sw}$ is in the image of $\mathrm{Nrd} : U(A,\sigma)(k) \to R^1_{K/k}G_m(k)$ and by exactness of the top row of (3.4.2), $\lambda = \delta(\lambda') = \text{point}$. □

Remark 3.5. One can replace the norm principle [17, Theorem 3.9] used in 3.4 above with [1, Theorem 1.1].

The orthogonal case

Our proof in this case makes use of the following result.

Proposition 3.6. *Let k be a field of characteristic different from 2 and let A be a central simple algebra over k of degree $\geqslant 3$ with orthogonal involution σ. Let $G = O^+(A,\sigma)$ and let L be a finite extension of k of odd degree. Then the canonical map*

$$H^1(k,G) \to H^1(L,G)$$

has trivial kernel.

Proof. We have the short exact sequence

$$1 \longrightarrow O^+(A,\sigma) \longrightarrow O(A,\sigma) \xrightarrow{\mathrm{Nrd}} \mu_2 \longrightarrow 1 \tag{3.6.1}$$

In the case A is split, $O(A,\sigma) = O(q)$ the orthogonal group of a quadratic form q, $O^+(A,\sigma) = O^+(q)$ and the reduced norm is the determinant. Springer's theorem [16, Chapter VII Theorem 2.7] gives $H^1(k, O(q)) \to H^1(L, O(q))$ has trivial kernel. That $H^1(k, O^+(q)) \to H^1(k, O(q))$ has trivial kernel follows from the observation that the determinant map $O(q)(k) \to \mu_2$ is onto. Combining these two results, 3.6 holds.

So assume A is not split. Then $O^+(A,\sigma)(k) = O(A,\sigma)(k)$ [14, 2.6, Lemma 1.b]. Since A admits an involution of the first kind and L/k is odd, A_L is not split and $O^+(A,\sigma)(L) = O(A,\sigma)(L)$.

Then (3.6.1) induces the following diagram with exact rows and commuting rectangles.

$$\begin{array}{ccccccc}
1 \longrightarrow & \mu_2 & \xrightarrow{\delta} & H^1(k, O^+(A,\sigma)) & \xrightarrow{i} & H^1(k, O(A,\sigma)) \\
& \downarrow h & & \downarrow f & & \downarrow g \\
1 \longrightarrow & \mu_2 & \xrightarrow{\delta} & H^1(L, O^+(A,\sigma)) & \xrightarrow{i} & H^1(L, O(A,\sigma))
\end{array} \tag{3.6.2}$$

Let $\lambda \in \ker(f)$. By the commutativity of the rightmost rectangle in (3.6.2), $g(i(\lambda)) = \text{point}$. Then [3, Theorem 2.1] gives $i(\lambda) = \text{point}$. By the exactness of the top row, there exists $\lambda' \in \mu_2$ such that $\delta(\lambda') = \lambda$. Since the left rectangle in 3.6.2 commutes, $\delta(h(\lambda')) = \text{point}$. Since h is the identity map and δ has trivial kernel, $\lambda' = 1$ and thus $\lambda = \delta(\lambda') = \text{point}$. □

Now we give the proof for absolutely simple, simply connected groups in the orthogonal case.

Theorem 3.7. *Let k be a field of characteristic different from 2 and let A be a central simple algebra over k of degree $\geqslant 4$ with orthogonal involution σ. Let $G = Spin(A,\sigma)$ and let L be a finite extension of k of odd degree. Then the canonical map*

$$H^1(k,G) \to H^1(L,G)$$

has trivial kernel.

Proof. The short exact sequence

$$1 \longrightarrow \mu_2 \xrightarrow{\ i\ } Spin(A,\sigma) \xrightarrow{\ \eta\ } O^+(A,\sigma) \longrightarrow 1 \tag{3.7.1}$$

induces the following commutative diagram with exact rows.

$$\begin{array}{ccccccc} O^+(A,\sigma)(k) & \xrightarrow{\delta} & H^1(k,\mu_2) & \xrightarrow{i} & H^1(k,Spin(A,\sigma)) & \xrightarrow{\eta} & H^1(k,O^+(A,\sigma)) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle j} \\ O^+(A,\sigma)(L) & \xrightarrow{\delta} & H^1(L,\mu_2) & \xrightarrow{i} & H^1(L,Spin(A,\sigma)) & \xrightarrow{\eta} & H^1(L,O^+(A,\sigma)) \end{array} \tag{3.7.2}$$

Choose $\lambda \in \ker(h)$. By commutativity of the rightmost rectangle in (3.7.2) $j(\eta(\lambda)) = \text{point}$. In particular, $\eta(\lambda) \in \ker(j)$ and by 3.6, $\eta(\lambda) = \text{point}$. By exactness of the top row, we may choose $\lambda' \in H^1(k,\mu_2)$ such that $i(\lambda') = \lambda$. By the commutativity of the central rectangle in (3.7.2), $i(g(\lambda')) = \text{point}$. So from exactness of the bottom row, we may choose $\lambda'' \in O^+(A,\sigma)(L)$ such that $\delta(\lambda'') = g(\lambda')$. Applying the norm map to both sides of this equality we find, $N_{L/k}(\delta(\lambda'')) = N_{L/k}(g(\lambda'))$. By restriction–corestriction the latter is $(\lambda')^{[L:k]}$. Let $\tilde{\lambda}$ be a representative of λ' in $k^*/(k^*)^2$. Since $[L:k]$ is odd, $\tilde{\lambda}^{[L:k]} = \tilde{\lambda}$ in $k^*/(k^*)^2$. In turn $[(\lambda')^{[L:k]}] = [\lambda']$ in $H^1(k,\mu_2)$. Thus $N_{L/k}(\delta(\lambda'')) = \lambda'$. Since $O^+(A,\sigma)$ is rational, [11, Théorème II.3.2] gives

$$N_{L/k}\big(\operatorname{im}\big(O^+(A,\sigma)(L) \xrightarrow{\ \delta\ } H^1(L,\mu_2)\big) \subset \operatorname{im}\big(O^+(A,\sigma)(k) \xrightarrow{\ \delta\ } H^1(k,\mu_2)\big)\big)$$

In particular λ' is in the image of $O^+(A,\sigma)(k) \to H^1(k,\mu_2)$. But then by exactness of the top row, $\lambda = i(\lambda') = \text{point}$. □

4. Absolutely simple adjoint groups of classical type

The main result of this section is the following.

Theorem 4.1. *Let k be a field of characteristic different from 2 and G an absolutely simple, adjoint, classical group over k. Let $\{L_i\}_{1\leqslant i\leqslant m}$ be a set of finite field extensions of k and let the greatest common divisor of the degrees of the extensions $[L_i : k]$ be d. If d is coprime to $S(G)$ the canonical map*

$$H^1(k,G) \to \prod_{i=1}^{m} H^1(L_i,G)$$

has trivial kernel.

The proofs in the unitary and symplectic cases are due to Barquero-Salavert [2]. So to prove 4.1 it is enough to consider the group of type ${}^1A_{n-1}$ and the orthogonal case. We consider these cases in 4.2 and 4.8 below.

Type ${}^1A_{n-1}$

Theorem 4.2. *Let k be a field of characteristic different from 2, A a central simple algebra of degree n over k and $G = PGL_1(A)$. Let $\{L_i\}_{1 \leqslant i \leqslant m}$ be a set of finite field extensions of k and let $\gcd([L_i : k]) = d$. If d is coprime to n then the canonical map*

$$H^1(k, G) \to \prod_{i=1}^{m} H^1(L_i, G)$$

has trivial kernel.

Proof. Consider the short exact sequence

$$1 \longrightarrow G_{\mathbf{m}} \longrightarrow GL_1(A) \longrightarrow PGL_1(A) \longrightarrow 1 \tag{4.2.1}$$

Since Hilbert's Theorem 90 gives $H^1(k, GL_1(A)) = 1$, the induced long exact sequences in Galois Cohomology produces the following commutative diagram with exact rows.

$$\begin{array}{ccccc} 1 & \longrightarrow & H^1(k, PGL_1(A)) & \xrightarrow{\delta} & H^2(k, G_{\mathbf{m}}) \\ & & \big\downarrow f & & g \big\downarrow \\ 1 & \longrightarrow & \prod H^1(L_i, PGL_1(A)) & \xrightarrow{\delta} & \prod H^2(L_i, G_{\mathbf{m}}) \end{array} \tag{4.2.2}$$

The pointed set $H^1(k, PGL_1(A))$ classifies isomorphism classes of central simple algebras of degree n over k and for $B \in H^1(k, PGL_1(A))$, $\delta(B) = [B][A]^{-1}$. Choose $B \in \ker(f)$. By commutativity of the diagram, $g(\delta(B)) =$ point in $\prod H^2(L_i, G_m)$.

Let A^o denote the opposite algebra of A and choose $B \otimes A^o$ a representative for the class $[B][A]^{-1}$ in $H^2(k, G_m)$. Let the exponent of $B \otimes A^o$ be s. Since by assumption $B \otimes A^o$ splits over each L_i, s divides each $[L_i : k]$. It follows that s divides d. Since the degree of $B \otimes A^o$ is n^2, s divides n^2.

Since by assumption n and d are coprime, $s = 1$, $B \otimes A^o$ is split and B is Brauer equivalent to A. Then since B and A are of the same degree, they are isomorphic and B is the point in $H^1(k, PGL_1(A))$. $\square$

The orthogonal case

The case in which G is of type B_n is a special case of 3.6. For the case in which G is of type D_n we begin by proving a general form of Scharlau's Norm Principle [16, Chapter 7, §4].

Lemma 4.3. *Let A be a central simple K-algebra with k-linear involution σ. Let L be a finite extension of k of odd degree and let g be a similitude of $(A, \sigma)_L$ with multiplier $\mu(g)$. Then $N_{L/k}(\mu(g))$ is the multiplier of a similitude of (A, σ).*

Proof. Let g be a similitude of $(A, \sigma)_L$. Let $\mu(g) = \sigma(g)g$ be the multiplier of g. By definition, the hermitian form $\langle \mu(g) \rangle_L$ is isomorphic to $\langle 1 \rangle_L$. In particular left multiplication by g gives an explicit isomorphism between the hermitian forms. We may identify $\langle \mu(g) \rangle_L$ with $\langle \mu(g) \rangle_L \otimes \langle 1 \rangle_L$ in $W(L) \otimes$

$W(A_L, \sigma_L)$. Since $[L : k(\mu(g))]$ is odd and $\langle \mu(g) \rangle_L \otimes \langle 1 \rangle_L \cong \langle 1 \rangle_L$ then $\langle \mu(g) \rangle_{k(\mu(g))} \otimes \langle 1 \rangle_{k(\mu(g))} \cong \langle 1 \rangle_{k(\mu(g))}$ [3, Corollary 1.4]. Let s be Scharlau's transfer map from $k(\mu(g)) \to k$ and let s_* be the induced transfer homorphism. Then $s_*(\langle \mu(g) \rangle_{k(a)} \otimes \langle 1 \rangle_{k(a)})$ is Witt equivalent to $\langle N_{k(\mu(g))/k}(\mu(g)) \rangle \otimes \langle 1 \rangle$ [21, Chapter 2, Lemma 5.8], [3, p. 362]. Since on the other hand $s_*(\langle 1 \rangle_{k(\mu(g))}) = \langle 1 \rangle$, then $\langle N_{k(\mu(g))/k}(\mu(g)) \rangle \otimes \langle 1 \rangle$ is Witt equivalent to 1. Since both are rank 1 hermitian forms, it follows from Witt's cancellation that they are in fact isomorphic which gives precisely that $N_{k(\mu(g))/k}(\mu(g))$ is the multiplier of a similitude of (A, σ). □

Remark 4.4. The result 4.3 above is [15, Proposition 12.21]. We give a proof here since only a partial sketch of the proof is given in [15].

We will also need a result on existence of improper similitudes.

Lemma 4.5. *Let k be a field of characteristic different from 2 and A a central simple algebra over k of even degree at least 4 with an orthogonal involution σ. Let L be a finite field extension of k of odd degree. If A is not split, then $GO^-(A, \sigma)(k)$ is nonempty if and only if $GO^-(A, \sigma)(L)$ is nonempty.*

Proof. If $g \in A$ is an improper similitude of A over k, then certainly g_L is an improper similitude of A_L over L. Conversely, choose $g \in A_L$ an improper similitude of A_L over L and let $\sigma(g)g = \mu(g)$. Then A_L Brauer equivalent to the quaternion algebra $(\delta, \mu(g))$ over L where δ is the discriminant of σ [15, Theorem 13.38]. From this we find $\mathrm{cor}(A_L)$ Brauer equivalent to $\mathrm{cor}((\delta, \mu(g)))$. Now $\mathrm{res}: H^2(k, \mu_2) \to H^2(L, \mu_2)$ certainly takes A to A_L and $\mathrm{cor}(\mathrm{res}(A)) = A$ since A is 2-torsion and $[L:k]$ is odd. On the other hand, $\mathrm{cor}((\delta, \mu(g))) = (\delta, N_{L/k}(\mu(g)))$. By 4.3 write $N_{L/k}(\mu(g))$ as $\mu(g')$ for g' a similitude of A over k. Thus A is Brauer equivalent to $(\delta, \mu(g'))$. If g' is a proper similitude then by [15, Proposition 13.38] $(\delta, \mu(g'))$ splits. But then A splits and we arrive at a contradiction. So g' is an improper similitude of A over k. □

We will also make use of the following two results:

Proposition 4.6. *Let k be a field of characteristic different from 2 and A a central simple algebra over k of degree at least 4 with an orthogonal involution σ. Let L be a finite extension of k of odd degree. Let $G = GO(A, \sigma)$ be the group of similitudes of (A, σ). Then the canonical map*

$$H^1(k, G) \to H^1(L, G)$$

has trivial kernel.

Proof. Let $G_0 = O(A, \sigma)$. We have the exact sequence

$$1 \to G_0 \to G \to G_m \to 1$$

where the map $G \to G_m$ takes each similitude a to its multiplier $\sigma(a)a$. In view of Hilbert's Theorem 90, the sequence yields the following commutative diagram with exact rows.

$$\begin{array}{ccccccc} k^* & \xrightarrow{\delta} & H^1(k, G_0) & \xrightarrow{i} & H^1(k, G) & \longrightarrow & 1 \\ \downarrow & & \downarrow{\scriptstyle r^*} & & \downarrow{\scriptstyle g} & & \\ L^* & \xrightarrow{\delta} & H^1(L, G_0) & \xrightarrow{i} & H^1(L, G) & \longrightarrow & 1 \end{array} \tag{4.6.1}$$

Let $\psi \in \ker(g)$. By the exactness of the top row of (4.6.1), there exists $\langle x\rangle \in H^1(k, G_0)$ such that $i(\langle x\rangle) = \psi$. Here $\langle x\rangle$ is a rank one hermitian form over (A, σ). Since commutativity of the right rectangle gives $i(r^*(\langle x\rangle)) = \text{point}$, exactness of the second row gives an $a \in L^*$ such that $r^*(\langle x\rangle) = \delta(a)$. We note that $\delta(a)$ is the isomorphism class of the rank one hermitian form $\langle a\rangle$ over $(A, \sigma)_L$.

Let $k(a)$ be the subfield of L generated by a over k. Since L is an odd degree extension of $k(a)$ and $\langle a\rangle_L \cong r^*(\langle x\rangle)_L$ then $\langle a\rangle_{k(a)} \cong r^*(\langle x\rangle)_{k(a)}$ [3, Corollary 1.4]. Let $s : k(a) \to k$ be the k-linear map given by $s(1) = 1$ and $s(a^j) = 0$ for all $1 \leqslant j < m$ where $m = [k(a) : k]$ and let s_* be the induced transfer homomorphism. Write $\langle a\rangle$ as $\langle a\rangle \otimes \langle 1\rangle_{k(a)}$ in $W(k(a)) \otimes W(A_{k(a)}, \sigma_{k(a)})$. Since $[k(a) : k]$ is odd, results of Bayer–Lenstra and Scharlau give that $s_*(\langle a\rangle \otimes \langle 1\rangle_{k(a)})$ is Witt equivalent to $\langle N_{k(a)/k}(a)\rangle \otimes \langle 1\rangle$ [21, Chapter 2, Lemma 5.8], [3, p. 362]. On the other hand, $s_*(r^*(\langle x\rangle)) = s_*(\langle 1\rangle \otimes \langle x\rangle)$ and since $[L : k]$ is odd, $s_*(\langle 1\rangle \otimes \langle x\rangle) \cong \langle x\rangle$ [21]. So $\langle N_{k(a)/k}(a)\rangle$ is Witt equivalent to $\langle x\rangle$ and since the two forms have dimension one over (A, σ), by Witt's cancellation for hermitian forms, $\langle N_{k(a)/k}(a)\rangle \cong \langle x\rangle$. Then $\langle x\rangle = \delta(N_{k(a)/k}(a))$ and thus $\psi = i(\langle x\rangle) = \text{point}$. □

Proposition 4.7. *Let k be a field of characteristic different from 2 and A a central simple algebra over k of degree at least 4 with an orthogonal involution σ. Let $G = GO^+(A, \sigma)$ and let L be a finite field extension of k of odd degree. Then the canonical map*

$$H^1(k, G) \to H^1(L, G)$$

has trivial kernel.

Proof. Consider the short exact sequence

$$1 \longrightarrow GO^+(A,\sigma) \xrightarrow{\ i\ } GO(A,\sigma) \xrightarrow{\ \eta\ } \mu_2 \longrightarrow 1 \tag{4.7.1}$$

where the map η takes $a \in GO(A, \sigma)$ to 1 if $\mathrm{Nrd}(a) = \mu(a)^{\deg(A)/2}$ and $\eta^{-1}(-1)$ is precisely $GO^-(A, \sigma)$.

In the case A is split, each hyperplane reflection gives an improper similitude. Thus $GO(A, \sigma)(k) \to \mu_2$ is onto and (4.7.1) induces the following commutative diagram with exact rows.

$$\begin{array}{ccccc}
1 & \longrightarrow & H^1(k, GO^+(A,\sigma)) & \xrightarrow{\ i\ } & H^1(k, GO(A,\sigma)) \\
 & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\
1 & \longrightarrow & H^1(L, GO^+(A,\sigma)) & \xrightarrow{\ i\ } & H^1(L, GO(A,\sigma))
\end{array} \tag{4.7.2}$$

Choose $\lambda \in \ker(g)$. Since the diagram (4.7.2) commutes and h has trivial kernel by 4.6, $i(\lambda) = \text{point}$. Then exactness of the top row of (4.7.2) gives $\lambda = \text{point}$.

In the case A is not split, we need only consider two scenarios. Firstly, suppose A and A_L both admit improper similitudes. Then $GO(A, \sigma)(k) \to \mu_2$ and $GO(A, \sigma)(L) \to \mu_2$ are both onto and the proof proceeds exactly as in the split case. Otherwise, by 4.5 neither admits an improper similitude. That is $GO^+(A, \sigma)(k) = GO(A, \sigma)(k)$, $GO^+(A, \sigma)(L) = GO(A, \sigma)(L)$ and (4.7.1) induces the following commutative diagram with exact rows.

$$\begin{array}{ccccccc}
1 & \longrightarrow & \mu_2 & \xrightarrow{\ \delta\ } & H^1(k, GO^+(A,\sigma)) & \xrightarrow{\ i\ } & H^1(k, GO(A,\sigma)) \\
 & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\
1 & \longrightarrow & \mu_2 & \xrightarrow{\ \delta\ } & H^1(L, GO^+(A,\sigma)) & \xrightarrow{\ i\ } & H^1(L, GO(A,\sigma))
\end{array} \tag{4.7.3}$$

Choose $\lambda \in \ker(g)$. Commutativity of the rightmost rectangle in 4.7.3 gives $i(\lambda) \in \ker(h)$. But by 4.6, this gives $i(\lambda) = \text{point}$. Then, by exactness of the top row of (4.7.3), $\exists \lambda' \in \mu_2$ such that $\delta(\lambda') = \lambda$. Commutativity of the left rectangle in (4.7.3) gives $\delta(f(\lambda')) = \text{point}$. From whence, since the bottom row of (4.7.3) is exact we find $f(\lambda') = 1$. But certainly f is the identity map. So in fact $\lambda' = 1$ and in turn, $\lambda = \delta(\lambda') = \text{point}$. □

We may now prove 4.1 for the absolutely simple group in the orthogonal case.

Theorem 4.8. *Let k be a field of characteristic different from 2 and A a central simple algebra over k of degree at least 4 with an orthogonal involution σ. Let $G = PGO^+(A, \sigma)$ and let L be a finite field extensions of k of odd degree. Then the canonical map*

$$H^1(k, G) \to H^1(L, G)$$

has trivial kernel.

Proof. Consider the short exact sequence

$$1 \longrightarrow G_m \longrightarrow GO^+(A, \sigma) \xrightarrow{\eta} PGO^+(A, \sigma) \longrightarrow 1 \tag{4.8.1}$$

by Hilbert's Theorem 90, this induces the following commutative diagram with exact rows.

$$\begin{array}{ccccccc} 1 \longrightarrow & H^1(k, GO^+(A, \sigma)) & \xrightarrow{\eta} & H^1(k, PGO^+(A, \sigma)) & \xrightarrow{\delta} & H^2(k, G_m) \\ & \downarrow f & & \downarrow g & & \downarrow h \\ 1 \longrightarrow & H^1(L, GO^+(A, \sigma)) & \xrightarrow{\eta} & H^1(L, PGO^+(A, \sigma)) & \longrightarrow & H^2(L, G_m) \end{array} \tag{4.8.2}$$

$H^1(k, PGO^+(A, \sigma))$ classifies k-isomorphism classes of triples (A', σ', ϕ') where A' is a central simple algebra over k of the same degree as A, σ' is an orthogonal involution on A' and ϕ' is an isomorphism from the center of the Clifford algebra of A' to the center of the Clifford algebra of A. For any such triple (A', σ', ϕ'), $\delta(A', \sigma', \phi') = [A'][A]^{-1}$ which is 2-torsion in the Brauer group since both A and A' admit involutions of the first kind. Then, since $[L : k]$ is odd, h is injective on the image of δ in $H^2(k, G_m)$.

So choose $(A', \sigma', \phi') \in \ker(g)$. By commutativity of the rightmost rectangle in (4.8.2), $h(\delta(A', \sigma', \phi')) = \text{point}$ and thus $\delta(A', \sigma', \phi') = \text{point}$. Then by the exactness of the top row of the diagram, there is a $\lambda' \in H^1(k, GO^+(A, \sigma))$ such that $\eta(\lambda') = (A', \sigma', \phi')$. By commutativity of the left rectangle of (4.8.2), $\eta(f(\lambda')) = \text{point}$ which by exactness of the bottom row, gives $f(\lambda') = \text{point}$. Then by 4.7, $\lambda' = \text{point}$ and thus $(A', \sigma', \phi') = \eta(\lambda') = \text{point}$. □

5. Quasisplit exceptional groups

The Rost invariant will be an important tool for the results in this section. For G absolutely simple, simply connected, the *Rost invariant* R_G is an invariant of G with values in $H^3(k, \mathbb{Q}/\mathbb{Z}(2))$ [10]. Notably, the Rost invariant generates the group of all normalized invariants of G with values in $H^3(k, \mathbb{Q}/\mathbb{Z}(2))$ [10, Theorem 9.11]. Let k be a field of characteristic different from 2 or 3. It is known that $R_G : H^1(k, G) \to H^3(k, \mathbb{Q}/\mathbb{Z}(2))$ has trivial kernel when G is a quasisplit exceptional group which is not of type E_8:

- The case where G is quasisplit of type ${}^{3,6}D_4$, E_6 or E_7 is due to Garibaldi [9,8] and Chernousov [6].
- Serre discussed the case where G is of type G_2 in [22, §8].
- The ingredients needed for the case of a split group of type F_4 are discussed in [22, §9]. One considers Rost's mod 3 invariant g_3 [20], and the Arason mod 2 invariants f_3 and f_5 [22, §9] on $H^1(k,G)$. It is known that g_3 vanishes on $J \in H^1(k,G)$ if and only if J is reduced [20] and further that if $g_3(J)=0$ then J is classified by f_3 and f_5. Given these results, that R_G has trivial kernel is a consequence of Springer's theorem [16, Chapter VII, Theorem 2.7].

The main result of this section is the following:

Theorem 5.1. *Let k be a field of characteristic different from* 2*. Let* $\{L_i\}_{1\leqslant i\leqslant m}$ *be a set of finite field extensions of k and let* $\gcd([L_i:k])=d$*. Let G be a quasisplit, absolutely simple exceptional k-group which is not of type* E_8*. If d is coprime to* $S(G)$*, then the canonical map*

$$H^1(k,G)\to\prod_{i=1}^{m}H^1(L_i,G)$$

has trivial kernel.

We begin by considering the simply connected case.

Proposition 5.2. *Let k be a field of characteristic different from* 2 *and* 3*. Let G be a quasisplit, absolutely simple exceptional k-group which is simply connected and is not of type* E_8*. Let* $\{L_i\}_{1\leqslant i\leqslant m}$ *be a set of finite field extensions of k such that* $\gcd([L_i:k])=d$*. If d is coprime to* $S(G)$*, then the canonical map*

$$H^1(k,G)\to\prod_{i=1}^{m}H^1(L_i,G)$$

has trivial kernel.

Proof. For G absolutely simple, simply connected, the Rost invariant R_G takes values in $(\mathbb{Z}/n_G\mathbb{Z})(2)$ where n_G is the Dynkin index of G [10]. The following diagram commutes:

$$\begin{array}{ccc} H^1(k,G) & \xrightarrow{R_G} & H^3(k,(\mathbb{Z}/n_G\mathbb{Z})(2)) \\ \downarrow f & & \downarrow g \\ \prod H^1(L_i,G) & \xrightarrow{R_G} & \prod H^3(L_i,(\mathbb{Z}/n_G\mathbb{Z})(2)) \end{array} \qquad (5.2.1)$$

Choose $\lambda\in\ker(f)$. Since $S(G)$ contains the prime divisors of n_G, and we have assumed that d is coprime to $S(G)$, a restriction–corestriction argument [12, Proposition 4.2.10] gives that g has trivial kernel. So by commutativity of (5.2.1), λ is in $\ker(R_G)$. Since R_G is known to have trivial kernel, we conclude that $\lambda=$ point. □

We return to the main goal of the section:

Theorem 5.3. *Let k be a field of characteristic different from* 2 *and* 3*. Let* $\{L_i\}_{1\leqslant i\leqslant m}$ *be a set of finite field extensions of k and let* $\gcd([L_i:k])=d$*. Let G be a quasisplit, absolutely simple exceptional k-group which is not of type* E_8*. If d is coprime to* $S(G)$*, then the canonical map*

$$H^1(k, G) \to \prod_{i=1}^{m} H^1(L_i, G)$$

has trivial kernel.

Proof. Since a group of type F_4 or G_2 is necessarily simply connected, that the result holds for these groups is a consequence of 5.2. In view of 5.2 we may assume that G is not simply connected. Then we have a short exact sequence

$$1 \longrightarrow \mu \longrightarrow G^{sc} \xrightarrow{\pi} G \longrightarrow 1 \tag{5.3.1}$$

where G^{sc} is a simply connected cover of G and μ is its center. Since G is by assumption quasisplit, then G^{sc} is quasisplit. So let T be the maximal, quasitrivial torus in G^{sc}.

As $\mu \subset T \subset G^{sc}$, the map $H^1(k, \mu) \to H^1(k, G^{sc})$ induced by the inclusion of μ in G^{sc} factors through the map $H^1(k, \mu) \to H^1(k, T)$ induced by the inclusion of μ in T. But since T is quasitrivial, $H^1(k, T)$ is trivial, and thus the image of the map $H^1(k, \mu) \to H^1(k, G^{sc})$ is trivial. Given this result, (5.3.1) induces the following commutative diagram with exact rows.

$$\begin{array}{ccccccc} 1 & \longrightarrow & H^1(k, G^{sc}) & \xrightarrow{\pi} & H^1(k, G) & \xrightarrow{\delta} & H^2(k, \mu) \\ & & \downarrow f & & \downarrow g & & \downarrow h \\ 1 & \longrightarrow & \prod H^1(L_i, G^{sc}) & \xrightarrow{\pi} & \prod H^1(L_i, G) & \xrightarrow{\delta} & \prod H^2(L_i, \mu) \end{array} \tag{5.3.2}$$

Choose $\lambda \in \ker(g)$. The prime divisors of the order of μ are contained in $S(G)$. Then since d is coprime to $S(G)$, d is coprime to the order of μ and a restriction–corestriction argument [12, Proposition 4.2.10] gives that h has trivial kernel. So by commutativity of the rightmost rectangle of (5.3.2), $\lambda \in \ker(\delta)$. By exactness of the top row of (5.3.2) choose $\lambda' \in H^1(k, G^{sc})$ such that $\pi(\lambda') = \lambda$. Commutativity of the left rectangle of (5.3.2) gives $f(\lambda') \in \ker(\pi)$ which is trivial by the exactness of the bottom row of (5.3.2). So $f(\lambda') = \text{point}$, from whence by 5.2, λ' is the point in $H^1(k, G^{sc})$. It is then immediate that $\lambda = \pi(\lambda')$ is the point in $H^1(k, G)$. □

Remark 5.4. One can avoid the restrictions on the characteristic k above by giving a proof in the flat cohomology sets $H^1_{\text{fppf}}(*, *)$ as defined in [25]. Since G is by assumption smooth, $H^1_{\text{fppf}}(k, G) = H^1(k, G)$.

6. Main result

Theorem 6.1. *Let k be a field of characteristic different from 2. Let $\{L_i\}_{1 \leqslant i \leqslant m}$ be a set of finite field extensions of k and let $\gcd([L_i : k]) = d$. Let G be an absolutely simple algebraic k-group which is not of type E_8 and which is either a simply connected or adjoint classical group or a quasisplit exceptional group. If d is coprime to $S(G)$, then the canonical map*

$$H^1(k, G) \to \prod_{i=1}^{m} H^1(L_i, G)$$

has trivial kernel.

Proof. This follows from 3.1, 4.1 and 5.1 above. □

Remark 6.2. By Shapiro's lemma [15, Lemma 29.6] and 6.1, one finds that the answer to **Q** is yes if k is a field of characteristic different from 2 and G is a simply connected or adjoint semisimple algebraic k-group which does not contain a simple factor of type E_8 and such that every exceptional simple factor of type other than G_2 is quasisplit.

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