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G-graphs: A new representation of groups

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Abstract

An important part of computer science is focused on the links that can be established between group theory and graph theory and graphs. CAYLEY graphs, that establish such a link, are useful in a lot of areas of sciences. This paper introduces a new type of graph associated with a group, the G-graphs, and presents many of their properties. We show that various characteristics of a group can be seen on its associated G-graph. We also present an implementation of the algorithm constructing these new graphs, an implementation that will lead to some experimental results. Finally we show that many classical graphs are G-graphs. The relations between G-graphs and CAYLEY graphs are also studied.

Keywords: Computational group theory; Graph representation of a group; Cayley graphs; G-graphs

1. Introduction

Group theory can be considered as the study of symmetry. A group is basically the collection of symmetries of some object preserving some of its structure; therefore many sciences have to deal with groups. It has been proved that graphs can be interesting tools for the study of groups. Groups linked with graphs have been arguably the most famous and productive area of algebraic graph theory (Biggs, 1974; Jones, 2002; Lauri and Scapellato, 2003; Lauri, 1997, 2003). A popular representation of groups by graphs is the CAYLEY graphs. These graphs were first used by A. Cayley in 1878 (Cayley, 1878, 1889) to construct pictorial representations of finite groups. With a group G and a set $S \subseteq G$ of generators a digraph called the CAYLEY

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graph is associated. The set of vertices of this graph is the set of elements of G and two vertices x, y are adjacent if and only if there exists $s \in S$ such that $y = sx$. If $S = S^{-1}$ the graph is undirected and if we choose for S a multi-set (repeating some generators) we get a CAYLEY *multi-graph*. Now a lot of work has been done on these graphs (Cooperman et al., 1991). The regularity and the underlying algebraic structure of CAYLEY graphs make them good candidates for applications such as optimizations on parallel architectures, or for the study of interconnection networks (Heydemann and Ducourthial, 1998), but can also be a limitation:

- Many interesting graphs are not CAYLEY graphs.
- CAYLEY graphs are always regular.
- CAYLEY graphs do not give much information about the group. For example take the cyclic group with an order equal to n. Assume that this group is generated by two generators; the CAYLEY graph associated with this group is a cycle with a length equal to n. No more information is given on this graph.
- It has been shown that two isomorphic groups give rise to two isomorphic CAYLEY graphs; the converse is not true (Morris, 1999), even for a subcategory of groups like abelian groups.

The purpose of this paper is to present some properties of a new type of graph – called a G-graph (Bretto and Laget, 2004; Bretto and Gillibert, 2004) – constructed from a group and to present an algorithm to construct it. A graph that can be used as a powerful tool, surmounting CAYLEY graphs' limitations.

G-graphs, like CAYLEY graphs, have both nice and highly regular properties. Consequently, these graphs can be used in any areas of science where CAYLEY graphs occur. Besides these graphs have the isomorphic property for abelian groups (two isomorphic abelian groups give rise to two isomorphic graphs and conversely). We also prove that much information about a given group can be deduced from its associated G-graph. The problem of the G-graph recognition is also studied. It occurs that many usual regular graphs, such as the cube, the hypercube, are some G-graphs. A list of the most common graphs that are G-graphs is established and some tools for detecting whether a graph is a G-graph are proposed. Most of the G-graphs in the final list are symmetric, but some semisymmetric graphs, such as the Ljubljana graph, are also G-graphs.

2. Basic definitions

2.1. Graph definitions

We define a graph $\Gamma = (V; E; \epsilon)$ as follows:

- V is the set of vertices and E is the set of edges.
- ϵ is a map from E to $P_2(V)$, where $P_2(V)$ is the set of subsets of V having one or two elements.

In this paper graphs are finite, i.e., sets V and E have finite cardinalities. For each edge a, we define $\epsilon(a) = [x; y]$ if $\epsilon(a) = \{x, y\}$ with $x \neq y$ or $\epsilon(a) = \{x\} = \{y\}$. If $x = y$, a is called a *loop*. The elements x, y are called extremities of a, and a is incident to x and y. The set $a \in E, \epsilon(a) = [x; y]\}$ is called a *multiedge* or *p-edge*, where p is the cardinality of the set. We define the *degree* of x by $d(x) = card(\{a \in E, x \in \epsilon(a)\})$.

Given a graph $\Gamma = (V; E; \epsilon)$, a *chain* is a non-empty graph $P = (V, E, \epsilon)$ with $V = \{x_0, x_1, \ldots, x_k\}$ and $E = \{a_1, a_2, \ldots, a_{k-1} a_k\}$, where x_i, x_{i+1} ($i \bmod k$) are extremities of a_i.

The elements of E must be distinct. The cardinality of E is the *length* of this chain. A graph is *connected* if, for all $x, y \in V$, there exists a chain from x to y.

$\Gamma' = (V'; E'; \epsilon')$ is a *subgraph* of Γ if it is a graph satisfying $V' \subseteq V$, $E' \subseteq E$ and ϵ' is the restriction from ϵ to E'. If $V' = V$ then Γ' is a *spanning subgraph*.

An *induced subgraph* generated by A, $\Gamma(A) = (A; U; \epsilon)$, with $A \subseteq V$ and $U \subseteq E$ is a subgraph such as $U = \{a \in E, \epsilon(a) = [x; y], x, y \in A\}$.

An induced subgraph such that any pair of vertices are adjacent is called a *clique*. Let $\Gamma = (V; E; \epsilon)$ be a graph; a *component* of Γ is a maximal connected induced subgraph.

Let $\Gamma_1 = (V_1; E_1; \epsilon_1)$ and $\Gamma_2 = (V_2; E_2; \epsilon_2)$ be two graphs, a *morphism* from $\Gamma_1 = (V_1; E_1; \epsilon_1)$ to $\Gamma_2 = (V_2; E_2; \epsilon_2)$ is a couple $(f, f^\#)$ where $f : V_1 \longrightarrow V_2$ is a map and $f^\# : E_1 \longrightarrow E_2$ is a map such that:

$$\text{if } \epsilon_1(a) = [x; y] \quad \text{then} \quad \epsilon_2(f^\#(a)) = [f(x); f(y)].$$

So (id_V, id_E) is a morphism from $G = (V; E; \epsilon)$ to G.

The *product* of two morphisms $(f, f^\#)$ and $(g, g^\#)$ is defined by: $(f, f^\#) \circ (g, g^\#) := (f \circ g, f^\# \circ g^\#)$. $(f, f^\#)$ is an *isomorphism* if there exists a morphism $(g, g^\#)$ from $\Gamma_2 = (V_2; E_2; \epsilon_2)$ to $\Gamma_1 = (V_1; E_1; \epsilon_1)$ such that $(g, g^\#) \circ (f, f^\#) = (id_{V_1}, id^\#_{E_1})$ and $(f, f^\#) \circ (g, g^\#) = (id_{V_2}, id^\#_{E_2})$. In this case we will define $(g, g^\#) = (f, f^\#)^{-1}$. So $(f, f^\#)$ is an isomorphism if and only if f is a bijection and $f^\#$ is a bijection. If there exists an isomorphism between Γ_1 and Γ_2 we will define $\Gamma_1 \simeq \Gamma_2$ and we will say that Γ_1 is *isomorphic* to Γ_2.

A graph $\Gamma = (V; E; \epsilon)$ is *k-partite* if there is a partition of V in k-parts such that any part does not contain any edge other than loops. We will write $\Gamma = (\sqcup_{i \in I} V_i; E; \epsilon)$, $card(I) = k$. A graph is *minimum k-partite*, $k \geq 2$, if it is not $(k-1)$-partite. It is easy to verify that for any graph Γ there exists k such that Γ is minimum k-partite. If a graph Γ is k-partite we will say that $(V_i)_{i \in \{1,2,\ldots,k\}}$ is a *k-representation* of Γ and we will call $(\Gamma, (V_i)_{i \in \{1,2,\ldots,k\}})$ a *k-graph*.

A k-graph morphism $(f, f^\#)$ is morphism from a k_1-graph $(\Gamma_1, (V_{1,i})_{i \in \{1,2,\ldots,k_1\}})$ to a k_2-graph $(\Gamma_2, (V_{2,j})_{j \in \{1,2,\ldots,k_2\}})$ verifying the following property:

For all $i \in \{1, 2, \ldots, k_1\}$, there exists $j \in \{1, 2, \ldots, k_2\}$ such that $f(V_{1,i}) \subseteq V_{2,j}$.

2.2. Group definitions

In this paper, groups are also finite. We denote the unit element by e. Let G be a group, and let $S = \{s_1, s_2, \ldots, s_k\}$, a non-empty set of elements of G. S is a *set of generators* from G if any element $\theta \in G$ can be written in the following way: $\theta = s_{i_1} s_{i_2} s_{i_3} \ldots s_{i_t}$ with $i_1, i_2, \ldots i_t \in \{1, 2, \ldots, k\}$. We say that G is *generated* by $S = \{s_1, s_2, \ldots, s_k\}$ and we write $G = \langle s_1, s_2, \ldots, s_k \rangle$. A group G is said to act in the left way on a space Ω when the following operation $(a, x) \longrightarrow a.x$ from $G \times \Omega$ to Ω verifies:

- $e.x = x$.
- $a.(b.x) = (a.b).x.$, $a, b \in G$ and $x \in \Omega$.

Let H be a subgroup of G; we use Hx instead of $H\{x\}$. The set Hx is called *right coset* of H in G. A subset T_H of G is said to be a *right transversal* for H if $\{Hx, x \in T_H\}$ is precisely the set of all cosets of H in G.

An S-group is a couple (G, S) where G is a finite group and S is a subset of G. An S-group morphism between (G_1, S_1) and (G_2, S_2) is a morphism f from G_1 to G_2 such that $f(S_1) \subseteq S_2$.

3. Group to graph process

3.1. Mathematical definition

Let (G, S) be a group with a set of generators $S = \{s_1, \; s_2, \; s_3 \ldots \; s_k\}$, $k \geq 1$. For any $s \in S$, we consider the left action of the subgroup $H = \langle s \rangle$ on G. So we have a partition $G = \bigsqcup_{x \in T_s} \langle s \rangle x$, where T_s is a right transversal of $\langle s \rangle$. The cardinality of $\langle s \rangle$ is $o(s)$, the order of the element s.

Let us consider the cycles

$$(s)x = (x, sx, s^2x, \ldots, s^{o(s)-1}x)$$

of the permutation $g_s \colon x \longmapsto sx$ of Σ_G. Hence $\langle s \rangle x$ is the support of the cycle $(s)x$. Notice that just one cycle of g_s contains the unit element e, namely $(s)e = (e, s, s^2, \ldots s^{o(s)-1})$. We define a graph denoted by $\Phi(G; S) = (V; E; \epsilon)$ in the following way:

- The vertices of $\Phi(G; S)$ are the cycles of g_s, $s \in S$, i.e., $V = \sqcup_{s \in S} V_s$ with $V_s = \{(s)x, x \in T_s\}$.
- For each $(s)x, (t)y \in V$, if $card(\langle s \rangle x \; \cap \langle t \rangle y) = p$, $p \geq 1$ then $\{\langle s \rangle x, \langle t \rangle y\}$ is a p-edge.

Thus, $\Phi(G; S)$ is a k-partite graph and any vertex has a $o(s)$-loop. We denote as $\widetilde{\Phi}(G; S)$ the graph $\Phi(G; S)$ without loops. By construction, one edge stands for one element of G. We can remark that one element of G labels several edges. Both graphs $\Phi(G; S)$ and $\widetilde{\Phi}(G; S)$ are called *graphs from groups* or *G-graphs* and we can say that the graphs are *generated* by the groups $(G; S)$. If $S = G$, the G-graph is called the *canonic graph*.

3.2. Algorithmic procedure

The following procedure constructs a graph from a group G and a subset S of G. A list of vertices and a list of edges represent the graph:

Procedure Group_To_Graph(G, S)
  Data:
    G a group
    $S = \{s_1, s_2, s_3, \ldots, s_k\}$, a subset of G
Cycles computing
  $L = \emptyset$
  for all a in S
    $l_2 = \emptyset$
    $g_s = \emptyset$
    for all x in G
      if x not in l_2 **then**
        $l_1 = \emptyset$
        for $k = 0$ to $k = Order(a) - 1$
          Add $(a^k) \times x$ to l_1
          Add $(a^k) \times x$ to l_2
        end for
        Add l_1 to g_s
      end if

end for
Add g_s to L
end for
Graph computing
for all s in L
Add s to V
for all s' in L
for all x in s
for all y in s'
if $x = y$ **then**
Add (s, s') to E
end if
end for
end for
end for
end for
Return (V, E)

3.3. Complexity and example

It is easy to see that the complexity of our implementation is $O(n^2 \times k^2)$ where n is the order of the group G and k is the cardinal of the family S.

Let G be the Klein group, the product of two cyclic groups of order 2. So $G = \{e, a, b, ab\}$ with $o(a) = 2$, $o(b) = 2$ and $ab = ba$. The set $S = \{a, b, ab\}$ is a family of generators of G. Let us compute the graph $\widetilde{\Phi}(G; S)$.

The cycles of the permutation g_a are

$$(a)e = (e, ae) = (e, a)$$
$$(a)b = (b, ab).$$

The cycles of the permutation g_b are

$$(b)e = (e, be) = (e, b)$$
$$(b)a = (a, ba) = (a, ab).$$

The cycles of the permutation g_{ab} are

$$(ab)e = (e, abe) = (e, ab)$$
$$(ab)a = (a, aba) = (a, b).$$

The graph $\widetilde{\Phi}(G; S)$ is isomorphic to the octahedral graph (see Fig. 1). The octahedral graph is a three-partite symmetric quartic graph.

4. Basic properties of G-graphs

Proposition 4.1. *Let (G, S) be a group with card$(S) = k$; the graph $\widetilde{\Phi}(G; S)$ is a k-graph.*

Proof. It is sufficient to show that $\widetilde{\Phi}(G; S)$ has a clique with a cardinality equal to k. Take a cycle of g_{s_1}, for example $(s_1)x$. This cycle contains an element (at least) which is contained in a cycle of g_{s_2}; this one is contained in a cycle of g_{s_3}, ... and this one is contained in a cycle of

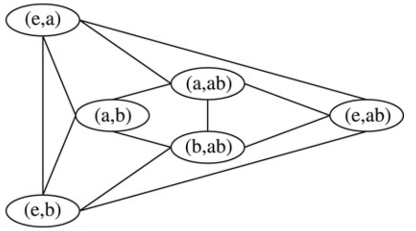

Fig. 1. The octahedral graph.

g_{s_k}. Hence this set of cycles forms a clique in $\widetilde{\Phi}(G; S)$ and the chromatic number of this graph is k. ■

Proposition 4.2. *Let $\Phi(G; S) = (\sqcup_{s\in S} V_s; E; \epsilon)$ be a G-graph with card$(S) = k$. We have the following properties:*

- *For all $v \in V_s$, $d(v) = o(s)(k + 1)$.*
- *For all $s \in S$ we have $\Sigma_{v\in V_s} d(v) = o(G)(k + 1)$.*

Proof. For each $v \in V_s$ we have $v = (s_i)x$; the cardinality of this cycle is $o(s_i)$. Moreover any element of $(s_i)x$ is exactly in one of the cycles of the permutation g_{s_j} for each $j \in \{1, 2, \dots, k\}$. Hence and from the definition of $\Phi(G; S)$ we have $d(v) = o(s)(k + 1)$. So we have

$$\sum_{v\in V_s} d(v) = \frac{o(G)}{o(s)} o(s)(k + 1) = o(G)(k + 1).$$

Consequently the edge number of $\Phi(G; S)$ is $\frac{k(k+1)o(G)}{2}$. ■

Proposition 4.3. *Let $\Phi(G; S) = (V; E; \epsilon)$ be a G-graph. This graph is connected if and only if S is a generator set of G.*

Proof. If $card(S) = 1$, the graph is connected if it has just one vertex $v = (e, s, s^2, \dots, s^{o(s)-1})$, so that the graph is connected if and only if $G = \langle s\rangle$. Assume that $card(S) \geq 2$. Let $(s)x \in V_s$ and $(s')y \in V_{s'}$. Because $G = \langle S\rangle$, there exists $s_1, s_2, s_3, \dots, s_n \in S$ such that $y = s_1 s_2 s_3 \dots s_n x$.

$$\begin{aligned}
&x \in \langle s\rangle x \ \cap \langle s_n\rangle x,\\
&s_n x \in \langle s_n\rangle x \ \cap \langle s_{n-1}\rangle s_n x,\\
&s_{n-1}s_n x \in \langle s_{n-1}\rangle s_n x \ \cap \langle s_{n-2}\rangle s_{n-1}s_n x,\\
&\dots\\
&s_2 \dots s_n x \in \langle s_2\rangle s_3 \dots s_n x \ \cap \langle s_1\rangle s_2 \dots s_n x,\\
&y \in \langle s_1\rangle s_2 \dots s_n x \ \cap \langle s'\rangle y.
\end{aligned}$$

Consequently there exists a chain from $(s)x \in V_s$ to $(s')y \in V_{s'}$. So $\Phi(G; S)$ is a connected graph.

Conversely, let $x \in G$. There exists $s_{i_1} \in S$ and $x_{i_1} \in T_{s_{i_1}}$ such that $x \in (s_{i_1})x_{i_1}$; hence $x = s_{i_1}^{t_{i_1}} x_{i_1}$. We have $e \in (s_{i_1})$. The graph is connected if there exists a chain from $x \in (s_{i_1})x_{i_1}$ to $e \in (s_{i_1})$; hence

$$x = s_{i_1}^{t_{i_1}} x_{i_1},\ x_{i_1} = s_{i_2}^{t_{i_2}} x_{i_2}, \dots,\ x_{i_{k-2}} = s_{i_{k-1}}^{t_{i_{k-1}}} x_{i_{k-1}},\ x_{i_{k-1}} = s_{i_k}^{t_{i_k}} x_{i_k}.$$

But $x_{i_k} = e$, so $x = s_{i_1}^{t_{i_1}} s_{i_2}^{t_{i_2}} \dots s_{i_k}^{t_{i_k}}$. ■

Corollary 4.4. *If k is odd and S is a generator set of G, then $\Phi(G; S)$ is eulerian.*

Proof. Indeed from proposition below the graph is connected; moreover for all vertices x of the graph we have $d(x) = p_s(k+1)$. If k is odd then $k+1$ is even, and consequently $d(x)$ is even. ■

Proposition 4.5. *Let h be a morphism between (G_1, S_1) and (G_2, S_2); then there exists a morphism, $\Phi(h)$, between $\Phi(G_1; S_1)$ and $\Phi(G_2; S_2)$.*

Proof. We define $\Phi(h) = \phi = (f, f^{\#})$ in the following way:

- $f \colon \sqcup_{s \in S_1} V_{1,s} \longrightarrow \sqcup_{s \in S_2} V_{2,s}$, with $(s)x \longmapsto (h(s))h(x)$.
- $f^{\#} \colon E_1 \longrightarrow E_2$, with $([(s)x, (t)y]; u) \longmapsto ([(h(s))h(x), (h(t))h(y)]; h(u))$.

It is easy to verify that ϕ is a morphism from $\Phi(G_1; S_1)$ to $\Phi(G_2; S_2)$. That leads to any group morphism giving rise to a graph morphism. ■

Moreover we have $\Phi(h \circ h') = \Phi(h) \circ \Phi(h')$, and $\Phi(id_{(G,S)}) = id_{\Phi(G,S)}$; hence if $(G_1; S_1) \simeq (G_2; S_2)$ then $\Phi(G_1; S_1) \simeq \Phi(G_2; S_2)$.

Theorem 4.6. *Let G_1 and G_2 be two abelian groups. These two groups are isomorphic if and only if $\Phi(G_1; G_1)$ and $\Phi(G_2; G_2)$ are isomorphic.*

Proof. From the preceding remark a group isomorphism gives rise to a graph isomorphism. Suppose that $\Phi(G_1; G_1)$ is isomorphic to $\Phi(G_2; G_2)$. These two graphs have the same sequence of degrees. Hence the two groups have the same number of elements of the same order. It is well known that two abelian groups are isomorphic if and only if they have the same number of elements of the same order. That leads to our assertion. ■

Theorem 4.7. *Let (G, S) be a group with $o(G) = n$. Assume that $S = G$. If for all d, $d|n$, there exist $\frac{n}{d}\phi(d)$ d-loops (where ϕ is the Euler characteristic), in $\Gamma = \Phi(G; G)$, then G is a cyclic group.*

Proof. $\forall\, x \in G,\ o(x) = d$, we have $\frac{n}{d}$ vertices with a d-loop; hence, if $o_d = card(\{x,\ o(x) = d\})$ then $\forall\ d|n$, $card(\{v \in V,\ \ v \text{ has a } d\text{-loop}\}) = \frac{n}{d}o_d$. Consequently the d-loop number is $\frac{n}{d}o_d$. By hypothesis one has $\frac{n}{d}o_d \le \frac{n}{d}\phi(d)$, so $n = \sum_{d|n} o_d \le \sum_{d|n} \phi(d) = n$. That leads to $o_d = \phi(d)$, and if $d = n$ then $o_n = \phi(n)$ and G is a cyclic group. ■

5. Infinite class of G-graphs

Proposition 5.1. *Let the dihedral group D_{2n} be the group of presentation*

$$\langle r, s \mid r^n = e, s^2 = e, sr = r^{n-1}s \rangle$$

For $S = \{r, s\}$, the graph $\widetilde{\Phi}(D_{2n}; S)$ of the dihedral group is the complete bipartite graph $K_{2,n}$.

See Fig. 2 for an example.

Proof. Let us compute the n-cycles of $V_r = \{(r)x;\ x \in G\}$. There are two of them:

$$(r)e = (e, r, r^2, \ldots, r^{n-1})$$
$$(r)s = (s, rs, r^2s, \ldots, r^{n-1}s).$$

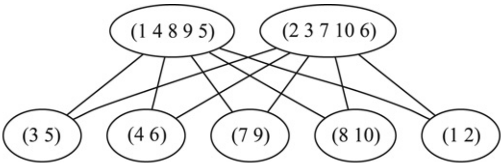

Fig. 2. $\widetilde{\Phi}(D_{10}; \{a, b\})$.

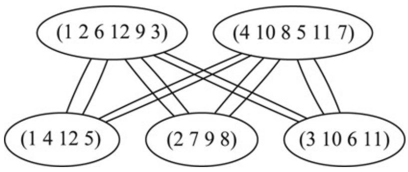

Fig. 3. $\widetilde{\Phi}(Q_3; \{a, b\})$.

And let us compute the 2-cycles of V_s. There are n of them:

$$(s)e = (e, s)$$
$$(s)r = (r, sr) = (r, r^{n-1}s)$$
$$(s)r^2 = (r^2, sr^2) = (r^2, r^{n-2}s)$$
$$\dots$$
$$(s)r^{n-1} = (r^{n-1}, sr^{n-1}) = (r^{n-1}, rs).$$

The cardinality of S is equal to 2, so $\widetilde{\Phi}(D_{2n}; S)$ is a bipartite graph. We must show that it is a complete graph, i.e. for all (x, y), x in V_r and y in V_s, there is exactly one edge between x and y.

For all y in V_s there is i in $\{0, 1, \dots, n-1\}$ such that $y = (s)r^i$. But we have

$$\langle s\rangle r^i = \{r^i, r^{n-i}s\}.$$

If $x = (r)s$, because $\langle r\rangle s = (s, rs, r^2s, \dots, r^{n-1}s)$, we have

$$\langle r\rangle s \cap \langle s\rangle r^i = \{r^{n-i}s\}.$$

If $x = (r)e$, because $\langle r\rangle e = (e, r, r^2, \dots, r^{n-1})$, we have

$$\langle r\rangle e \cap \langle s\rangle r^i = \{r^i\}.$$

So, for all x in V_r and y in V_s, there is exactly one edge between x and y. ■

Proposition 5.2. *Let the generalized quaternion group Q_n be the group of presentation*

$$\langle a, b \mid a^{2n} = e, b^2 = a^n, ab = ba^{2n-1}\rangle.$$

For $S = \{a, b\}$, the graph $\widetilde{\Phi}(Q_n; S)$ of the generalized quaternion group is the complete double-edged bipartite graph $K^2_{2,n}$.

See Fig. 3 for an example.

Proof. First notice that $ba = a^{2n-1}b$, $b^2a = a^na = a^{n+1}$ and $b^3a = ba^{n+1} = a^{2n-(n+1)}b = a^{n-1}b$. More generally we can say

$$ba^i = a^{2n-i}b$$

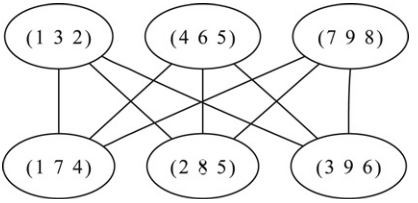

Fig. 4. $\widetilde{\Phi}(C_3 \times C_3; \{a, b\})$.

$$b^2 a^i = a^{n+i}$$
$$b^3 a^i = a^{n-i} b.$$

Let us compute the $2n$-cycles of V_a. There are 2 of them:

$$(a)e = (e, a, a^2, \dots, a^{2n-1})$$
$$(a)b = (b, ab, a^2 b, \dots, a^{2n-1} b).$$

Let us compute the 4-cycles of V_b. There are n of them:

$$(b)e = (e, b, b^2, b^3) = (e, b, a^n, a^n b)$$
$$(b)a = (a, ba, b^2 a, b^3 a) = (a, a^{2n-1} b, a^{n+1}, a^{n-1} b)$$
$$(b)a^2 = (a^2, ba^2, b^2 a^2, b^3 a^2) = (a^2, a^{2n-2} b, a^{n+2}, a^{n-2} b)$$
$$\dots$$
$$(b)a^{n-1} = (a^{n-1}, ba^{n-1}, b^2 a^{n-1}, b^3 a^{n-1}) = (a^{n-1}, a^{n+1} b, a^{2n-1}, ab).$$

The cardinality of S is equal to 2, so $\widetilde{\Phi}(Q_n; S)$ is a bipartite graph. We must show that for all $x \in V_a$, for all $y \in V_b$, there are exactly two edges between x and y.

For all $y \in V_b$ there exists $i \in \{0, 1, \dots, n-1\}$ such that $y = (b)a^i$. So we have

$$(b)a^i = (a^i, ba^i, b^2 a^i, b^3 a^i) = (a^i, a^{2n-i} b, a^{n+i}, a^{n-i} b)$$

a^i and a^{n+1} are in $\langle a \rangle e$. $a^{2n-i} b$ and $a^{n-i} b$ are in $\langle a \rangle b$. So, for all $y \in V_b$, there are exactly two edges between y and $(a)e$, and two edges between y and $(a)b$. ■

Proposition 5.3. *Let $C_n \times C_k$ be the product of two cyclic groups. Such a product is generated by two elements, a and b, with $a^n = b^k = e$. More precisely, $C_n \times C_k$ is the group of presentation*

$$\langle a, b \mid a^n = e, b^k = e, ab = ba \rangle.$$

For $S = \{a, b\}$, the graph $\widetilde{\Phi}(C_n \times C_k; S)$ of the product of two cyclic groups, is the complete bipartite graph $K_{n,k}$..

See Fig. 4 for an example.

6. The bipartite case

Recall that a graph is *simple* if it has neither loops nor multiedges. Let $\Gamma = (V_1, V_2; E)$ be a bipartite simple graph; $\varphi = (f, f^{\#}) \in Aut_p(\Gamma)$ (where $Aut_p(\Gamma)$ stands for the set of automorphisms such that $f(V_i) = V_j,\ i, j \in \{1, 2\}$) give rise to a bijection of E:

$$e = [x, y] \in E \mapsto \varphi(e) = [f(x), f(y)] \in E.$$

So $Aut_p(\Gamma)$ acts on E, on V_1, on V_2.

A k-partite graph $\Gamma = (\bigsqcup_{i \in I} V_i; E; \varepsilon)$ is *semi-regular* if for every $i \in I : x, y \in V_i \Rightarrow d(x) = d(y)$.

Theorem 6.1. (1) *Let:* $(G; S)$ *with* $S = \{s, t\}$ *and* $\langle s \rangle \cap \langle t \rangle = \{id\}$, $\widetilde{\Phi}(G; S) = (V_s \sqcup V_t; E; \varepsilon)$. *For* $g \in G$ *we define* $\delta(g) = (\delta_g, \delta_g^{\#})$ *by:*

- $\delta_g((s)x) = (s)xg^{-1}, s \in S, x \in G;$
- *if* $e = ([(s)x, (t)y], u) \in E : \delta_g^{\#}(e) = ([(s)xg^{-1}, (t)yg^{-1}], ug^{-1})$.

Hence we have the following properties:

(a) $\widetilde{\Phi}(G; S)$ *is a simple, bipartite, semi-regular connected graph.*
(b) $\triangle = \{\delta(g), g \in G\}$ *is a subgroup of* $Aut_p \widetilde{\Phi}(G; S)$.
(c) $\triangle$ *acts transitively on* V_s, *on* V_t *and on* E.
(d) *For every* $v \in V_s \sqcup V_t$, $Stab_{\triangle}(v)$ *is a cyclic group.*

Conversely suppose $\Gamma = (V_1 \sqcup V_2; E; \varepsilon)$ *is a simple, bipartite, semi-regular connected graph with a subgroup* $\triangle$ *of* $Aut_p \Gamma$ *such that:*

(i) $\triangle$ *is acting transitively on* V_1 *and on* V_2.
(ii) *For every* $v \in V_1 \sqcup V_2$, $Stab_{\triangle}(v)$ *is a non-trivial cyclic group.*
(iii) $\triangle$ *is acting transitively on* E.
(iv) *There is an edge* $\{x_1, x_2\}$ *in* Γ *such that* $Stab_{\triangle}(x_1)$ *and* $Stab_{\triangle}(x_2)$ *are different and*

$$\frac{o(G)}{|Stab_{\triangle} x_1|} = |V_1|, \quad \frac{o(G)}{|Stab_{\triangle} x_2|} = |V_2|.$$

Then there exists $(G; S)$ *such that* $\Gamma \simeq_p \widetilde{\Phi}(G; S)$.

Proof. A 2-edge between $(s)x$ and $(t)y$ implies $card(\langle s \rangle \cap \langle t \rangle) \geq 2$. We have $\delta_{g-1}(V_s) \subset V_s$; and $g \mapsto \delta(g)$ is a morphism. If $e = ([(s)x, (t)y], u), e' = [(s)x', (t)y', u'] \in E$, then $s^i x = t^j y = u, s^{i'} x' = t^{j'} y' = u'$ for $i, j, i', j' \in \mathbf{N}$: we verify that for $g = u'^{-1}u$, we have $\delta_g^{\#}(e) = e'$, so $\triangle$ acts transitively (and regularly) on E. $\text{Stab}_{\triangle}((s)id) = \langle \delta(s) \rangle$ and the other $\text{Stab}_{\triangle}$s are conjugate with this one.

Conversely, we choose $G = \triangle$ and for $i = 1, 2$ we fix $x_i \in V_i$ and σ_i a generator of $\text{Stab}_{\triangle}(x_i)$. Let $S = \{\sigma_1, \sigma_2\}$; with a vertex $(\sigma_1)\rho$ of $\widetilde{\Phi}(G; S)$ we associate the vertex $\rho^{-1}(x_1)$ of V_1: this gives a p-isomorphism between $\widetilde{\Phi}(G; S)$ and Γ. ■

Recall that the *Cayley graph* $\text{Cay}(G; A)$ associated with a group G and $A = A^{-1} \subset G$ has for vertices the elements of G, with an edge between x and y if and only if there exists $a \in A$ such that $y = ax$. The *line graph* $L(\Gamma)$ associated with a simple graph Γ has for vertices the edges of Γ, two vertices being adjacent if and only if the corresponding edges in Γ are adjacent.

Proposition 6.2. *Let* $S = \{s, t\}$ *with* $\langle S \rangle = G$ *and* $\langle s \rangle \cap \langle t \rangle = \{e\}$. *Then* $\Gamma = \widetilde{\Phi}(G; S)$ *is a simple graph and* $L(\Gamma) \simeq Cay(G; A)$ *where* $A = (\langle s \rangle \cup \langle t \rangle) \backslash \{e\}$.

Proof. Every element of the set V of vertices of $L(\Gamma)$ is an edge $a = ([(s)x, (t)y], u)$ of Γ with $\langle s \rangle x \cap \langle t \rangle y = \{u\}$ so $\theta : G \to V,\ \theta(u) = a$ is a bijection; if $a' = ([(s)x, (t)y'], u') \in V$ we have $u = s^i x, u' = s^{i'} x$ (with $i, i' \in \mathbf{N}$) and hence $u' = s^{i'-i} u$ with $a = s^{i'-i} \in A$, and u and u' are adjacent in $\text{Cay}(G; A)$; the converse is easy. ■

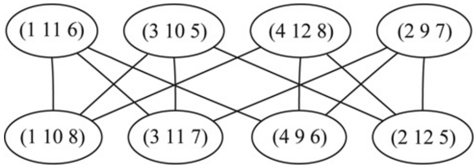

Fig. 5. $\widetilde{\Phi}(A_4; S)$.

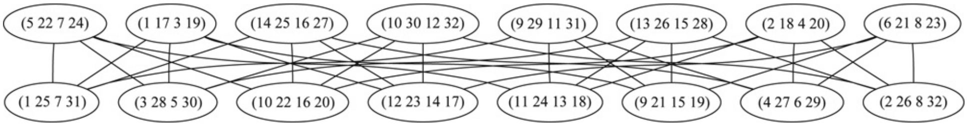

Fig. 6. $\widetilde{\Phi}$(SmallGroup(32, 6); S).

7. Experimental results

7.1. How to recognize a G-graph

Given a G-graph Γ, an interesting problem is how to find a group G and a family S such that $\widetilde{\Phi}(G; S)$ is isomorphic to Γ. If both G and S exist, we say that Γ is a G-graph. Here, we use the `SmallGroups` library from GAP (Groups, Algorithms, and Programming (The GAP Team, 2002)). This library gives us access to all groups of certain small orders. The groups are sorted by their orders and they are listed up to isomorphism. Currently, the library contains the groups of order at most 2000 except 1024 (423 164 062 groups). In this section, we prove that many usual graphs are G-graphs and we exhibit their corresponding groups.

The cube — Let us consider the skeleton of a cube. It is a graph with 8 vertices and 12 edges. All vertices are of degree 3 and the graph is bipartite. Suppose the cube is a G-graph $\widetilde{\Phi}(G; S)$. Then the corresponding group G is of order 12 and is generated by a family S of cardinality 2, because the graph is bipartite. The alternate group with 12 elements, A_4, a subgroup of S_4, is generated by the two cycles $(1, 2, 3)$ and $(1, 3, 4)$. Let S be the family $\{(1, 2, 3), (1, 3, 4)\}$. If we compute the graph $\widetilde{\Phi}(A_4; S)$ with our algorithm we find the graph depicted in Fig. 5.

It is easy to check that this graph is isomorphic to the cube. Thus, the cube is a G-graph as expected.

The hypercube — Let us consider the skeleton of a hypercube of dimension 4. It is a graph with 16 vertices and 32 edges. All vertices are of degree 4 and the graph is bipartite. Suppose the hypercube is a G-graph $\widetilde{\Phi}(G; S)$. Then the corresponding group G is of order 32 generated by a family S of cardinal 2, because the graph is bipartite. The order of the elements of the family S must be 4 because the vertex degree is 4. If we look at the library `SmallGroups` we find 51 groups of order 32. Only seven groups of order 32 can be generated by two elements of order 4: the groups number 2, 6, 10, 11, 13, 14 and 20. If we compute the corresponding graphs with our algorithm we find that `SmallGroup(32,6)` matches (see Fig. 6).

7.2. A short list of G-graphs

Many common graphs are G-graphs. Here is a short list of well-known graphs that are G-graphs. The corresponding groups are indicated, most of the time by a reference to the `SmallGroups` library.

- Bipartite complete graphs ($G = C_n \times C_k$, $S = \{(1, 0), (0, 1)\}$)
- Cycles of even length ($G = D_n$, S is constituted with two symmetries)
- The octahedral graph ($G = C_2 \times C_2$, $S = \{(1, 0), (0, 1), (1, 1)\}$)
- The cuboctahedral graph
  ($G = C_2 \times C_2 \times C_2$, $S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$)
- The square (G is the Klein's group, $G = \{e, a, b, ab\}$, and $S = \{a, b\}$)
- The cube ($G = A_4$, $S = \{(1, 2, 3), (1, 3, 4)\}$)
- The hypercube ($G =$ `SmallGroup(32,6)`, $S = \{f1, f1 * f2\}$)
- The 2×2 grid on a torus ($G = Q_2$, $S = \{a, b\}$)
- The 3×3 grid on a torus ($G = D_6$, $S = \{s \in G; Ordre(S) = 2\}$)
- The 4×4 grid on a torus ($G =$ `SmallGroup(32,6)`, $S = \{f1, f1 * f2\}$)
- The Heawood graph ($\langle a, b \mid a^7 = b^3 = e, ab = baa\rangle$, $S = \{b, ba\}$)
- The Pappus graph
  ($G = \langle a, b, c \mid a^3 = b^3 = c^3 = e, ab = ba, ac = ca, bc = cba\rangle$, $S = \{b, c\}$)
- The Mobius–Kantor graph ($G =$ `SmallGroup(24,3)`, $S = \{f1, f1 * f2\}$)
- The Gray graph ($G =$ `SmallGroup(81,7)`, $S = \{f1, f2\}$)
- The Ljubljana graph ($G =$ `SmallGroup(168,43)`, $S = \{f1, f1 * f2 * f4\}$)

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