On the weight hierarchy of the semiprimitive codes

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Abstract

An irreducible cyclic \((n, k)\) code is said to be semiprimitive if \(n = (2^k - 1)/N\) where \(N > 2\) divides \(2^j + 1\) for some \(j \geq 1\). The complete weight hierarchy of the semiprimitive codes is determined when \(k/2j\) is odd. In the other cases, when \(k/2j\) is even, some partial results on the generalized Hamming weights of the semiprimitive codes are obtained. We apply the above results to find the generalized Hamming weight of some classes of dual codes of primitive BCH codes with designed distance \(N + 2\) when \(k/2j\) is odd.

1. Introduction

Let \(F = GF(2^k)\) be a finite field with \(2^k\) elements and let \(\psi\) be a generator of the multiplicative group \(F^* = F \setminus \{0\}\).

Let \(h(x) \in GF(2)[x]\) be an irreducible polynomial of degree \(k\) and period \(n\). Then any irreducible \((n, k)\) code \(C\) over \(GF(2)\) can be described as

\[ C = \{ c(a) | c(a) = (Tr(a), Tr(a\beta), \ldots, Tr(a\beta^{n-1})) , a \in F \} \]

where \(\beta\) is a zero of \(h(x)\) and \(Tr(x)\) denotes the trace function from \(GF(2^k)\) to \(GF(2)\). Note that \(k\) is the multiplicative order of 2\(\pmod{n}\).

An irreducible cyclic code is said to be semiprimitive if \(\beta = \psi^N\) where \(N > 2\) and \(N|2^j + 1|2^k - 1\) for some integer \(j \geq 1\). Observe that in this case \(k\) is even and \(2j|k\). The length of the code \(C\) is \(n = (2^k - 1)/N\).

For \(N|2^k - 1\) we define

\[ P_l = \{ f \in F \mid f = \psi^l, l \equiv i \pmod{N} \} \]

Then \(P_0\) is the set of nonzero \(N\)th powers in \(F\) and \(P_l = \psi^lP_0\) for \(0 \leq i \leq N - 1\).

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Baumert and McEliece [1] computed the weight distribution of the semiprimitive codes as given in the following result.

**Lemma 1.** Let $C$ be a semiprimitive code where $N > 2$ and $N|2^j + 1|2^k - 1$ for some $j \geq 1$. Then $C$ is a two-weight code and the Hamming weight, $w(c(a))$, of a nonzero codeword $c(a)$ is given by

$$
w(c(a)) = \begin{cases} 
(2^{k-1} - (−1)^{(k/2)j+1}(N - 1)2^{(k/2)−1})/N & \text{if } a \in P_0, \\
(2^{k-1} - (−1)^{(k/2)j}2^{(k/2)−1})/N & \text{if } a \in F^* \setminus P_0,
\end{cases}
$$

where $P_0$ is the set of nonzero $N$th powers in $F^*$.

There is a vectorspace isomorphism from $F$ to $C$ given by $a \rightarrow c(a)$. It is useful to observe that in the case when $k/2j$ is odd the vectors of $C$ of minimum weight correspond exactly to $a \in P_0$ while the maximum weight vectors correspond to $a \in F^* \setminus P_0$. In the case when $k/2j$ is even the situation is the opposite.

2. The weight hierarchy

For any code $D$, let $\chi(D)$ be the support of $D$, i.e., the set of positions where not all of the codewords of $D$ are zero. The $r$th generalized Hamming weight of a code $C$ is defined by

$$d_r(C) = \min \{|\chi(D)||D\text{ is an } r\text{-dimensional subcode of } C\}.$$

The weight hierarchy of a code $C$ is the set of generalized Hamming weights $\{d_r(C)\}, 1 \leq r \leq k$.

The weight hierarchy has been determined for the Golay code, Reed-Muller codes by Wei [13], for codes meeting the Griesmer bound by Helleseth et al. [6].

For the BCH codes very little is known. It has been shown by Feng et al. [5] that $d_2 = 8$ for all binary primitive double-error-correcting codes. Van der Geer and van der Vlugt [11] proved that $d_3 = 10$ for all binary primitive double-error-correcting BCH codes and $d_2 = 11$ in the triple-error-correcting case. Kabatianski [8] proved that $d_2 = 3t + 2$ for all sufficiently long $t$-error-correcting primitive BCH codes.

The generalized Hamming weights of the dual of the BCH codes are studied by Chung [3], Duursma et al. [4], and van der Geer and van der Vlugt [10, 12].

A simple, but very useful observation by Helleseth and Kumar [7] and van der Geer and van der Vlugt [12] is that for any $r$-dimensional subcode $D$ of $C$ it holds that

$$|\chi(D)| = \frac{1}{2^{r-1}} \sum_{d \in D} w(d). \quad (1)$$

Hence, to find the $r$th generalized Hamming weight of a code, it is enough to find the smallest sum of the weights for any $r$-dimensional subcode. In particular, if we can
find an $r$-dimensional subcode where all nonzero codewords have minimum weight $d$, this subcode will have support size equal to $d_r = (2^r - 1)d/2^{r-1}$.

**Lemma 2.** Let $C$ be a semiprimitive code where $N > 2$ and $N|2^j + 1|2^k - 1$ for some $j \geq 1$. Let $m = m(N, k)$ be the largest divisor of $k$ such that $\gcd(N, 2^m - 1) = 1$. Let $k_0$ be the greatest odd divisor of $k/2j$. Then $m \geq k_0j$ and for $1 \leq r \leq m$ the generalized Hamming weight $d_r$ of $C$ equals

$$d_r = (2^r - 1)d/2^{r-1}$$

where $d$ is the minimum distance of $C$.

**Proof.** Since $\gcd(N, 2^m - 1) = 1$ and $m|k$, it follows that all the elements in the subfield $GF(2^m)$ of $F$ are $N$th powers. Therefore the nonzero vectors of the $m$-dimensional subspace $V_0 = GF(2^m)$ are contained in $P_0$. Further, the nonzero vectors of the $m$-dimensional subspace $V_1 = \psi GF(2^m)$ are contained in $P_1$ which is a subset of $F^*/P_0$. Hence, from Lemma 1 it follows that there exist a subspace $V = V_0$ or $V = V_1$ such that

$$D = \{c(a) | a \in V\}$$

is an $m$-dimensional subcode of $C$ which contains $2^m - 1$ nonzero vectors of minimum weight $d$. Therefore for $1 \leq r \leq m$, it follows from (1) that

$$d_r = (2^r - 1)d/2^{r-1}.$$
Since $V_i$ is a vector space of dimension $m$, any $r$-dimensional subspace $D_r$ of $GF(2^m)$ contains at least $2^r - m - 1$ vectors from $V_i^*$ for all $i$, $0 \leq i \leq 2^m$. Hence, any $r$-dimensional subspace of $GF(2^m)$ contains at least $(N-1)((2m+1)/N)(2^r - m - 1)$ elements from $GF(2^m)^* \setminus P_0$, i.e., the elements from $V_i^*$ where $0 \leq i \leq 2^m$ and $i \not\equiv 0 \pmod{N}$. Since the mapping $a \rightarrow c(a)$ gives an isomorphism from $GF(2^m)$ to $C$, it follows from Lemma 1 that any $r$-dimensional subcode of $C$ has at least $(N-1)((2m+1)/N)(2^r - m - 1)$ codewords of the maximum weight in the two-weight code $C$.

To find $d_r$, for $r > m$, it is therefore sufficient to find an $r$-dimensional subcode of $C$ with exactly $(N-1)((2m+1)/N)(2^r - m - 1)$ codewords of the maximum weight and the remaining nonzero codewords of minimum weight in the subcode.

Let $U$ be an $r$-dimensional subspace of $GF(2^m)$ containing $GF(2^m)$. Then $U = \langle u_1, u_2, \ldots, u_{2^m}, f_1, \ldots, f_m \rangle$ where $u_i \in GF(2^m) \setminus GF(2^m)$, $f_j \in GF(2^m)$ for $1 \leq i \leq r-m$, $1 \leq j \leq m$. Then $U$ is a disjoint union of $GF(2^m)$ and $2^r - m - 1$ cosets of the form $u + GF(2^m)$ where $u \in U \setminus GF(2^m)$.

We will show that each such coset $u + GF(2^m)$ where $u \in U \setminus GF(2^m)$ contains $(N-1)2^{m-1}$ elements from $GF(2^m)^* \setminus P_0$ and $(2^m - N - 1)/N$ elements from $P_0$. This follows since for fixed $u \in U \setminus GF(2^m)$ then $u + f \in V_i$ has exactly one solution $f \in GF(2^m)$ for any $i = 1, 2, \ldots, 2^m$. Suppose, $u + f_1 \in V_i$ and $u + f_2 \in V_i$, then $f_1 + f_2 \in V_i$ which implies $f_1 = f_2$ since $1 \leq i \leq 2^m$. Further, since $GF(2^m)$ contains $2^m - 1$ nonzero elements from $P_0$, it follows that the $r$-dimensional subcode $D$ of $C$ given by

$$D = \{ c(u) \mid u \in U \}$$

has the following weight distribution:

- $2^{m-N+1}(2^r - m - 1) + 2^m - 1$ words of weight $d = (2^{2m-1} - (N-1)2^{m-1})/N$,
- $(N-1)2^{m-N+1}(2^r - m - 1)$ words of weight $d + 2^{m-1} = (2^{2m-1} + 2^{m-1})/N$,
- 1 word of weight 0.

In particular, this subcode $D$ contains as few maximum weight codewords as possible. Hence, from (1) we can find $d_r$ by summing over all the weights in $D$, which leads to

$$d_r = \frac{1}{2^{r-1}} \sum_{d \in D} w(d) = (2^r - 1)d/2^{r-1} + \frac{(N-1)(2^m + 1)(1 - 2^{m-r})}{N}$$

which implies the result in the theorem. □

This result was obtained in a different way by van der Vlugt [9].

It is also straightforward to obtain bounds on $d_r$ in the case when $k/2j$ is even, using this method and construct subcodes with as many words as possible of the minimum
weight \(d\). We leave further details to the reader. We have, however, so far been unable to find the complete weight hierarchy in the case where \(k/2j\) is even.

It is interesting to note that the results on the weight hierarchy for the semiprimitive codes when \(k/2j\) are odd, can be applied to find some of the generalized Hamming weights of some dual of BCH codes with designed distance \(N + 2\). No similar connections hold in the case when \(k/2j\) is even. In fact, the following theorem extends some results in Duursma et al. [4], and van der Geer and van der Vlugt [10] on the weight hierarchy of some duals of the BCH codes.

**Theorem 4.** Let \(N > 2\), \(N|2^j + 1|2^k - 1\) and \(k/2j\) odd for some \(j \geq 1\). The dual code of a primitive BCH code of length \(2^k - 1\) and designed distance \(N + 2\) has generalized Hamming weights

\[
d_r = (2^r - 1)d/2^{r-1}
\]

for \(1 \leq r \leq k/2\), where \(d = 2^k - 1 - (N - 1)2^{(k/2)-1}\).

**Proof.** From the Carlitz–Uchiyama [2] bound it follows that for the weight of any codeword in the dual of the primitive BCH code of designed distance \(2t + 1\) it holds that

\[
2^{k-1} - (t - 1)2^{(k/2)} \leq d \leq 2^{k-1} + (t - 1)2^{(k/2)}.
\]

It is important to note that repeating every codeword \(N\) times in a semiprimitive code of length \(n = (2^k - 1)/N\) is a subcode of the dual of a BCH code of designed distance \(N + 2 = 2t + 1\), since a typical codeword in the repeated semiprimitive code is

\[
c(a) = (Tr(a), Tr(a\psi^N), \ldots, Tr(a\psi^{(2^k - 2)N})),
\]

where \(\psi\) is a primitive element in \(GF(2^k)\).

In particular in the semiprimitive case when \(k/2j\) is odd the minimum distance of the repeated semiprimitive code is

\[
2^{k-1} - (t - 1)2^{k/2} = 2^{k-1} - (N - 1)2^{(k/2)-1}
\]

which therefore equals the minimum distance of the dual of the BCH code of designed distance \(N + 2\). Hence, if the semiprimitive code contains a subspace of dimension \(m = m(N, k) = k/2\) of vectors of minimum weight, the same is true for the BCH code. This completes the proof. \(\Box\)

**Remark.** One should observe that combining the arguments in Theorem 4 with Theorem 3 give an upper bound for \(d_r\) for the duals of the BCH codes above also in the case \(k/2 < r \leq k\).

3. Conclusions

We have completely determined the weight hierarchy of the semiprimitive codes in the case \(N > 2\), \(N|2^j + 1|2^k - 1\) and \(k/2j\) odd for some \(j \geq 1\). Clearly, one may apply
this method to obtain bounds for $d_r$ also in the case $k/2j$ is even by constructing subcodes with many vectors of minimum weight $d$. To completely determine the weight hierarchy in the other case when $k/2j$ is even is still an open problem.

A code $C$ is said to satisfy the chain condition if there is a chain of subcodes $D_1 \subset D_2 \subset \cdots \subset D_k = C$ such that $\dim D_i = i$ and $|\chi(D_i)| = d_i$ for $1 \leq i \leq k$. It follows directly from the proof of Theorem 3 that the chain condition holds for these codes.

Finally, we would like to remark that it is straightforward to generalize the results above to nonbinary semiprimitive codes and to the corresponding duals of the nonbinary BCH codes.

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References