Discrete Mathematics 16 (1976) 9-12.
© North-Holland Publishing Company

r-TUPLE COLORINGS OF UNIQUELY COLORABLE GRAPHS

Dennis P. GELLER*

Human Sciences and Technology Group, School of Advanced Technology, State University of New York, Binghamton, NY 13901, U.S.A.

Received 14 August 1974
Revised 8 December 1975

An r-tuple coloring of a graph is one in which r colors are assigned to each point of the graph so that the sets of colors assigned to adjacent points are always disjoint. We investigate the question of whether a uniquely n-colorable graph can receive an r-tuple coloring with fewer than nr colors. We show that this cannot happen for $n = 3$ and $r = 2$ and that for a given n and r, to establish the conjecture that no uniquely n-colorable graph can receive an r-tuple coloring from fewer than nr colors, it suffices to prove it for only a finite set of uniquely n-colorable graphs.

An ***r*-tuple coloring** of a graph G is an assignment to each point of G of an unordered r-tuple of distinct colors such that the r-tuples assigned to two adjacent points have no colors in common. A completely equivalent formulation [2, 3] is to say that an r-tuple coloring of G is a coloring, in the usual sense, of the lexicographic product $G[K_r]$; in fact, we will denote the least number of colors which which one can achieve an r-tuple coloring of G by $\chi(G[K_r])$.

Clearly $\chi(G[K_r]) \le r\chi(G)$. A lower bound, as shown in [2], is $\chi(G[K_r]) \geq \chi(G) + 2r - 2$; the two bounds agree when G is bipartite. It is also known that for all values of $\chi(G) > 2$ and $r \geq 2$ examples can be found where the upper bound is not sharp [2]. In this note we show that if G is uniquely colorable [1] then the lower bound cannot hold for pair colorings and then briefly discuss the more general question of whether the upper bound is in fact always sharp for such graphs.

A family $\{C_1, \ldots, C_n\}$ *covers* a graph G *at least* (*exactly*) r *times* if each point of G belongs to at least (exactly) r of the C_i. Thus, given an n-coloring of G, the n color classes form a family (of independent sets) which covers G exactly once.

Lemma 1. *Let G be uniquely n-colorable. If a collection of n independent sets covers G at least once, then it covers G exactly once.*

Proof. Let $\{C_1, \ldots, C_n\}$ cover G at least once, and let $u \in C_1 \cap C_2$. Note that no set C_i is wholly contained in any collection of the others, for otherwise we could cover G at least once with $n - 1$ sets, yielding $\chi(G) \le n - 1$. But now, consider the following collections:

*Present address: Massachusetts Computer Associates, 26 Princess Street, Wakefield, MA, U.S.A.

$$\mathbf{A} = \left\{C_1, C_2 - C_1, C_3 - (C_1 \cup C_2), \ldots, C_n - \bigcup_1^{n-1} C_i\right\},$$

$$\mathbf{B} = \left\{C_2, C_1 - C_2, C_3 - (C_1 \cup C_2), \ldots, C_n - \bigcup_1^{n-1} C_i\right\}.$$

These are each families of n disjoint independent sets whose union is the point set $V(G)$ of G; i.e., each is a coloring of G. However, the colorings are different, as in $\mathbf{B}$ the point u is colored the same as all points in $C_2 - C_1$, while it is colored differently from these in $\mathbf{A}$. This contradicts the hypothesis that G was uniquely n-colorable.

Theorem 2. *If G is uniquely n-colorable, $n > 2$, then $\chi(G[K_2]) \geq n + 3$.*

Proof. Suppose that $\chi(G[K_2]) = n + 2$. Let $\{C_1, \ldots, C_{n+2}\}$ be a coloring of $G[K_2]$, and let $\{A_1, \ldots, A_{n+2}\}$ be the corresponding projections of the C_i onto G. Then each A_i is an independent set and the collection $\{A_i\}$ covers G twice. Note first that no n of the $\{A_i\}$ can cover G at least once. For, if, for example, $\{A_1, \ldots, A_n\}$ covered G at least once it would cover G exactly once by the lemma. But this would mean that $\{A_{n+1}, A_{n+2}\}$ covered G once, implying that $n = \chi(G) \leq 2$, a contradiction.

Thus, for every i and j, $A_i \cap A_j \neq \emptyset$. We proceed to develop two different colorings of G. Define the sets B_i, $i = 1, \ldots, n$ by

$$B_1 = A_1,$$

$$B_2 = A_2 - A_1,$$

$$B_3 = A_3 - (A_1 \cup A_2),$$

$$\vdots$$

$$B_{n-1} = A_{n-1} - (A_1 \cup \ldots \cup A_{n-2}),$$

$$B_n = (A_n \cup A_{n+1} \cup A_{n+2}) - (A_1 \cup A_2 \cup \ldots \cup A_{n-1}).$$

The B_i cover G.

They are also independent. This is clear for $B_1, \ldots, B_{n-1}$. To see that B_n is also independent note that

$$B_n = (A_n \cap A_{n+1}) \cup (A_n \cap A_{n+2}) \cup (A_{n+1} \cap A_{n+2})$$

and, therefore, that any two points of B_n have common membership in some A_i, each of which is independent. Then, by Lemma 1, the B_i must be disjoint.

Thus, the $\{B_i\}$ yield a coloring of G. But a different coloring of G results when we define $B'_1 = A_2$, $B'_2 = A_1 - A_2$, and $B'_i = B_i$ for $i > 2$. This contradicts the unique colorability of G.

Corollary 3. *If G is uniquely 3-colorable, then $\chi(G[K_2]) = 6$.*

Although this theorem offers minimal support for it, it is tempting to consider the following conjecture:

(*) If G is uniquely n-colorable, then $\chi(G[K_r]) = nr$.

Actually, we can consider (*) to be a family of conjectures, one for each value of $n > 2$ and $r \geq 2$. For $n = 2$ it is already a theorem, and for a given n and all r it is true for those uniquely n-colorable graphs which contain n-cliques. Furthermore, for a given n and r, the truth of (*) depends only on its validity for small graphs.

Theorem 4. *For a given n and r there is a number $f(n, r)$ such that if (*) holds for graphs G with $p(G) \leq f(n, r)$, then (*) holds for all G.*

Proof. By induction. Let G be uniquely n-colorable, where $p = p(G)$ satisfies

$$\binom{\{p/n\}}{2} > \binom{nr-1}{r}$$

and assume that the result holds for graphs with fewer than p points; i.e., that for such graphs G' if G' is uniquely n-colorable then $\chi(G'[K_r]) = nr$. Consider the unique coloring of G from n colors; if two points u and v receive the same color then the result G' of identifying them is also uniquely n-colorable. Furthermore, if in some r-tuple coloring of G, u and v receive the same r-tuple of colors, then $\chi(G'[K_r]) = \chi(G[K_r])$; the chromatic number cannot decrease since the identification of the pseudo-vertices $u[K_r]$ and $v[K_r]$ is a homomorphism, and does not increase since the fact that u and v receive the same r-tuple guarantees that the resulting coloring is valid for G'.

Now, let the coloring of G partition the points into sets $\{V_1, \ldots, V_n\}$. In any of the V_i, we could choose any two points to identify, as above; in fact, there are $\binom{|V_i|}{2}$ possible choices. In particular, by picking the largest of the V_i we are assured of $\binom{\{p/n\}}{2}$ possible choices.

We wish to make a choice which will lead to a graph G' with $\chi(G'[K_r]) = \chi(G[K_r])$. Assume that $\chi(G[K_r]) \leq nr - 1$; then there are at most $\binom{nr-1}{r}$ distinct r-tuples possible. Thus, if we choose p such that

$$\binom{\{p/n\}}{2} > \binom{nr-1}{r}$$

there must be two points in the largest V_i which receive the same r-tuple of colors in some r-tuple coloring. By identifying them, as above, we get a uniquely n-colorable graph G' such that $\chi(G'[K_r]) = \chi(G[K_r])$. But, since G' is smaller than G the induction hypothesis yields $\chi(G[K_r]) = nr$.

A graph G need not be uniquely n-colorable for $\chi(G[K_r])$ to be equal to nr. Stahl [4] has pointed out the following:

Theorem 5 (Stahl). *If the independence number of G satisfies* $\beta_0(G)\chi(G) = p(G)$, *then* $\chi(G[K_r]) = r\chi(G)$.

Proof. Let $\chi(G) = n$ and let $\{C_1, \ldots, C_m\}$ be a coloring of $G[K_r]$. Let A_i be the projection of C_i on G. Then each A_i is independent, so that $|A_i| \leq \beta_0(G)$. Furthermore,

$$rp = \sum_{i=1}^{m} C_i \leq m\beta_0(G).$$

But, since $\beta_0(G) = p/n$, $rp \leq mp/n$, so that $rn \leq m$.

However, since $\chi(G) = n$, $m = \chi(G[K_r]) \leq nr$.

References

[1] G. Chartrand and D.P. Geller, On uniquely colorable planar graphs, J. Combin. Theory 6 (1969) 271–278.

[2] D.P. Geller and S. Stahl, The chromatic number and other functions of the lexicographic product of graphs, J. Combin. Theory 19 (1975) 87–95.

[3] S. Stahl, n-tuple colorings and associated graphs, to appear.

[4] S. Stahl, personal communication.