Inequalities for Two Simplices

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We establish in this paper an inequality for two simplices, which combine altitudes and edge-lengths of one simplex with distances from an interior point to its facets of the other simplex, and give some applications thereof.

1. INTRODUCTION

The well-known Neuberg–Pedoe inequality is the first inequality involving two triangles [14–16]. Following Pedoe, a number of inequalities for two triangles have been established [12; 13, XII]. Yang Lu and Zhang Jingzhong in [18] generalized the Neuberg–Pedoe inequality to \( \mathbb{R}^n \). The research inspired by Yang and Zhang on geometric inequalities for two high-dimensional simplices has been extensive [4, 7–10, 17]. The main aim of this paper is to establish a new inequality involving two simplices which combine edge-lengths and altitudes of one simplex with distances from an interior point to its facet of the other simplex. As applications, we obtain some other inequalities for a simplex, and we also sharpen some inequalities of L. Fejes Tóth and L. Gerber.

We use the following notations throughout this paper. Let \( \Omega \) be an \( n \)-dimensional simplex in \( \mathbb{R}^n \) with vertices \( A_0, A_1, \ldots, A_n \) (i.e., \( \Omega = \langle A_0, A_1, \ldots, A_n \rangle \)) and of volume \( V \), and let \( I, O, \) and \( G \) be the incenter, circumcenter, and centroid of \( \Omega \), respectively. Let \( \Omega_i = \langle A_0, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n \rangle \) be its facet which lies in a hyperplane \( \pi_i \), \( S_i \) the facet area of \( \Omega_i \) (i.e., its \((n-1)\)-dimensional volume), \( d_i \) the distance from an
interior point \( P \) of \( \Omega \) to \( \pi_i \), and \( h_i \) the altitude of \( \Omega \) from the vertex \( A_i \), i.e., the distance from \( A_i \) to \( \pi_i \).

Our main results are the following three theorems.

**Theorem 1.** Let \( \Omega = \langle A_0, A_1, \ldots, A_n \rangle \) and \( \Omega' = \langle A'_0, A'_1, \ldots, A'_n \rangle \) be two \( n \)-simplices, \( h_i \) the altitude of \( \Omega \) from the vertex \( A_i \), \( a_{ij} = |A_iA_j| \) and \( d_i' \) the distance from an interior point \( P' \) of \( \Omega' \) to the facet \( \Omega' = \langle A'_0, \ldots, A'_{i-1}, A'_{i+1}, \ldots, A'_n \rangle \). Then

\[
\sum_{0 \leq i < j \leq n} \frac{d_i'd_j'}{a_{ij}^2} \leq \frac{1}{4} \left( \frac{1}{\sum_{i=0}^{n} h_i} \right)^2,
\]

and equality holds if and only if \( \Omega \) is regular and \( P' \) is the incenter of \( \Omega' \).

Let \( G' \) be the centroid of \( \Omega' \) and \( d_i'(G') \) the distance from \( G' \) to \( \pi_i \). Taking \( P' = G' \) in Theorem 1 and noting the obvious geometric fact that

\[
\frac{d_i'(G')}{h_i} = \frac{1}{n+1},
\]

we obtain

**Theorem 2.** Let \( \Omega = \langle A_0, A_1, \ldots, A_n \rangle \) and \( \Omega' = \langle A'_0, A'_1, \ldots, A'_n \rangle \) be two \( n \)-simplices with altitudes \( h_0, h_1, \ldots, h_n \) and \( h'_0, h'_1, \ldots, h'_n \), respectively. \( a_{ij} = |A_iA_j| \) \((0 \leq i < j \leq n)\). Then

\[
\sum_{0 \leq i < j \leq n} \frac{h_i'h_j'}{a_{ij}^2} \leq \frac{1}{4} \left( \frac{1}{\sum_{i=0}^{n} h_i} \right)^2,
\]

and equality holds if and only if \( \Omega \) is regular and \( I' = G' \).

By Cauchy's inequality and (1.2), we have

**Theorem 3.** Under the hypotheses in Theorem 2, we have

\[
\sum_{i=0}^{n} \frac{a_{ij}^2}{h_i'h_j} \geq (n+1)^2 \left( \sum_{i=0}^{n} \frac{h_i'}{h_i} \right)^{-2},
\]

and equality holds if and only if \( \Omega \) is regular and \( I' = G' \).

We will give the proof of Theorem 1 in Section 3, while in Section 2 we will show some applications of the above theorems.
2. SOME INEQUALITIES FOR A SIMPLEX

From Theorem 1 we can derive the following two interesting inequalities for a simplex.

THEOREM 4. Let $d_i$ be the distance from an interior point $P$ of $\Omega = \langle A_0, A_1, \ldots, A_n \rangle$ to $\pi_i$ $(i = 0, 1, \ldots, n)$. Then

\begin{equation}
\sum_{0 \leq i < j \leq n} \frac{d_id_j}{a_{ij}^2} \leq \frac{1}{4}, \tag{2.1}
\end{equation}

\begin{equation}
\sum_{0 \leq i < j \leq n} \frac{a_{ij}^2}{d_id_j} \geq n^2(n + 1)^2, \tag{2.2}
\end{equation}

and equalities hold if and only if $\Omega$ is regular and $P = I$.

Proof. Taking $\Omega' = \Omega$ in (1.1) and noting the obvious fact that

$$\sum_{i=0}^{n} \frac{d_i}{h_i} = 1,$$

we obtain the inequality (2.1). By Cauchy's inequality, we have

\begin{equation}
\left( \sum_{0 \leq i < j \leq n} \frac{a_{ij}^2}{d_id_j} \right) \left( \sum_{0 \leq i < j \leq n} \frac{d_id_j}{a_{ij}^2} \right) \geq \left( \frac{(n + 1)n}{2} \right)^2. \tag{2.3}
\end{equation}

Inequality (2.2) follows from (2.1) and (2.3).

COROLLARY 1. Let $d_0, d_1, \ldots, d_n$ be the distances from an interior point $P$ of $\Omega$ to its $(n + 1)$ facets and $R$ the circumradius of $\Omega$. Then

\begin{equation}
\left( \prod_{i=0}^{n} d_i \right)^{2/(n+1)} \leq \frac{1}{n^2} \left( R^2 - |OG|^2 \right), \tag{2.4}
\end{equation}

\begin{equation}
\sum_{0 \leq i < j \leq n} \sqrt{d_id_j} \leq \frac{n + 1}{2} R, \tag{2.5}
\end{equation}

and equalities hold if and only if $\Omega$ is regular and $P$ is the center of $\Omega$. 
Proof. According to (2.1) and Cauchy’s inequality, we get
\[
\frac{1}{4} \left( \sum_{0 \leq i < j \leq n} a_{ij}^2 \right) \geq \left( \sum_{0 \leq i < j \leq n} a_{ij} \right) \left( \sum_{0 \leq i < j \leq n} \frac{d_i d_j}{a_{ij}} \right) \geq \left( \sum_{0 \leq i < j \leq n} \sqrt{d_i d_j} \right)^2.
\]
(2.6)

Noting the well-known formula
\[
\sum_{0 \leq i < j \leq n} a_{ij}^2 = (n + 1)^2 (R^2 - |OG|^2),
\]
(2.7)
from (2.6), we derive
\[
\sum_{0 \leq i < j \leq n} \sqrt{d_i d_j} \leq \frac{n + 1}{2} (R^2 - |OG|^2)^{1/2}.
\]
(2.8)
Inequalities (2.4) and (2.5) follow from (2.8). \(\blacksquare\)

Remark. It is easy to see that (2.4) and (2.5) are two sharpenings of the Gerber’s inequality [3].
\[
\prod_{i=1}^{n} d_i \leq \left( \frac{R}{n} \right)^{n+1}.
\]
Taking \(P = I\) in (2.4) yields at once that
\[
R^2 \geq (nr)^2 + |OG|^2,
\]
which is a sharpening of the famous inequality of L. Fejes Tóth [2]: \(R \geq nr\). This also is an analogy of the interesting inequality of M. S. Klamkin [5],
\[
R^2 \geq (nr)^2 + |OI|^2.
\]

Corollary 2. Let \(h_0, h_1, \ldots, h_n\) be the altitudes of \(\Omega = \langle A_0, A_1, \ldots, A_n \rangle\), \(a_{ij} = |A_i A_j|\). Then
\[
\sum_{0 \leq i < j \leq n} \frac{h_i h_j}{a_{ij}^2} \leq \frac{(n + 1)^2}{4}, \quad (2.9)
\]
\[
\sum_{0 \leq i < j \leq n} \frac{a_{ij}^2}{h_i h_j} \geq n^2, \quad (2.10)
\]
and equalities hold if and only if \(\Omega\) is regular.
Proof. Inequalities (2.9) and (2.10) follow by taking $\Omega' = \Omega$ in (1.2) and (1.3).

Remark. Taking $n = 2$ in (2.10), we obtain an inequality of Z. Mitrovic, which is regarded as a more elegant inequality for a triangle in [13, Appendix 1, 6.7]. Hence (2.10) is a generalization to several dimensions of Mitrovic’s inequality.

**Corollary 3.** For an $n$-dimensional simplex $\Omega = \langle A_0, A_1, \ldots, A_n \rangle$, let $a_{ij} = |A_i A_j|$. Then

\[
\sum_{0 \leq i < j \leq n} \frac{1}{a_{ij}^2} \leq \frac{1}{4r^2},
\]

(2.11)

and equality holds if and only if $\Omega$ is regular.

Proof. Taking for $\Omega'$ in (1.2) a regular simplex and noting the known fact that

\[
\sum_{i=0}^{n} \frac{1}{h_i} = \frac{1}{r}
\]

we obtain (2.11).

Remark. The inequality (2.11) is a generalization of Walker’s inequality (see [13]) to higher-dimensional simplices. For the other proof of (2.11) the reader is referred to [7].

3. THE PROOF OF THEOREM 1

To prove Theorem 1, we need the following two lemmas.

**Lemma 1.** Let $x_0, x_1, \ldots, x_n$ be $(n + 1)$ real constants, and $\theta_{ij} (= \theta_{ji})$ denote the internal dihedral angle between facets $\Omega_i$ and $\Omega_j$ of an $n$-dimensional simplex $\Omega$. Then

\[
\sum_{i=0}^{n} x_i^2 \geq 2 \sum_{0 \leq i < j \leq n} x_i x_j \cos \theta_{ij},
\]

(3.1)

and equality holds if and only if $(x_0, x_1, \ldots, x_n) = \lambda (S_0, S_1, \ldots, S_n)$ where $\lambda$ is any real constant number.
Proof. Let \( A = (c_{ij})_{(n+1) \times (n+1)} \), where
\[
c_{ij} = \begin{cases} 
1 & \text{if } i = j \\
-\cos \theta_{ij} & \text{if } i \neq j.
\end{cases}
\]
Then the matrix \( A \) is positive semidefinite [11, 19]. Therefore
\[
XAX' \geq 0
\]
holds for any \( X = (x_0, x_1, \ldots, x_n) \in \mathbb{R}^n \).
Inequality (3.1) follows from (3.2).
In the following we show the equality in (3.1) holds if and only if
\[
X = \lambda(S_0, S_1, \ldots, S_n).
\]
Assume that (3.3) holds. Since (see [1])
\[
\sum_{i=0}^{n} S_i^2 = 2 \sum_{0 \leq i < j \leq n} S_i S_j \cos \theta_{ij}
\]
we have
\[
XAX' = \lambda^2 \left( \sum_{i=0}^{n} S_i^2 - 2 \sum_{0 \leq i < j \leq n} S_i S_j \cos \theta_{ij} \right) = 0.
\]
Hence the equality in (3.1) holds. Conversely, put
\[
\Phi(X) = \sum_{i=0}^{n} x_i^2 - 2 \sum_{0 \leq i < j \leq n} x_i x_j \cos \theta_{ij}.
\]
Then \( \Phi(X) \geq 0 \) holds for any \( X \in \mathbb{R}^n \). So assume \((x_0, x_1, \ldots, x_n) \in \mathbb{R}^n \)
such that the equality in (3.1) holds, namely \( \Phi((x_0, x_1, \ldots, x_n)) = 0 \); then
\((x_0, x_1, \ldots, x_n)\) is a minimum point of \( \Phi \). Hence we have
\[
\frac{\partial \Phi}{\partial x_i} = 0 \quad (i = 0, 1, \ldots, n).
\]
Namely
\[
x_i - \sum_{j=0 \atop j \neq i}^{n} x_j \cos \theta_{ij} = 0 \quad (i = 0, 1, \ldots, n).
\]
Equation (3.4) can be rewritten in the form
\[
AX' = 0.
\]
Since det|\(A| = 0\) and rank \(A = n\) (see [19]), the system of fundamental solutions of (3.5) has only a non-vanishing vector. On the other hand, noting the known fact (see [19])

\[
S_i = \sum_{j=0, j\neq i}^n S_j \cos \theta_{ij},
\]

we find that \((S_0, S_1, \ldots, S_n)\) is a solution vector of (3.5). Hence any solution vector of (3.5) satisfies

\[
(x_0, x_1, \ldots, x_n) = \lambda(S_0, S_1, \ldots, S_n),
\]
as desired.

**Lemma 2 [11].** Let \(V\) be the volume of \(\Omega\) and \(\theta_{ij}\) the internal dihedral angle between \(\Omega_i\) and \(\Omega_j\), and let \(V_{ij}\) denote the volume of the \((n - 2)\)-dimensional simplex \(\langle A_0, A_{i-1}, A_{i+1}, \ldots, A_{j-1}, A_{j+1}, \ldots, A_n \rangle\). Then

\[
\frac{S_i S_j \sin \theta_{ij}}{V_{ij}} = \frac{nV}{n - 1}.
\]

**Proof of Theorem 1.** Let \(T_{ij}\) (0 \(\leq i < j \leq n\)) be the bisection of \(\Omega\) at the dihedral angle \(\theta_{ij}\) and the \((n - 1)\)-dimensional volume \(|T_{ij}|\), and let \(E_{ij}\) be the intersection of edge \(A_i A_j\) with \(T_{ij}\). Then \(\Omega\) is divided into two \(n\)-simplex \(\Sigma_1 = \langle A_0, \ldots, A_{i-1}, E_i, A_{i+1}, \ldots, A_n \rangle\) and \(\Sigma_2 = \langle A_0, \ldots, A_{i-1}, E_j, A_{j+1}, \ldots, A_n \rangle\) by \(T_{ij}\). It follows that

\[
V = V(\Sigma_1) + V(\Sigma_2), \tag{3.6}
\]
where \(V(\Sigma_1)\) and \(V(\Sigma_2)\) are the volumes of \(\Sigma_1\) and \(\Sigma_2\), respectively. By Lemma 2 and (3.6), we have

\[
\frac{(n - 1)}{nV_{ij}} S_i S_j \sin \theta_{ij} = \frac{(n - 1)}{nV_{ij}} S_j |T_{ij}| \sin \frac{\theta_{ij}}{2} + \frac{(n - 1)}{nV_{ij}} S_i |T_{ij}| \sin \frac{\theta_{ij}}{2}.
\]

This clearly implies

\[
|T_{ij}| = \frac{2S_i S_j}{S_i + S_j} \cos \frac{\theta_{ij}}{2}.
\]

Therefore

\[
|T_{ij}| \leq \sqrt{S_i S_j} \cos \frac{\theta_{ij}}{2}. \tag{3.7}
\]
From (3.7) and Lemma 1, we infer
\[
\sum_{0 \leq i < j \leq n} d'_i d'_j |T_{ij}| \leq \frac{1}{2} \sum_{0 \leq i < j \leq n} (d'_i S_i)(d'_j S_j)(1 - \cos \theta_{ij})
\leq \frac{1}{4} \left( \sum_{i=0}^{n} d'_i S_i \right)^2.
\tag{3.8}
\]

On the other hand, we have
\[
a_{ij}|T_{ij}| = |A_i E_{ij}| |T_{ij}| + |A_j E_{ij}| |T_{ij}|
\geq nV(\Sigma_i) + nV(\Sigma_j)
= nV.
\tag{3.9}
\]

Combining (3.8) and (3.9), we obtain
\[
(nV)^2 \sum_{0 \leq i < j \leq n} \frac{d'_i d'_j}{a_{ij}^2} \leq \frac{1}{4} \left( \sum_{i=0}^{n} d'_i S_i \right)^2.
\tag{3.10}
\]

By the formula $h, S_i = nV$, inequality (1.1) follows immediately from (3.10).

It is clear that equality in (1.1) holds if and only if $d'_0 = d'_1 = \cdots = d'_n$, $S_0 = S_1 = \cdots = S_n$, and $a_{ij} \perp T_{ij}$. This also is equivalent to saying that $P' = G'$ and $\Omega$ is regular. \qed

REFERENCES