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Generalized filter models ☆

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Abstract

In this paper, starting from filters which are a natural generalization of intersection filters (Barendregt et al., J. Symbolic Logic 48 (1983) 931–940), the existence of filter models and filter semimodels for the λ-calculus is investigated. The construction of filters is based on a Z-semilattice of types in which the subsets having infimum are given by a collection Z, called subset system. The set of representable functions is characterized in the obtained domain. In the case where the properties of the subset system Z guarantee the existence of a filter model, the proof of soundness and completeness of the associated natural Z-type assignment system is routine. © 2000 Elsevier Science B.V. All rights reserved.

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0. Introduction

Curry's system [9] for type inference of untyped λ-terms is one of the simplest systems to describe functional behaviour of λ-terms. Curry's original theory uses, besides type variables, the type constructor $\rightarrow$ to form types like $\sigma \rightarrow \tau$. They are intended to describe the behaviour of λ-terms which map elements of σ to elements of τ. Curry's system however, is not completely satisfying from a semantical point of view: it is not complete with respect to any immediately obvious semantics. In fact, the set of types of a λ-term is not closed under β-conversion, unless the (Eq_β)-rule is added to the system. Moreover, some λ-terms which are meaningful as functional programs are not typable.

For these reasons some extensions to Curry's system have been introduced. Among these extensions, one widely studied is the intersection type system [2, 4, 6, 13],

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introduced to describe the functional behaviour of solvable λ-terms. In this system the Curry types have been enhanced by introducing a constant (denoted by T in the present paper) as a universal type, the new type constructor $\wedge$, called intersection, and a type inclusion relation $\leqslant$. A type of the form $\sigma \wedge \tau$ is the type of λ-terms having both the type σ and the type τ. This system enjoys some nice properties: the (Eq_β)-rule is admissible, and the two classes of solvable and normalizing λ-terms are characterized. More interesting, from the point of view of the present work, is the fact that the set of types that can be assigned to a λ-term is closed under $\wedge$ and $\leqslant$, so it turns out to be a filter. This fact allows the $\wedge$-filter model to be built, a λ-model in which the interpretation of a λ-term is given by the set of types that can be assigned to it. The construction of this model is explicitly done in [2]. In [5] the approach is more semantical, in the sense that two functions F and G (for application and abstraction [1]) are defined and used to interpret λ-terms. Also in this latter approach the interpretation of a λ-term is the filter of the types derivable for it. The minimal type inclusion relation $\leqslant$ can be parameterized by an arbitrary relation $\Sigma \subseteq \mathbf{T} \times \mathbf{T}$ ($\mathbf{T}$ is the set of types) to obtain the type theory $\mathscr{T}_\Sigma = \langle \mathbf{T}, \leqslant_\Sigma \rangle$, where $\leqslant_\Sigma$ is the minimal inclusion relation containing Σ. In [7] it is proved that the type theory $\mathscr{T}_\Sigma$ yields a filter model if and only if the (Eq_β)-rule is admissible. This parameterization is also used in [5] to study the class of the functions representable over the filter domains and to characterize the properties of $\leqslant_\Sigma$ giving filter domains in which all continuous functions are representable.

Other extensions of Curry's types have been proposed in the literature: firstly polymorphic types ($\forall$-types) [12, 22] in which the universal quantifier $\forall$ allows types such as $\forall\varphi.\sigma$ to be constructed. The type $\forall\varphi.\sigma$ describes λ-terms having the types $\sigma[\varphi := \tau]$ for all types τ. More recently, polymorphic-intersection types ($\forall\wedge$-types) [15], constructed by using both $\wedge$ and $\forall$ type constructors, and infinite-intersection types (ω-types) [17], where countable intersection is allowed, have been proposed. An interesting open problem is if it is possible to generalize filter models for these type systems. A partial answer to this question is given in [15], in which a filter model for $\forall\wedge$-types is defined; unfortunately this model cannot be made the basis of a completeness proof.

The aim of the present paper is to introduce a unifying setting in which a generic type theory can be embedded and the conditions leading to filter λ-models can be found.

As the order relation of type inclusion is essential for the definition of filter, the sets of types considered here are not given in a constructive way. Instead of exhibiting a formal language describing types, in this paper the set of types is any partial order having a largest element and closed for arrow type construction; the sets of types having an infimum are explicitly given by a collection Z of subsets of types, called subset system.

In Section 1, the Z-type theory and the related Z-filters are defined; for each choice of the subset system Z, a particular type theory is obtained. The conditions under which the subset system makes the construction of a filter λ-model possible are given. Soundness and completeness of the Z-type assignment, induced by the Z-type theory, are proved in Section 2.

Section 3 is devoted to characterizing, with respect to continuity, the set of functions representable in the Z-filter domain.

Section 4, finally, analyzes some well-known type theories, in terms of Z-type theories.

1. Z-Filter models

The definitions of λ-model and λ-semi-model used in the present paper are essentially the ones of [15, 20] and [8]. The starting point is a *monotonic applicative structure* $\langle D, \bullet\rangle$, where D is a *poset* (a set partially ordered by a binary relation $\leqslant$) having at least two elements and $\bullet$ is a *binary monotonic operation of application*. The set of functions from D to D is ordered pointwise (i.e. $f \sqsubseteq g$ if for all $d \in D$ $f(d) \leqslant g(d)$).

Definition 1.1. (i) $(D \to D)_{\mathrm{M}}$ denotes the set of *monotonic* functions from D to D, i.e.: $(D \to D)_{\mathrm{M}} = \{f : D \to D \mid \forall d \in D\ \forall e \in D\ [d \leqslant e \Rightarrow f(d) \leqslant f(e)]\}$.

(ii) $(D \to D)_{\mathrm{R}}$ denotes the set of functions *representable* over $\langle D, \bullet\rangle$, i.e.: $(D \to D)_{\mathrm{R}} = \{f : D \to D \mid \exists d \in D\ [\forall e \in D\ [d \bullet e = f(e)]]\}$.

Notice that, owing to the monotonicity of application, one has: $(D \to D)_{\mathrm{R}} \subseteq (D \to D)_{\mathrm{M}}$.

Let Λ be the set of λ-terms. An *environment* ρ is a valuation of λ-term variables in D; the set of all environments is denoted by **Env**. Also environments are considered pointwise ordered, so $\rho \sqsubseteq \rho'$ means that for all variables $x : \rho(x) \leqslant \rho'(x)$.

The choice of two monotonic functions $F : D \to (D \to D)$ and $G : (D \to D) \to D$, where $(D \to D)$ is some collection of monotonic functions from D to D, allows a mapping $[\![-]\!]_{(-)} : \Lambda \to (\mathbf{Env} \to D)$ to be defined. When the choice of the subset $(D \to D)$ assures that the function $[\![-]\!]_{(-)}$ is a total function, an interpretation for every λ-term is obtained.

In the sequel $\lambda\!\!\lambda e \in D.\ P[e]$ denotes the function that associates with an element a of D, the value $P[e := a]$.

Definition 1.2. Let $\mathscr{M} = \langle D, F, G\rangle$ be a tuple where D is a poset, F and G two monotonic functions $F : D \to (D \to D)$, $G : (D \to D) \to D$.

(i) The function $[\![-]\!]^{\mathscr{M}}_{(-)} : \Lambda \to (\mathbf{Env} \to D)$ is defined as follows:

1. $[\![x]\!]^{\mathscr{M}}_{\rho} = \rho(x)$,
2. $[\![M_1 M_2]\!]^{\mathscr{M}}_{\rho} = F([\![M_1]\!]^{\mathscr{M}}_{\rho})([\![M_2]\!]^{\mathscr{M}}_{\rho})$,
3. $[\![\lambda x.M]\!]^{\mathscr{M}}_{\rho} = G(\lambda\!\!\lambda e \in D.[\![M]\!]^{\mathscr{M}}_{\rho[x:=e]})$.

(ii) $\mathscr{M} = \langle D, F, G\rangle$ is an *ordered interpretation* of Λ if $[\![-]\!]^{\mathscr{M}}_{(-)}$ is a total function, i.e. for all environments ρ and all λ-terms M, $\lambda\!\!\lambda e \in D.[\![M]\!]^{\mathscr{M}}_{\rho[x:=e]} \in (D \to D)$.

(iii) An ordered interpretation is a *λ-semi-model* if $F \circ G \sqsubseteq \mathrm{id}_{(D \to D)}$; it is a *$\lambda$-model* if $F \circ G = \mathrm{id}_{(D \to D)}$.

The following Lemma states some useful properties of the function $[\![-]\!]^{\mathscr{M}}_{(-)}$.

Lemma 1.3. *In any ordered interpretation the following hold*:
(i) *if* $\rho(x)=\rho'(x)$ *for all* $x\in FV(M)$, *then* $[\![M]\!]^{\mathscr{M}}_{\rho}=[\![M]\!]^{\mathscr{M}}_{\rho'}$
(ii) *if* $y\notin FV(M)$, *then* $[\![\lambda x.M]\!]^{\mathscr{M}}_{\rho}=[\![\lambda y.M[x:=y]]\!]^{\mathscr{M}}_{\rho}$
(iii) $[\![M[x:=N]]\!]^{\mathscr{M}}_{\rho}=[\![M]\!]^{\mathscr{M}}_{\rho[x:=[\![N]\!]^{\mathscr{M}}_{\rho}]}$
(iv) $[\![M]\!]^{\mathscr{M}}_{\rho}$ *is monotonic in* ρ.

Proof. By structural induction on M (see [20]). □

Lemma 1.3 allows it to be pointed out that λ-semimodels model β-reduction, in the sense that if $M\to_{\beta}N$, then $[\![M]\!]^{\mathscr{M}}_{\rho}\leqslant[\![N]\!]^{\mathscr{M}}_{\rho}$, for all ρ, and that λ-models model β-conversion.

An interesting class of models is that of filter models, introduced in [2, 5]. Filter models are based on a type theory which is essentially a lower semilattice $\langle X,\leqslant\rangle$ with a largest element, and elements $\sigma\to\tau$ when σ and τ are in X; the infimum for a finite subset $\{\sigma_1,\sigma_2,\dots,\sigma_n\}$ of X is denoted by $\sigma_1\wedge\sigma_2\wedge\cdots\wedge\sigma_n$. With this type theory it is possible to associate an algebraic complete lattice consisting of all filters over X, partially ordered by set inclusion. In [5, 21] filter domains which are models of λ-calculus are characterized. The goal of this paper is to generalize this construction to obtain λ-models, starting from a new type theory, that is again a poset $\langle X,\leqslant\rangle$ with largest element T and elements $\sigma\to\tau$ for every σ and τ in X, but the subsets of X having infimum are given, in a parametrized way, as a collection Z. In this way a general framework is obtained, in which different type theories can be embedded. In the sequel the concept of subset system, inspired by that one defined in [3, 24], is used to define a generalized notion of type theory and of filter. The λ-calculus notations of [1] are used. The set theory framework supposed is the classical ZFC (Zermelo–Fraenkel set theory with the axiom of choice).

Definition 1.4. A *subset system* on a set S is any collection Z of its subsets such that for every $s\in S$, the singleton $\{s\}\in$Z; the elements of Z are called Z-*sets* of S.

Given a set S, the set of non-empty subsets, the set of finite, non-empty subsets, and the set of countable, non-empty subsets are examples of subset systems on S, extensively used in the literature.

If S is clear from the context, the expression "R is a Z-set" instead of "R is a Z-set of S" is used.

Definition 1.5. Let $\langle P,\leqslant\rangle$ be a poset and Z a subset system on P; $\langle P,\leqslant,\mathrm{Z}\rangle$ is a Z-*semilattice* if:

(a) $R\in Z$ implies that R has an infimum p in $P: p=\bigsqcap R$.

(b) if $R\in Z$ and every $r\in R$ is the infimum of some Z-set $V^{(r)}$, i.e. $r=\bigsqcap V^{(r)}$, then $\bigcup\{V^{(r)}\mid r\in R\}\in Z$.

Owing to condition (b) of Definition 1.5 and to the properties of infimum, for the operator $\sqcap$ a kind of associativity holds; in fact, if R is a Z-set of P such that every $r \in R$ is the infimum of some Z-set $V^{(r)}$, then $\sqcap R = \sqcap \bigcup \{V^{(r)} \mid r \in R\}$.

Structures we are interested in are exactly those sets of types that are Z-semilattices with respect to a partial order relation and a subset system Z.

In this study the attention is focussed to the order relation between types, instead of to the way in which types are formed. $\to$ is the only constructor explicitly requested.

Definition 1.6. A Z-*type theory* $\mathscr{T}_Z$ is a structure $\langle \mathbf{T}, \leqslant, Z, \top, \to \rangle$, where $\langle \mathbf{T}, \leqslant, Z \rangle$ is a Z-semilattice having $\top$ as largest element and $\to$ is a binary function on $\mathbf{T}$, satisfying the following condition:

(1) $\sqcap \{\sigma \to \tau \mid \sigma \to \tau \in R\} \leqslant \xi \neq \top$ implies $\xi = \sqcap \{\mu \to \nu \mid \mu \to \nu \in K\}$ and for every $\mu \to \nu$ of K there is a nonempty set $S \subset R$ such that:
 $-\{\sigma \mid \sigma \to \tau \in S\}$ and $\{\tau \mid \sigma \to \tau \in S\}$ are Z-sets and
 $-\mu \leqslant \sqcap \{\sigma \mid \sigma \to \tau \in S\}$, $\sqcap \{\tau \mid \sigma \to \tau \in S\} \leqslant \nu$.

Condition (1) of Definition 1.6, that generalizes Condition C3 of [5] and Condition B of [21], plays an essential role in the construction of the λ-semimodel, as it will be clear in the proof of Lemma 1.12. It imposes quite strong constraints on relationships between the partial order on $\mathbf{T}$ and *functional types*. By functional types one means those elements of $\mathbf{T}$ that are either *arrow-types* (i.e. types obtained by the application of the binary function $\to$) or types equal to the infimum of a Z-set of arrow-types. Firstly, Condition (1) says that every type ξ greater than or equal to a funtional type $\sqcap \{\sigma \to \tau \mid \sigma \to \tau \in R\}$ must be a functional type too, i.e. $\xi = \sqcap \{\mu \to \nu \mid \mu \to \nu \in K\}$. Moreover, it requires that the left and right sides of any element of K are related in the partial order relation to the left and right sides of elements of R. One can note that this relation in the simple case of arrow-types, looks as a kind of converse of the usual contravariant rule on $\to$ (if $\mu \leqslant \sigma$ and $\tau \leqslant \nu$, then $\sigma \to \tau \leqslant \mu \to \nu$).

In the sequel $\sigma \geqslant \tau$ is sometimes used for $\tau \leqslant \sigma$.

Definition 1.7. (i) A Z-*filter* on the Z-type theory $\mathscr{T}_Z$ is any subset $d \subseteq \mathbf{T}$ such that
$-\top \in d$;
$-$if $\sigma \in d$ and $\sigma \leqslant \tau$, then $\tau \in d$;
$-$ for every $S \in Z$ if $S \subseteq d$, then $\sigma = \sqcap S \in d$.
(ii) If $A \subseteq \mathbf{T}$, $\uparrow A$ is the minimal Z-filter on $\mathscr{T}_Z$ that includes A.
(iii) $\mathscr{F}_Z$ is the set of Z-filters on $\mathscr{T}_Z$.

In the sequel the abbreviation $\uparrow \sigma$ for $\uparrow \{\sigma\}$ is used.

Lemma 1.8. (i) $\tau \in \uparrow \sigma$ *if and only if* $\sigma \leqslant \tau$.
(ii) $A \subseteq d$ *implies* $\uparrow A \subseteq d$.

Proof. Obvious by the definition of Z-filters. $\square$

According to the definition of filters it immediately follows that if $A=\emptyset$, then $\uparrow A = \uparrow \mathsf{T}$ and if $\sigma \leqslant \tau$, then $\uparrow \tau \subseteq \uparrow \sigma$.

Lemma 1.8 gives two properties of filters. In particular (ii) provides a useful proof tool; in fact to prove $\uparrow A = \uparrow B$ it is sufficient to show $A \subseteq \uparrow B$ and $B \subseteq \uparrow A$.

The following lemma gives some insights on the structure of the set of filters.

Lemma 1.9. *In* $\langle \mathscr{F}_{\mathrm{Z}}, \subseteq \rangle$ *the following hold*:
(i) $\uparrow \mathsf{T}$ *and* $\mathbf{T}$ *are the minimal and maximal elements, respectively.*
(ii) *For* $X \subseteq \mathscr{F}_{\mathrm{Z}}, \bigsqcup X = \uparrow \cup X$ *and* $\bigsqcap X = \cap X$, *hence* $\langle \mathscr{F}_{\mathrm{Z}}, \subseteq \rangle$ *is a complete lattice.*

Proof. Obvious. □

The set of filters $\mathscr{F}_{\mathrm{Z}}$ is turned into an applicative structure by defining an application $\bullet$ between Z-filters.

Definition 1.10. For $d_1, d_2 \in \mathscr{F}_{\mathrm{Z}}$ we define the application $\bullet$ as:
$d_1 \bullet d_2 = \uparrow \{\{\tau \in \mathbf{T} \mid \exists \sigma \in d_2$ and $\sigma \to \tau \in d_1\}$.

It is easy to prove that the application is monotonic in both the arguments.

Definition 1.11. (i) Let $\mathbf{F} : \mathscr{F}_{\mathrm{Z}} \to (\mathscr{F}_{\mathrm{Z}} \to \mathscr{F}_{\mathrm{Z}})_{\mathrm{M}}$ and $\mathbf{G} : (\mathscr{F}_{\mathrm{Z}} \to \mathscr{F}_{\mathrm{Z}})_{\mathrm{M}} \to \mathscr{F}_{\mathrm{Z}}$ be two functions defined as usual [5]:

$$\mathbf{F}(d) = \lambda e.d \bullet e, \quad \mathbf{G}(f) = \uparrow \{\sigma \to \tau \mid \tau \in f(\uparrow \sigma)\}.$$

(ii) Let $\mathscr{N} = \langle \mathscr{F}_{\mathrm{Z}}, \mathbf{F}, \mathbf{G} \rangle$.

$\mathbf{F}$ and $\mathbf{G}$ are monotonic functions. The following Lemma shows that $\mathbf{F} \circ \mathbf{G} \sqsubseteq \mathrm{id}_{(\mathscr{F}_{\mathrm{Z}} \to \mathscr{F}_{\mathrm{Z}})_{\mathrm{M}}}$ proving in this way that, whatever the choice of the subset system Z may be, the collection of monotonic functions on $\mathscr{F}_{\mathrm{Z}}$ gives rise to a λ-semimodel. Note that the proof of this lemma is founded on Condition 1 of Definition 1.6, indeed it assures that if $\mu \to \nu \in \mathbf{G}(f)$, then $\nu \in f(\uparrow \mu)$, for every monotonic function f.

To obtain a λ-model one needs some more conditions on the subset system Z. The conditions of Definition 1.14 are proved to be sufficient to this goal.

Lemma 1.12. (i) $\lambda\!\!\lambda e [\![M]\!]^{\mathscr{N}}_{\rho[x:=e]} \in (\mathscr{F}_{\mathrm{Z}} \to \mathscr{F}_{\mathrm{Z}})_{\mathrm{M}}$.
(ii) *Let* $f \in (\mathscr{F}_{\mathrm{Z}} \to \mathscr{F}_{\mathrm{Z}})_{\mathrm{M}}$. *If* $\mu \neq \mathsf{T} \in \mathbf{G}(f)$, *then there exists a* Z-*set* $H \subseteq \{\sigma \to \tau \mid \tau \in f(\uparrow \sigma)\}$ *such that* $\mu = \bigsqcap H$.
(iii) $\mathbf{F} \circ \mathbf{G} \sqsubseteq \mathrm{id}_{(\mathscr{F}_{\mathrm{Z}} \to \mathscr{F}_{\mathrm{Z}})_M}$.

Proof. (i) By Lemma 1.3(iv).

(ii) By induction on the definition of filters. The case $\mu \in \{\sigma \to \tau \mid \tau \in f(\uparrow \sigma)\}$ is trivial. For the case $\mu = \bigsqcap R$ use Condition (b) of Definition 1.5. The case $\mu \geqslant \nu$ is still to be considered. By induction $\nu = \bigsqcap K$, $K \subseteq \{\sigma \to \tau \mid \tau \in f(\uparrow \sigma)\}$, hence $\bigsqcap K \leqslant \mu$.

By Condition (1) of Definition 1.6, $\mu = \sqcap\{\alpha \to \beta \mid \alpha \to \beta \in R\}$ and for every $\alpha \to \beta \in R$ there is a non-empty set $S \subseteq K$ such that:
– $\{\sigma \mid \sigma \to \tau \in S\}$ and $\{\tau \mid \sigma \to \tau \in S\}$ are Z-sets,
– $\alpha \leqslant \sqcap\{\sigma \mid \sigma \to \tau \in S\}$, $\sqcap\{\tau \mid \sigma \to \tau \in S\} \leqslant \beta$.
For every $\sigma \to \tau$ of S, the fact that $\tau \in f(\uparrow \sigma)$ implies $\tau \in f(\uparrow \sqcap\{\sigma \mid \sigma \to \tau \in S\})$, since $\uparrow \sigma \subseteq \uparrow \sqcap\{\sigma \mid \sigma \to \tau \in S\}$ and f is monotonic. So $\sqcap\{\tau \mid \sigma \to \tau \in S\} \in f\{\uparrow \sqcap\{\sigma \mid \sigma \to \tau \in S\})$ and for every $\alpha \to \beta \in R : \beta \in f(\uparrow \sqcap\{\sigma \mid \sigma \to \tau \in S\}) \subseteq f(\uparrow \alpha)$, hence

$$R \subseteq \{\sigma \to \tau \mid \tau \in f(\uparrow\sigma)\}.$$

(iii) One must prove that $(\mathbf{F} \circ \mathbf{G})(f)(e) \subset f(e)$, i.e. that $\uparrow\{\sigma \to \tau \mid \tau \in f(\uparrow\sigma)\} \bullet e \subseteq f(e)$. By Lemma 1.8(ii) it is sufficient to prove that for any element v such that there is an element $\mu \in e$ with $\mu \to v \in \uparrow \{\sigma \to \tau \mid \tau \in f(\uparrow \sigma)\}$, one has $v \in f(e)$. $\mu \to v \in \uparrow\{\sigma \to \tau \mid \tau \in f(\uparrow\sigma)\}$ implies that there is a Z-set K such that $\mu \to v = \sqcap\{\xi \to \theta \mid \xi \to \theta \in K\}$ and $\theta \in f(\uparrow\xi)\}$ by (ii). By Condition (1) of Definition 1.6, there exists a Z-set $H \subseteq K$ such that $\mu \leqslant \sqcap\{\xi \mid \xi \to \theta \in H\}$ and $\sqcap\{\theta \mid \xi \to \theta \in H\} \leqslant v$. For all $\xi \to \theta \in H : \theta \in f(\uparrow \xi)$. By monotonicity of $f : f(\uparrow \xi) \subseteq f(\uparrow \sqcap\{\xi \mid \xi \to \theta \in H\}) \subseteq f(e)$, so $\sqcap\{\theta \mid \xi \to \theta \in H\} \in f(e)$ and $v \in f(e)$. □

By restricting the functions $\mathbf{F}$ and $\mathbf{G}$ of Definition 1.11 to the subset of representable functions: $\mathbf{F}_\mathrm{R}$ and $\mathbf{G}_\mathrm{R}$, the expected well behaviour is obtained, in the sense that $(\mathbf{F}_\mathrm{R} \circ \mathbf{G}_\mathrm{R})(f) = f$.

Lemma 1.13. *Let* $\mathbf{F}_\mathrm{R} : \mathscr{F}_\mathrm{Z} \to (\mathscr{F}_\mathrm{Z} \to \mathscr{F}_\mathrm{Z})_\mathrm{R}$ *and* $\mathbf{G}_\mathrm{R} : (\mathscr{F}_\mathrm{Z} \to \mathscr{F}_\mathrm{Z})_\mathrm{R} \to \mathscr{F}_\mathrm{Z}$ *be the restrictions of* $\mathbf{F}$ *and* $\mathbf{G}$, *respectively*, *to the set of functions representable in* $\mathscr{F}_\mathrm{Z}$. *Then* $\mathbf{F}_\mathrm{R} \circ \mathbf{G}_\mathrm{R} = \mathrm{id}_{(\mathscr{F}_\mathrm{Z} \to \mathscr{F}_\mathrm{Z})_\mathrm{R}}$.

Proof. Because of Lemma 1.12(iii) and the fact that $(\mathscr{F}_\mathrm{Z} \to \mathscr{F}_\mathrm{Z})_\mathrm{R} \subseteq (\mathscr{F}_\mathrm{Z} \to \mathscr{F}_\mathrm{Z})_\mathrm{M}$, it is sufficient to prove: $\tau \in f(e)$ implies $\tau \in (\mathbf{F}_\mathrm{R} \circ \mathbf{G}_\mathrm{R})(f)(e)$, for any representable function f. Let $d \in \mathscr{F}_\mathrm{Z}$ be the filter representing f. By Lemma 1.8(ii), it is sufficient to show that $v \in (\mathbf{F}_\mathrm{R} \circ \mathbf{G}_\mathrm{R})(f)(e)$ for any v such that there is a $\mu \in e$ with $\mu \to v \in d$.

$$\begin{aligned} \mu \to v \in d &\Rightarrow v \in d \bullet \uparrow \mu \\ &\Rightarrow \mu \to v \in \uparrow \{\sigma \to \tau \mid \tau \in d \bullet \uparrow \sigma\} \\ &\Rightarrow v \in \uparrow \{\sigma \to \tau \mid \tau \in f(\uparrow\sigma)\} \bullet e \text{ (since } \mu \in e) \\ &\Rightarrow v \in (\mathbf{F}_\mathrm{R} \bullet \mathbf{G}_\mathrm{R})(f)(e). \qquad \square \end{aligned}$$

Unfortunately, however, the definition of Z-type theory does not guarantee to represent as many functions as necessary to obtain an ordered interpretation of Λ (Definition 1.2(ii)). In the sequel conditions are given, under which the Z-type theory assures that all the functions $\lambda e.[\![M]\!]^{\mathscr{N}}_{\rho[x:=e]}$ are representable.

Definition 1.14. Let Z be a subset system on a set P.
(i) Z is *U-closed* if $R_1, R_2 \in \mathrm{Z}$ implies $R_1 \cup R_2 \in \mathrm{Z}$, i.e. Z is a model of pairing.

(ii) Z is *Z-preserving* if for any function $f: P \to P$, $S \in Z$ implies $\{f(s) \mid s \in S\} \in Z$, i.e. Z is a model of replacement.

Notice that the set of finite, non-empty subsets of a set is both U-closed and Z-preserving.

In the sequel $f(S)$ is used as short for $\{f(s) \mid s \in S\}$ and UZ-preserving is used for U-closed and Z-preserving. Moreover, by stretching the definition, a Z-type theory is said UZ-preserving (U-closed, Z-preserving) if its subset system Z is UZ-preserving (U-closed, Z-preserving).

Lemma 1.15. *Let* $e = \uparrow\{\sigma \to \tau \mid \tau \in [\![M]\!]^{\mathscr{N}}_{\rho[x:=\uparrow\sigma]}\}$.

(i) $\mu \to \nu \in e$ *implies* $\nu \in [\![M]\!]^{\mathscr{N}}_{\rho[x:=\uparrow\mu]}$.

(ii) *If the subset system Z is Z-preserving, then* $\nu \neq \top \in e \bullet d$ *implies that there exists an element* μ *of* d *such that* $\mu \to \nu \in e$.

(iii) *If the subset system Z is UZ-preserving, then* $\lambda\!\!\lambda a[\![M]\!]^{\mathscr{N}}_{\rho[x:=a]} \in (\mathscr{F}_Z \to \mathscr{F}_Z)_R$.

Proof. (i) By Lemma 1.12(i) and (ii).

(ii) By induction on the definition of filters, using (i). The fact that the subset system Z is Z-preserving is needed in the case $\nu = \bigsqcap R$.

(iii) It is sufficient to prove that the function $\lambda\!\!\lambda a.[\![M]\!]^{\mathscr{N}}_{\rho[x:=a]}$ is represented by the filter $\uparrow\{\sigma \to \tau \mid \tau \in [\![M]\!]^{\mathscr{N}}_{\rho[x:=\uparrow\sigma]}\}$, i.e. that for every $d \in \mathscr{F}_Z$: $\uparrow\{\sigma \to \tau \mid \tau \in [\![M]\!]^{\mathscr{N}}_{\rho[x:=\uparrow\sigma]}\} \bullet d = (\lambda\!\!\lambda a.[\![M]\!]^{\mathscr{N}}_{\rho[x:=a]})(d)$.

($\subseteq$) Obvious, by (i), (ii) and Lemma 1.3(iv).

($\supseteq$) The proof is by induction on M.

– $M = x$ or $M = y$. Obvious.

– $M = M_1M_2$. By Lemma 1.8(ii), it is sufficient to prove that for any ν such that there is a $\mu \in [\![M_2]\!]^{\mathscr{N}}_{\rho[x:=d]}$ and $\mu \to \nu \in [\![M_1]\!]^{\mathscr{N}}_{\rho[x:=d]}$, one has $\nu \in \uparrow\{\sigma \to \tau \mid \tau \in [\![M_1M_2]\!]^{\mathscr{N}}_{\rho[x:=\uparrow\sigma]}\} \bullet d$.

$$\begin{aligned}
&\mu \in [\![M_2]\!]^{\mathscr{N}}_{\rho[x:=d]}\\
&\quad\Rightarrow \mu \in \uparrow\{\sigma \to \tau \mid \tau \in [\![M_2]\!]^{\mathscr{N}}_{\rho[x:=\uparrow\sigma]}\} \bullet d \quad \text{(by induction)}\\
&\quad\Rightarrow \exists\delta \in d[\delta \to \mu \in \uparrow\{\sigma \to \tau \mid \tau \in [\![M_2]\!]^{\mathscr{N}}_{\rho[x:=\uparrow\sigma]}\}] \quad \text{(by (ii))}\\
&\quad\Rightarrow \mu \in [\![M_2]\!]^{\mathscr{N}}_{\rho[x:=\uparrow\delta]} \quad \text{(by (i))}.
\end{aligned}$$

Analogously one can show that there is an element $\gamma \in d$ such that $\mu \to \nu \in [\![M_1]\!]^{\mathscr{N}}_{\rho[x:=\uparrow\gamma]}$. By the U-closure of the subset system Z, there is a type $\xi = \bigsqcap\{\gamma, \delta\}$, hence, by the monotonicity property (Lemma 1.3(iv)), one has $\mu \in [\![M_2]\!]^{\mathscr{N}}_{\rho[x:=\uparrow\xi]}$ and $\mu \to \nu \in [\![M_1]\!]^{\mathscr{N}}_{\rho[x:=\uparrow\xi]}$, so $\nu \in [\![M_1M_2]\!]^{\mathscr{N}}_{\rho[x:=\uparrow\xi]}$.

$$\begin{aligned}
&\nu \in [\![M_1M_2]\!]^{\mathscr{N}}_{\rho[x:=\uparrow\xi]}\\
&\quad\Rightarrow \xi \to \nu \in \uparrow\{\sigma \to \tau \mid \tau \in [\![M_1M_2]\!]^{\mathscr{N}}_{\rho[x:=\uparrow\sigma]}\}\\
&\quad\Rightarrow \nu \in \uparrow\{\sigma \to \tau \mid \tau \in [\![M_1M_2]\!]^{\mathscr{N}}_{\rho[x:=\uparrow\sigma]}\} \bullet d \quad \text{(since } \xi \in d).
\end{aligned}$$

$-M = \lambda y.M_1$ $[\![\lambda y.M_1]\!]^{\mathcal{N}}_{\rho[x:=d]} = \uparrow\{\sigma \to \tau \mid \tau \in [\![M_1]\!]^{\mathcal{N}}_{\rho[x:=d,y:=\uparrow\sigma]}\}$ by Definition 1.10. So, by Lemma 1.8(ii), it is sufficient to prove that $\{\sigma \to \tau \mid \tau \in [\![M_1]\!]^{\mathcal{N}}_{\rho[x:=d,y:=\uparrow\sigma]}\} \subseteq \uparrow\{\alpha \to \beta \mid \beta \in [\![\lambda y.M_1]\!]^{\mathcal{N}}_{\rho[x:=\uparrow\alpha]}\} \bullet d$.

$$\begin{aligned}
&\tau \in [\![M_1]\!]^{\mathcal{N}}_{\rho[x:=d,y:=\uparrow\sigma]}\\
&\quad\Rightarrow \tau \in \uparrow\{\gamma \to \delta \mid \delta \in [\![M_1]\!]^{\mathcal{N}}_{\rho[x:=\uparrow\gamma,y:=\uparrow\sigma]}\} \bullet d \quad \text{(by induction)}\\
&\quad\Rightarrow \exists\mu \in d[\mu \to \tau \in \uparrow\{\gamma \to \delta \mid \delta \in [\![M_1]\!]^{\mathcal{N}}_{\rho[x:=\uparrow\gamma,y:=\uparrow\sigma]}\}] \quad \text{(by (ii))}\\
&\quad\Rightarrow \tau \in [\![M_1]\!]^{\mathcal{N}}_{\rho[x:=\uparrow\mu y:=\uparrow\sigma]} \quad \text{(by (i))}\\
&\quad\Rightarrow \sigma \to \tau \in [\![\lambda y.M_1]\!]^{\mathcal{N}}_{\rho[x:=\uparrow\mu]}\\
&\quad\Rightarrow \mu \to (\sigma \to \tau) \in \uparrow\{\alpha \to \beta \mid \beta \in [\![\lambda y.M_1]\!]^{\mathcal{N}}_{\rho[x:=\uparrow\alpha]}\}\\
&\quad\Rightarrow \sigma \to \tau \in \uparrow\{\alpha \to \beta \mid \beta \in [\![\lambda y.M_1]\!]^{\mathcal{N}}_{\rho[x:=\uparrow\alpha]}\} \bullet d. \qquad \square
\end{aligned}$$

By Lemmas 1.12, 1.13 and 1.15 immediately we have

Theorem 1.16. (i) $\mathcal{N} = \langle \mathcal{F}_Z, \mathbf{F}, \mathbf{G}\rangle$ *is a* λ*-semimodel.*

(ii) *If the* Z*-type theory is* UZ*-preserving, then* $\mathcal{R} = \langle \mathcal{F}_Z, \mathbf{F}_R, \mathbf{G}_R\rangle$ *is a* λ*-model.*

2. Z-type assignment

Each type theory $\mathcal{T}_Z$ induces a type assignment for the set Λ, which can be defined by means of a natural deduction system. As for the $\wedge$-type assignment, the set of types that can be assigned to each λ-term turns out to be a Z-filter. As a consequence of this result, one has, on one hand, that the Z-filter model can be used to prove completeness of the Z-type assignment, and, on the other hand, that the Z-filter model can be syntactically defined by means of the Z-type assignment instead of via functions **F** and **G**.

As usual, a *statement* is an expression of the form $M:\sigma$, where M (the *subject*) is a λ-term and σ (the *predicate*) is an element of **T**. A *basis* is a set of statements with only variables as subjects. The subjects in a basis do not need to be distinct. $B\backslash x$ is used to indicate the basis obtained from B by deleting the statements whose subject is x.

Definition 2.1. (i) The Z-*type assignment* $\vdash^Z$ induced by the Z-type theory $\mathcal{T}_Z$ is defined by the following natural deduction system:

Axioms

(T) $\vdash^Z M : \mathsf{T}$ (one for any $M \in \wedge$)

(Var) $B \vdash^Z X : \sigma$ if $x : \sigma \in B$

Rules

$$(\to \mathrm{I})\quad \frac{B\backslash x \cup \{x:\sigma\} \vdash^{Z} M:\tau}{B \vdash^{Z} \lambda x.M:\sigma\to\tau}$$

$$(\to \mathrm{E})\quad \frac{B_1 \vdash^{Z} M:\sigma\to\tau,\ B_2 \vdash^{Z} N:\sigma}{B_1\cup B_2 \vdash^{Z} MN:\tau}$$

$$(\leqslant)\quad \frac{B_1 \vdash^{Z} M:\sigma,\ \sigma\leqslant\tau}{B\vdash^{Z} M:\tau}$$

$$(\mathrm{ZI})\quad \frac{B_\sigma \vdash^{Z} M:\sigma \text{ for all } \sigma \text{ of a } Z\text{-set } S}{\bigcup\{B_\sigma \mid \sigma\in S\}\vdash^{Z} M:\sqcap S}$$

$$(\mathrm{ZE})\quad \frac{B\vdash^{Z} M:\sqcap S}{B\vdash^{Z} M:\sigma \text{ for all } \sigma\in S}$$

(ii) A deduction $\mathscr{D}$ is a set of statements arranged as a tree according to the deductions rules, whose root is called the *end-statement* of $\mathscr{D}$ and whose leaves are the premises of $\mathscr{D}$.

(iii) $B \vdash^{Z} M:\sigma$ denotes that $M:\sigma$ is derivable from the basis B; $\mathscr{D}:B \vdash^{Z} M:\sigma$ indicates that $\mathscr{D}$ is a deduction showing $B\vdash^{Z} M:\sigma$. $B\restriction M$ indicates the basis obtained from B by considering only the statements whose subjects are the variables occurring free in the term M : $B\restriction M=\{x:\sigma \mid x:\sigma\in B \text{ and } x\in FV(M)\}$.

The rule (ZE) is redundant, since it is directly derivable from rule ($\leqslant$).

If the Z-sets are infinite, the (ZI)-rule is an infinitary rule. To use transfinite induction on the structure of deductions with every statement of a deduction $\mathscr{D}$ an ordinal number $O(\mathscr{D})$ is associated as follows:

– 0 is associated with each premise of $\mathscr{D}$;
– the ordinal associated with the conclusion of a rule is greater than all ordinals associated with its premises.

The ordinal $O(\mathscr{D})$ is the ordinal associated with the end statement of $\mathscr{D}$.

The usual properties of type assignments can be proved also for the Z-type assignment system.

Lemma 2.2. *Let $\mathscr{T}_Z$ be a Z-type theory.*
(i) $B\backslash x\cup\{x:\sigma\}\vdash^{Z} x:\tau$ *iff* $\sigma\leqslant\tau$.
(ii) *If for all* $\sigma,\tau: B\backslash x\cup\{x:\sigma\}\vdash^{Z} M:\tau$ *implies* $B\backslash x\cup\{x:\sigma\}\vdash^{Z} N:\tau$, *then for all* $\mu: B\vdash^{Z}\lambda x.M:\mu$ *imples* $B\vdash^{Z}\lambda x.N:\mu$.
(iii) $B\vdash^{Z} M:\tau$ *iff* $B\restriction M\vdash^{Z} M:\tau$.
(iv) *If* $B\backslash x\cup\{x:\sigma\}\vdash^{Z} M:\tau$ *and* $y\notin FV(M)$, *then* $B\backslash y\cup\{y:\sigma\}\vdash^{Z} M[x:=y]:\tau$.
(v) $B\vdash^{Z} MN:\tau\neq\top$ *iff* $\tau\geqslant\sqcap S$ *and for all* $\sigma\in S$ *there is a type* $\mu^{(\sigma)}$ *such that* $B\vdash^{Z} M:\mu^{(\sigma)}\to\sigma$ *and* $B\vdash^{Z} N:\mu^{(\sigma)}$.

Proof. By (possibly transfinite) induction on the structure of deductions. □

Lemma 2.3. $B\vdash^{Z}\lambda x.M:\tau\neq\top$ *iff* $\tau\geqslant\sqcap\{\mu\to\nu \mid \mu\to\nu\in S\}$ *and, for all* $\mu\to\nu\in S: B\backslash x\cup\{x:\mu\}\vdash^{Z} M:\nu$.

Proof. (If) Trivial.

(Only if) By induction on the structure of deduction. The last rule cannot be ($\rightarrow$ E). The cases ($\rightarrow$ I) and ($\leqslant$) are trivial; for (ZI) use Condition (b) of Definition 1.5. □

As a consequence of Lemmas 2.2 and 2.3, it is possible to prove the Subject Reduction Theorem for the Z-type assignment. The proof of this Theorem, however simple, is here omitted because the completeness result of Theorem 2.12 provides an indirect proof of the following:

Theorem 2.4 (Subject Reduction Theorem). *If* $M \rightarrow_\beta N$ *and* $B \vdash^Z M : \sigma$, *then* $B \vdash^Z N : \sigma$.

In order to obtain the completeness result of the Z-type assignment, the notions of type interpretation in a model (semimodel) and of semantics satisfability are needed.

Definition 2.5. (i) Let $\langle D, F, G\rangle$ be a λ-semimodel or a λ-model. A function $\mathscr{V} : \mathbf{T} \rightarrow \mathscr{P}(D)$ is a *type interpretation in D* for $\langle \mathbf{T}, \leqslant\rangle$ iff it satisfies the following conditions:
(1) $\mathscr{V}(\sigma)$ is an upper closed non-empty set,
(2) $\mathscr{V}(\mathsf{T}) = D$,
(3) $\mathscr{V}(\sigma \rightarrow \tau) \subseteq \{d \in D \mid \forall e \in \mathscr{V}(\sigma)[F(d)(e) \in \mathscr{V}(\tau)]\}$,
(4) $\mathscr{V}(\sigma \rightarrow \tau) \supseteq \{G(f) \mid f \in (D \rightarrow D)$ and $\forall e \in \mathscr{V}(\sigma)[f(e) \in \mathscr{V}(\tau)]\}$,
(5) $\mathscr{V}(\bigsqcap S) = \bigcap\{\mathscr{V}(\sigma) \mid \sigma \in S\}$.

(ii) Let $\mathscr{M} = \langle D, F, G\rangle$ be a λ-semimodel (λ-model), ρ an environment and $\mathscr{V}$ a type interpretation in D.

$$\mathscr{M}, \rho, \mathscr{V} \models M : \tau \quad \text{iff} \quad [\![M]\!]_\rho^{\mathscr{M}} \in \mathscr{V}(\tau),$$
$$\mathscr{M}, \rho, \mathscr{V} \models B \quad \text{iff} \quad M, \rho, \mathscr{V} \models x : \sigma \text{ for all } x : \sigma \in B,$$
$$\mathscr{M}, \mathscr{V}, B \models M : \tau \quad \text{iff} \quad \text{for all } \rho : \mathscr{M}, \rho, \mathscr{V} \models B \text{ implies } \mathscr{M}, \rho, \mathscr{V} \models M : \tau.$$

(iii) A type interpretation $\mathscr{V}$ *respects* $\leqslant$ iff $\sigma \leqslant \tau$ implies $\mathscr{V}(\sigma) \subseteq \mathscr{V}(\tau)$.

$$\mathscr{M}, B \models^{\leqslant} M : \tau \quad \text{iff} \quad \text{for all } \mathscr{V} \text{ respecting } \leqslant : \mathscr{M}, \mathscr{V}, B \models M : \tau$$
$$B \models^{\leqslant}_{\mathrm{sm}} M : \tau \quad \text{iff} \quad \text{for all } \lambda\text{-semimodels } \mathscr{M} : \mathscr{M}, B \vdash^{\leqslant} M : \tau$$
$$B \models^{\leqslant} M : \tau \quad \text{iff} \quad \text{for } \lambda\text{-models } \mathscr{M} : \mathscr{M}, B \models^{\leqslant} M : \tau.$$

Definition 2.6. (i) $\vdash^Z$ *is semi-sound* iff $B \vdash^Z M : \tau$ implies $B \models^{\leqslant}_{\mathrm{sm}} M : \tau$; $\vdash^Z$ is *sound* iff $B \vdash^Z M : \tau$ implies $B \models^{\leqslant} M : \tau$.

(ii) $\vdash^Z$ is *semi-complete* iff $B \models^{\leqslant}_{\mathrm{sm}} M : \tau$ implies $B \vdash^Z M : \tau$; $\vdash^Z$ is *complete* iff $B \models^{\leqslant} M : \tau$ implies $B \models^Z M : \tau$.

The upper closed condition on type interpretation interprets the essence of the λ-semimodel definition; however, it is a redundant requirement for the soundness proof of those type assignment systems for which the β-reduction rule

$$\frac{B \vdash^Z M : \tau \quad M \rightarrow_\beta N}{B \vdash^Z N : \tau}$$

is admissible. That is the case in the Z-type assignment system. In fact, in this case the provability of the Subject Reduction Theorem assures that $[\![M]\!]^{\mathscr{M}}_\rho \in \mathscr{V}(\sigma)$ implies $[\![N]\!]^{\mathscr{M}}_\rho \in \mathscr{V}(\sigma)$ if $M \rightarrow_\beta N$; on the other hand in the soundness proofs of type assignment systems for which β-reduction rule is not an admissible rule, the upper closed condition is necessary. Finally, the result of Lemma 2.13 on simple type interpretations does make use of this assumption.

Lemma 2.7. (i) *If* $\vdash^Z$ *is semisound, then* $\vdash^Z$ *is sound.*
(ii) *If* $\vdash^Z$ *is complete, then* $\vdash^Z$ *is semicomplete.*

Proof. Obvious; notice that $B \models^{\leqslant}_{\text{sm}} M : \tau$ implies $B \models^{\leqslant} M : \tau$. $\square$

Lemma 2.8 (Soundness). (i) *if* $B \vdash^Z M : \tau$, *then* $B \models^{\leqslant}_{\text{sm}} M : \tau$.
(ii) *if* $B \vdash^Z M : \tau$, *then* $B \models^{\leqslant} M : \tau$.

Proof. (i) By (possibly transfinite) induction on the structure of deductions. For the axioms the proof is trivial. For $(\rightarrow \text{I})$:

$$\frac{B \backslash x \cup \{x : \sigma\} \vdash^Z M : \tau}{B \vdash^Z \lambda x.M : \sigma \rightarrow \tau}$$

if $\mathscr{M}, \rho, \mathscr{V} \models B$, for every $e \in \mathscr{V}(\sigma)$, also $\mathscr{M}, \rho[x := e], \mathscr{V} \models B \backslash x \cup \{x : \sigma\}$, hence, by induction, $[\![M]\!]^{\mathscr{M}}_{\rho[x:=e]} \in \mathscr{V}(\tau)$; Condition (4) of Definition 2.5(i) assures that $G(\lambda\!\!\lambda e.[\![\mathscr{M}]\!]^{\mathscr{M}}_{\rho[x:=e]}) = [\![\lambda x.M]\!]^{\mathscr{M}}_\rho \in \mathscr{V}(\sigma \rightarrow \tau)$. For $(\rightarrow \text{E})$ and (ZI), Conditions (3) and (5) of Definition 2.5(i) are used, respectively. The $(\leqslant)$-rule is trivially satisfied by the hypothesis that $\mathscr{V}$ respects $\leqslant$.
(ii) By (i) and Lemma 2.7(i). $\square$

In order to prove the semicompleteness for $\vdash^Z$, the λ-semimodel $\mathscr{N}$, introduced in Theorem 1.16(i), can be used, by showing that $\{\sigma \mid B \vdash^Z M : \sigma\}$ is a Z-filter and that, for nay M, $[\![M]\!]^{\mathscr{N}}_\rho$ is given by the set of types derivable for M from a suitable basis.

Definition 2.9. (i) Let $B_\rho = \{x : \sigma \mid \sigma \in \rho(x)\}$ for any environment ρ.
(ii) Let $\rho_B(x) = \uparrow \{\sigma \mid x : \sigma \in B\}$ for any basis B.

Lemma 2.10. (i) $\{\sigma \mid B \vdash^Z M : \sigma\}$ *is a Z-filter.*
(ii) $[\![M]\!]^{\mathscr{N}}_\rho = \{\sigma | B_\rho \vdash^Z M : \sigma\}$

Proof. (i) Obvious by the definition of Z-filters.

(ii) By induction on the structure of λ-terms. Use Lemma 2.3 in the case $M=\lambda x.M_1$. □

Lemma 2.11. (i) $\mathscr{V}_Z(\tau)=\{d\in\mathscr{F}_Z|\tau\in d\}$ *is a type interpretation.*
(ii) $\mathscr{V}_Z$ *respects* $\leqslant$.
(iii) $B\vdash^Z M:\tau$ *iff* $B_{\rho_B}\vdash^Z M:\tau$.
(iv) $\mathscr{N},\rho_B,\mathscr{V}_Z\models B$.

Proof. (i) and (ii) Routine.
(iii) By induction on derivation; notice that $B\vdash^Z x:\sigma$ *iff* $\sigma\in\uparrow\{\tau\,|\,x:\tau\in B\}$.
(iv) Immediate by (i) and Definition 2.5(ii). □

Lemmas 2.10 and 2.11 can be re-phrased for $\mathscr{R}=\langle\mathscr{F}_Z,\mathbf{F}_R,\mathbf{G}_R\rangle$, in case $\mathscr{R}$ is a λ-model. From that follows:

Theorem 2.12 (Completeness). (i) $B\vdash^Z M:\tau$ *iff* $B\models^{\leqslant}_{\rm sm} M:\tau$.
(ii) *If the Z-type theory is* UZ-*preserving, then* $B\vdash^Z M:\tau$ *iff* $B\models^{\leqslant} M:\tau$.

Proof. (i) (If) By Lemmas 2.10, 2.11(iii) and (iv).
(Only if) By Lemma 2.8(i).
(ii) (If) As in (i), using the λ-model $\mathscr{R}=\langle\mathscr{F}_Z,\mathbf{F}_R,\mathbf{G}_R\rangle$.
(Only if) By Lemma 2.8(ii). □

Another, often used, notion of type interpretation is that one of simple type interpretation (see, for example, [19]). A type interpretation is *simple* iff

$$\mathscr{V}(\sigma\to\tau)=\{d\in D\,|\,\forall e\in\mathscr{V}(\sigma)[F(d)(e)\in\mathscr{V}(\tau)]\}.$$

Simple type interpretations lead naturally to the definition of simple semantics satisfability ${}^{\rm s}\!\models^{\leqslant}$: $B\;{}^{\rm s}\!\models^{\leqslant} M:\tau$ iff for all models $\mathscr{M}$, all environments ρ and all simple type interpretations V respecting $\leqslant$: if $\mathscr{M},\rho,\mathscr{V}\models B$, then $\mathscr{M},\rho,\mathscr{V}\models M:\tau$. The simple semantics satisfability for semimodels (${}^{\rm s}\!\models^{\leqslant}_{\rm sm}$) can be defined analogously.

Lemma 2.13. *Let* $\langle D,F,G\rangle$ *be a λ-model or a λ-semimodel for which* $G\circ F\sqsubseteq \mathrm{id}_D$, *then any type interpretation in D is a simple type interpretation.*

Proof (*By definition of type interpretation*). $\mathscr{V}(\sigma\to\tau)\subseteq\{d\in D\,|\,\forall e\in\mathscr{V}(\sigma)[F(d)(e)\in\mathscr{V}(\tau)]\}$; so it is sufficient to prove $\{d\in D\,|\,\forall e\in\mathscr{V}(\sigma)[F(d)(e)\in\mathscr{V}(\tau)]\}\subseteq\mathscr{V}(\sigma\to\tau)$. Condition (4) of Definition 2.5(i) assures that if for every $e\in\mathscr{V}(\sigma)$ $F(d)(e)\in\mathscr{V}(\tau)$, then $G(F(d))\in\mathscr{V}(\sigma\to\tau)$; by the hypothesis $G(F(d))\leqslant d$ and by the upper closed condition of type interpretation $d\in\mathscr{V}(\sigma\to\tau)$. □

The soundness and the semisoundness of the Z-type assignment, with respect to simple semantics, can easily be proved by induction on the structure of the deduction.

Lemma 2.14 (Soundness for simple semantics). (i) *if* $B \vdash^{Z} M : \tau$, *then* $B\ {}^{s}\!\models^{\leqslant}_{sm} M : \tau$.
(ii) *if* $B \vdash^{Z} M : \tau$, *then* $B^{s} \models^{\leqslant} M : \tau$.

Mitchell [19] proved that simple type interpretations validate the following rule (simple rule):

$$\frac{B \vdash^{Z} \lambda x.Mx : \sigma \to \tau}{B \vdash^{Z} M : \sigma \to \tau} \qquad x \notin FV(M).$$

In the Z-type-assignment the simple rule is not admissible, as the two following counterexamples prove.

(1) Let $\sigma \leqslant \sigma'$ and $\tau' \leqslant \tau$. $\{y : \sigma' \to \tau'\} \vdash^{Z} \lambda x.yx : \sigma \to \tau$, but one cannot deduce $\{y : \sigma' \to \tau'\} \vdash^{Z} y : \sigma \to \tau$ unless $\sigma' \to \tau' \leqslant \sigma \to \tau$ can be proved.

(2) Let R be a Z-set. $\{y : \sigma \to \tau \mid \tau \in R\} \vdash^{Z} \lambda x.yx : \sigma \to \sqcap R$, but one cannot deduce $\{y : \sigma \to \tau \mid \tau \in R\} \vdash^{Z} y : \sigma \to \sqcap R$ unless $\{\sigma \to \tau \mid \tau \in R\}$ is a Z-set and $\sqcap \{\sigma \to \tau \mid \tau \in R\} \leqslant \sigma \to \sqcap R$.

For this reason, in order to obtain the completeness, it is mandatory to require two more conditions linking the partial order relation $\leqslant$ between types and the type constructor $\to$.

Condition (2) (contravariance)

$$\text{if } \sigma \leqslant \sigma' \text{ and } \tau' \leqslant \tau, \text{ then } \sigma' \to \tau' \leqslant \sigma \to \tau.$$

Condition (3)

$$\text{if } R \text{ is a Z-set, then also } \{\sigma \to \tau \mid \tau \in R\} \text{ is a Z-set and}$$
$$\sqcap \{\sigma \to \tau \mid \tau \in R\} \leqslant \sigma \to \sqcap R.$$

Lemma 2.15. *Let the partial order relation* $\leqslant$ *of a* Z*-type theory satisfy Conditions* (2) *and* (3).
(i) *The simple rule is admissible for the* Z*-type assignment.*
(ii) *For the* λ*-semimodel* $\mathcal{N} = \langle \mathcal{F}_Z, \mathbf{F}, \mathbf{G} \rangle : \mathbf{G} \circ \mathbf{F} \sqsubseteq \mathrm{id}_{\mathcal{F}_Z}$.

Proof. (i) By Lemmas 2.3, 2.2(i) and (v).

(ii) One must prove that for every $d \in \mathcal{F}_Z : \mathbf{G}(\mathbf{F}(d)) \subseteq d$.

$\mathbf{G}(\mathbf{F}(d)) = \uparrow\{\alpha \to \beta \mid \beta \in d \bullet \uparrow\alpha\}$, so by Lemma 1.8(ii) it is sufficient to prove $\{\alpha \to \beta \mid \beta \in d \bullet \uparrow\alpha\} \subseteq d$. This fact is immediate by noticing that, if the partial order relation $\leqslant$ satisfies Conditions (2) and (3), $d \bullet \uparrow\alpha = \{\nu \mid \exists \mu \in \uparrow\alpha \text{ and } \mu \to \nu \in d\}$. By the fact that $\mu \in \uparrow\alpha$ implies $\alpha \leqslant \mu$ and by Condition (2) the proof is done. □

Theorem 2.16 (Completeness for simple semantics). *Let the partial order relation* $\leqslant$ *of a* Z*-type theory satisfy Conditions* (2) *and* (3).
(i) $B \vdash^{Z} M : \tau$ *iff* $B\ {}^{s}\!\models^{\leqslant}_{sm} M : \tau$.
(ii) *If the* Z*-type theory is* UZ*-preserving, then* $B \vdash^{Z} M : \tau$ *iff* $B\ {}^{s}\!\models^{\leqslant} M : \tau$.

Proof. (i) (If) By Lemmas 2.12(i), 2.13 and 2.15(ii).

(Only if) By Lemma 2.14(i).

(ii) (If) By Lemmas 2.12(ii), 2.13 and 2.15(ii), since if $\mathbf{G} \circ \mathbf{F} \sqsubseteq \mathrm{id}_{\mathscr{F}_Z}$, then $\mathbf{G}_R \circ \mathbf{F}_R \sqsubseteq \mathrm{id}_{\mathscr{F}_Z}$.

(Only if) By Lemma 2.14(ii). □

The definition of a λ-model can be given in syntactic terms, in the style of Hindley-Longo (see, for example, [14]). For λ-semimodels a syntactical definition can be found in [20]. Lemma 2.10(ii) characterizes the interpretation of λ-terms by means of the Z-type assignment system, and gives a way to obtain, directly, a syntactical definition of the λ-(semi)model $\langle \mathscr{F}_Z, \mathbf{F}, \mathbf{G} \rangle$.

The results of Section 1 can be obviously stated starting from the syntactical definition of λ-(semi)model. In particular, one can point out that Theorem 1.16 can be proved in a simpler way, since, if the Z-type theory is UZ-preserving, the set of assumptions for a term variable used in any deduction is a Z-set.

Definition 2.17. An *ordered syntactic interpretation* of the λ-calculus is a triple: $\mathscr{M} = \langle D, \bullet, [\![-]\!]_{(-)} \rangle$, where $\langle D, \bullet \rangle$ is a monotonic applicative structure, and $[\![-]\!]_{(-)} : \Lambda \rightarrow (\mathbf{Env} \rightarrow D)$ satisfies the following conditions:

(1) $[\![x]\!]_\rho = \rho(x)$

(2) $[\![MN]\!]_\rho = [\![M]\!]_\rho \bullet [\![N]\!]_\rho$,

(3) if for all $d \in D [\![M]\!]_{\rho[x:=d]} \leqslant [\![N]\!]_{\rho[x:=d]}$, then $[\![\lambda x.M]\!]_\rho \leqslant [\![\lambda x.N]\!]_\rho$,

(4) if $y \notin FV(M)$, then $[\![\lambda x.M]\!]_\rho = [\![\lambda y.M[x:=y]]\!]_\rho$,

(5) if $\rho(x) = \rho'(x)$ for all $x \in FV(M)$, then $[\![M]\!]_\rho = [\![M]\!]_{\rho'}$.

(ii) A *syntactical λ-semimodel* is an ordered syntactic interpretation in which:

(6) $[\![\lambda x.M]\!]_\rho \bullet d \leqslant [\![M]\!]_{\rho[x:=d]}$.

(iii) A *syntactical λ-model* is an ordered syntactic interpretation in which:

(6′) $[\![\lambda x.M]\!]_\rho \bullet d = [\![M]\!]_{\rho[x:=d]}$.

To re-write in syntactic terms the result of Theorem 1.16, the following definition is introduced:

Definition 2.18. $B \restriction_{\mathscr{D}} x$ indicates the set of those statements in B having x as subject that are actually used in the deduction $\mathscr{D} : B \vdash^Z M : \sigma; B \restriction_{\mathscr{D}} x$ is defined by (possibly transfinite) induction on $\mathscr{D}$.

$B \restriction_{\mathscr{D}} x = \emptyset$: if

- $O(\mathscr{D}) = 0$ and the applied axiom is (T), or an axiom for a variable different from x, or
- the last applied rule in $\mathscr{D}$ is $(\rightarrow \mathrm{I})$ and the cancelled assumption is for the variable x;

$B \restriction_{\mathscr{D}} x = \{x : \sigma\}$: if

- $O(\mathscr{D}) = 0$ and the used axiom is $B \vdash^Z x : \sigma$;

$B \restriction_{\mathscr{D}} x = B \restriction_{\mathscr{D}1} x$: if

- the last applied rule in $\mathscr{D}$ is $(\leqslant)$ and $\mathscr{D}_1$ is the deduction of the premise, or

– the last applied rule is $(\to \mathrm{I})$ and the cancelled assumption is for a term variable different from x;
  $B\restriction_{\mathscr{D}} x = B\restriction_{\mathscr{D}1} x \cup B\restriction_{\mathscr{D}2} x$: if
– the last applied rule in $\mathscr{D}$ is $(\to \mathrm{E})$ and $\mathscr{D}_1$ and $\mathscr{D}_2$ are the deductions of the premises;
  $B\restriction_{\mathscr{D}} x = \bigcup_i (B\restriction_{\mathscr{D}i} x)$: if
– the last applied rule in $\mathscr{D}$ is (ZI) and $\mathscr{D}_i$ are the deductions of the premises.

Lemma 2.19. *Let $\mathscr{T}_{\mathrm{Z}}$ be a* UZ*-preserving* Z*-type theory. For all deductions $\mathscr{D}$, for all term variables x, $\{\sigma \mid \{x:\sigma\} \in B\restriction_{\mathscr{D}} x\} \neq \emptyset$ implies that the set $\{\sigma \mid \{x:\sigma\} \in B\restriction_{\mathscr{D}} x\}$ is a* Z*-set.*

Proof (*By* (*possibly transfinite*) *induction on the structure of deduction*). If the last applied rule is (ZI) the Z-preserving hypothesis and Condition (b) of Definition 1.5 are needed, whereas the U-closed hypothesis is needed when the last applied rule is $(\to E)$. □.

Theorem 2.20. *Let $\mathscr{F}_{\mathrm{Z}}$ be a* Z*-type theory and $[\![M]\!]_\rho = \{\sigma \mid B_\rho \vdash^{\mathrm{Z}} M:\sigma\}$.*
(i) *$\langle \mathscr{F}_{\mathrm{Z}}, \bullet, [\![-]\!]_{(-)}\rangle$ is a λ-semimodel.*
(ii) *If the* Z*-type theory is* UZ*-preserving then $\langle \mathscr{F}_{\mathrm{Z}}, \bullet, [\![-]\!]_{(-)}\rangle$ is a λ-model.*

Proof. (i) Clauses (1)–(6) of Definition 2.17 are proved easily.
(ii) To prove $(6')$ use Lemma 2.19. □

3. Z-continuous functions and representable functions

Continuity plays a central role in the study of λ-calculus models and, more generally, in the domain theory. Foundational in this framework has been the work of Scott [23] on continuous lattices, that has yielded the notion of Scott-continuous function as function preserving the suprema of directed sets.

The set of the (Scott-)continuous functions on a poset is a canonical subset usually considered to obtain a λ-model, as required in Definition 1.2; so, in the sequel, the notion of continuity is generalized to Z-semilattices; moreover, the relationships between the set of the functions continuous with respect to this notion and that one of functions representable on a Z-semilattice, are investigated.

In order to consider representable functions, one has to carry out, in a uniform way, the notion of subset system from the set of types to the set of filters and to the set of functions. To this aim the definition of subset function is introduced.

Definition 3.1. (i) A *subset function* z is a function-class, defined on the class of sets $\mathscr{S}$, which associates with each set S a set $\mathrm{z}(S)$ of its subsets, such that:
(a) for every $s \in S$, the singleton $\{s\} \in \mathrm{z}(S)$
(b) if $R \in \mathrm{z}(\mathrm{z}(S))$, then $\bigcup R \in \mathrm{z}(S)$

The elements of $\mathrm{z}(S)$ are called z-*sets* of S.

(ii) A subset function z is *strong* if for every function $f : S \to S'$ $(S, S' \in \mathscr{S})$, if $R \in \mathrm{z}(S)$, then $f(R) \in \mathrm{z}(S')$.

The definition of subset function allows one to insert Condition (b) of Definition 1.5, there given on Z-semilattices, in the more general environment of sets. Obviously, if z is a strong subset function, also the subset system $\mathrm{z}(S)$ is $\mathrm{z}(S)$-preserving, for every set S. The fact that the subset function z is strong allows one to prove some interesting properties of the filter models.

Definition 3.2. Let $\mathbf{P}$ and $\mathbf{P}'$ be posets and z a strong subset function.

(i) A non-empty set $X \subseteq \mathbf{P}$ is z-*directed* if for every $S \in z(\mathbf{P})$, such that $S \subseteq X$, there is an upper bound in X, i.e. an element $x \in X$ such that for all $s \in S, s \leqslant x$.

(ii) An element $p \in \mathbf{P}$ is z-*compact* in $\mathbf{P}$ if for every z-directed set $X \subseteq \mathbf{P}$: $p \leqslant \bigsqcup X$ implies that there exists an element x in X such that $p \leqslant x$.

(iii) A poset $\mathbf{P}$ is z-*algebraic* if for every $p \in \mathbf{P}$, the set $\{p' \mid p' \leqslant p$ and p' z–compact$\}$ is z-directed and $p = \bigsqcup \{p' \mid p' \leqslant p$ and p' z–compact$\}$.

(iv) A function $f : \mathbf{P} \to \mathbf{P}'$ is z-*continuous* iff, for every z-directed set $X \subseteq \mathbf{P}$: $f(\bigsqcup X) = \bigsqcup \{f(x) \mid x \in X\}$.

Let z be a strong subset function and $\langle \mathbf{T}, \leqslant \rangle$ a poset such that $\langle \mathbf{T}, \leqslant, \mathrm{Z}(\mathbf{T}), \mathsf{T}, \to \rangle$ is a $\mathrm{z}(\mathbf{T})$-type theory. In the following Lemma the lattice $\langle \mathscr{F}_{\mathrm{z}(\mathbf{T})'} \subseteq \rangle$ is proved to be z-algebraic.

Lemma 3.3. *In* $\langle \mathscr{F}_{\mathrm{z(T)}'} \subseteq \rangle$ *the following hold*:

(i) *for every* z-*directed set* $X \subseteq \mathscr{F}_{\mathrm{z}(\mathbf{T})}$: $\bigsqcup X = \bigcup X$.

(ii) *for every* $d \in \mathscr{F}_{\mathrm{z}(\mathbf{T})}$ *the set* $\{\uparrow \sigma \mid \uparrow \sigma \subseteq d\}$ *is a* z-*directed set and* $d = \bigcup \{\uparrow \sigma \mid \uparrow \sigma \subseteq d\}$.

(iii) $\langle \mathscr{F}_{\mathrm{z(T)}'} \subseteq \rangle$ *is* z-*algebraic*:

Proof. (i) By Definition 1.7(ii) and Lemma 1.9(ii), it is sufficient to prove that $\bigcup X$ is a $\mathrm{z}(\mathbf{T})$-filter. The only interesting case is when $S \subseteq \bigcup X$ is a z-set of $\mathbf{T}$; in this case $\bigsqcap S \in \bigcup X$ has to be proved.

Let $f_S : \mathbf{T} \to \mathscr{F}_{\mathrm{z}(\mathbf{T})}$ be a function defined by $f_S(\sigma) \equiv$ **if** $\sigma \in S$ **then** $x^{(\sigma)}$ **else** T, where $x^{(\sigma)} \in X$ is a filter containing σ; notice that f_S is not unique. By the fact that the subset function z is strong follows that $f_S(S) \subseteq X$ is a z-set of $\mathscr{F}_{\mathrm{z}(\mathbf{T})}$; since X is z-directed, in X there is a filter d' such that for all $\sigma \in S$, $f_S(\sigma) \subseteq d'$; obviously $S \subseteq d'$ and, by the definition of filters, $\bigsqcap S \in d'$.

(ii) First we prove that, for every $d \in \mathscr{F}_{\mathrm{z(T)}}$, the set $\{\uparrow\sigma \mid \uparrow\sigma \subseteq d\}$ is z-directed. Let $X \subseteq \{\uparrow \sigma \mid \uparrow \sigma \subseteq d\}$ be a z-set of $\mathscr{F}_{\mathrm{z(T)}}$ and let $f : \mathscr{F}_{\mathrm{z(T)}} \to \mathbf{T}$ be a function defined by $f(e) \equiv$ **if** $e = \uparrow\sigma$ **then** σ **else** T, again, since the subset function z is strong, $\{f(x) \mid x \in X\}$ is a z-set of $\mathbf{T}$, so there exists a $\xi = \bigsqcap \{f(x) \mid x \in X\}$ which implies $\uparrow\xi \subseteq d$ and for every $\uparrow\sigma \in X$, $\uparrow\sigma \subseteq \uparrow\xi$. Now we prove $d = \bigcup \{\uparrow\sigma \mid \uparrow\sigma \subseteq d\}$: $\tau \in d$

implies $\uparrow\tau \subseteq d$, so $\tau \in \bigcup\{\uparrow\sigma \mid \uparrow\sigma \subseteq d\}$; for the converse: $\tau \in \bigcup\{\uparrow\sigma \mid \uparrow\sigma \subseteq d\}$ implies $\tau \in \uparrow\sigma$ for some $\uparrow\sigma \subseteq d$, that is $\tau \in d$.

(iii) It is sufficient to prove that the z-compact elements of $\mathscr{F}_{\mathrm{z(T)}}$ are exactly those of the form $\uparrow\sigma$, for $\sigma \in \mathbf{T}$. $\uparrow\sigma \subseteq \bigcup X$ for some z-directed X implies $\exists d \in X$ such that $\sigma \in d$, hence $\uparrow\sigma \subseteq d$. Now let d be a z-compact element of $\mathscr{F}_{\mathrm{z(T)}}$. From (ii) one knows that $d = \bigcup\{\uparrow\sigma \mid \uparrow\sigma \subseteq d\}$, which implies $d \subseteq \uparrow\sigma$ for some $\sigma \in d$ (because of z-compactness), but $\sigma \in d$ implies $\uparrow\sigma \subseteq d$, so one has $\uparrow\sigma = d$. $\square$

The following Lemmas 3.4 and 3.5 state that, if the subset function z is strong, the z-continuous functions are monotonic and representable, i.e. $(\mathscr{F}_{\mathrm{z(T)}} \to \mathscr{F}_{\mathrm{z(T)}})_{\mathrm{Cz}} \subseteq (\mathscr{F}_{\mathrm{z(T)}} \to \mathscr{F}_{\mathrm{z(T)}})_{\mathrm{R}} \subseteq ((\mathscr{F}_{\mathrm{z(T)}} \to \mathscr{F}_{\mathrm{z(T)}})_{\mathrm{M}}$ where $(\mathscr{F}_{\mathrm{z(T)}} \to \mathscr{F}_{\mathrm{z(T)}})_{\mathrm{Cz}}$ denotes the set of z-continuous functions on $\mathscr{F}_{\mathrm{z(T)}}$. Finally, Theorem 3.6 shows that a function is representable if and only if it is z-continuous: $(\mathscr{F}_{\mathrm{z(T)}} \to \mathscr{F}_{\mathrm{z(T)}})_{\mathrm{Cz}} = (\mathscr{F}_{\mathrm{z(T)}} \to \mathscr{F}_{\mathrm{z(T)}})_{\mathrm{R}}$. In such a way one proves that the set of z-continuous functions is a good choice for the collection by means of which a filter λ-model can be defined (Definition 1.2).

Lemma 3.4. (i) *Let X be a* z*-directed set of filters. For every monotonic function f from $\mathscr{F}_{\mathrm{z(T)}}$ to $\mathscr{F}_{\mathrm{z(T)}}$, the set $\{f(x) \mid x \in X\}$ is* z*-directed.*

(ii) *Every* z*-continuous function f from $\mathscr{F}_{\mathrm{z(T)}}$ to $\mathscr{F}_{\mathrm{z(T)}}$ is a monotonic function.*

Proof. (i) Let $B \subseteq \{f(x) \mid x \in X\}$ be a z-set of $\mathscr{F}_{\mathrm{z(T)}}$. Since the subset function z is strong, one can construct (possibly by using the axiom of choice) a z-set $A \subseteq X$ such that $x \in A$ if and only if $f(x) \in B$. Because of X is z-directed, in X there exists an element x^* that is an upper bound of A. So, by monotonicity of f, $f(x^*)$ is an upper bound of B.

(ii) Let d and e be filters, such that $d \subseteq e$. Obviously $\{\uparrow\sigma \mid \uparrow\sigma \subseteq d\} \subseteq \{\uparrow\sigma \mid \uparrow\sigma \subseteq e\}$. Then the proof is done by Lemma 3.3(i) and (ii) and definition of $\bigsqcup$. $\square$

Lemma 3.5. (i) *A monotonic function f from $\mathscr{F}_{\mathrm{z(T)}}$ to $\mathscr{F}_{\mathrm{z(T)}}$ is a* z*-continuous function if and only if for every $\sigma \in f(d)$ there exists a type $\tau \in d$ such that $\sigma \in f(\uparrow\tau)$.*

(ii) *A* z*-continuous function f from $\mathscr{F}_{\mathrm{z(T)}}$ to $\mathscr{F}_{\mathrm{z(T)}}$ is represented by the filter* $\mathbf{G}(f) = \uparrow\{\sigma \to \tau \mid \tau \in f(\uparrow\sigma)\}$.

Proof. (i) (If) One has to prove that for every z-directed set of filters $X: f(\bigsqcup X) = \bigsqcup\{f(x) \mid x \in X\}$.

By monotonicity of the function f and by definition of $\bigsqcup$, follows immediately that $\bigsqcup\{f(x) \mid x \in X\} \subseteq f(\bigsqcup X)$.

To prove that $f(\bigsqcup X) \subseteq \bigsqcup\{f(x) \mid x \in X\}$ notice that be Lemma 3.3(i) it is sufficient to show that $f(\bigcup X) \subseteq \bigcup\{f(x) \mid x \in X\}$.

$$
\begin{aligned}
\sigma \in f(\textstyle\bigcup X) &\Rightarrow \exists \tau \in \textstyle\bigcup X \;[\sigma \in f(\uparrow\tau)] \quad \text{(by hypothesis)}\\
&\Rightarrow \exists x \in X \;[\tau \in x \text{ and } \sigma \in f(\uparrow\tau)]\\
&\Rightarrow \sigma \in \textstyle\bigcup\{f(x) \mid x \in X\} \quad (\text{since } f(\uparrow\tau) \subseteq f(x))
\end{aligned}
$$

(Only if) Let f be a z-continuous function from $\mathscr{F}_{z(T)}$ to $\mathscr{F}_{z(T)}$.

$f(d)=f(\bigsqcup\{\uparrow\tau \mid \uparrow\tau\subseteq d\})$ implies $f(d)=\bigcup\{f(\uparrow\tau)\mid\uparrow\tau\subseteq d\}$ by z-continuity of f and Lemma 3.3(i). So $\sigma\in f(d)$ implies that there exists a $\tau\in d$ such that $\sigma\in f(\uparrow\tau)$.

(ii) One must prove that $f(e)=\mathbf{G}(f)\bullet e=(\mathbf{F}\circ\mathbf{G})(f)(e)$, for every filter e.

$$\begin{aligned}(\subseteq)\ \tau\in f(e) &\Rightarrow \exists\sigma\in e\,[\tau\in f(\uparrow\sigma)] \quad \text{(by (i))}\\ &\Rightarrow \sigma\to\tau\in\mathbf{G}(f)\\ &\Rightarrow \tau\in\mathbf{G}(f)\bullet e\end{aligned}$$

($\supseteq$) Immediate by Lemma 1.12(iii). □

Theorem 3.6. (i) *Application in* $\mathscr{F}_{z(T)}$ *is* z-*continuous.*

(ii) $(\mathscr{F}_{z(T)}\to\mathscr{F}_{z(T)})_R=(\mathscr{F}_{z(T)}\to\mathscr{F}_{z(T)})_{Cz}$.

Proof. (i) z-continuity is proved in the second argument only; the proof for the first argument is similar. Let $X\subseteq\mathscr{F}_{z(T)}$ be a z-directed set. By Lemmas 3.3(i) and 3.4(i). $\bigsqcup X=\bigcup X$ and $\bigsqcup\{d\bullet x\mid x\in X\}=\bigcup\{d\bullet x\mid x\in X\}$. By Lemma 1.8(ii) it is sufficient to prove $\tau\in\bigcup\{d\bullet x\mid x\in X\}$, for all $\sigma\in\bigcup X$ such that $\sigma\to\tau\in d$.

$$\begin{aligned}\sigma\in\bigcup X \text{ and } \exists\tau\,[\sigma\to\tau\in d] &\Rightarrow \exists x\in X\,[\sigma\in x]\,\&\,\exists\tau\,[\sigma\to\tau\in d]\\ &\Rightarrow \exists x\in X\,[\tau\in d\bullet x]\\ &\Rightarrow \tau\in\bigcup\{d\bullet x\mid x\in X\}\end{aligned}$$

The converse is proved in a similar way.

(ii) (If) By (i).

(Only if) By Lemma 3.5(ii). □

An alternative way to prove the result of Theorem 3.6(ii) is that of investigating the structure of the poset $\langle(\mathscr{F}_{z(T)}\to\mathscr{F}_{z(T)})_{Cz},\sqsubseteq\rangle$.

It has been proved (see, for example, [1]) that the set of Scott-continuous functions from D to D: $(D\to D)_c$ is an algebraic c.p.o. if D is an algebraic c.p.o. Moreover, the compact elements of $(D\to D)_c$ are exactly the functions $\bigsqcup\{f_{de}\mid f_{de}\in S\}$, where S is a finite set, and f_{de} are step functions defined by means of the compact elements of D. By using an approach analogous to that of [5], it is quite easy to prove that $\langle(\mathscr{F}_{z(T)}\to\mathscr{F}_{z(T)})_{Cz},\sqsubseteq\rangle$ is a z-algebraic complete lattice.

By defining the step functions:

$$f_{\uparrow\sigma\uparrow\tau}(d)=\begin{cases}\uparrow\tau & \text{if } \uparrow\sigma\subseteq d,\\ \uparrow\mathsf{T} & \text{otherwise.}\end{cases}$$

One can prove [18] that the z-compact elements of $\langle(\mathscr{F}_{z(T)}\to\mathscr{F}_{z(T)})_{Cz},\sqsubseteq\rangle$ are exactly the functions $\bigsqcup\{f_{\uparrow\sigma\uparrow\tau}\mid f_{\uparrow\sigma\uparrow\tau}\in S\}$, where the set S is a z-set in $(\mathscr{F}_{z(T)}\to\mathscr{F}_{z(T)})_{Cz}$. By the fact that every z-compact element $\bigsqcup\{f_{\uparrow\sigma\uparrow\tau}\mid f_{\uparrow\sigma\uparrow\tau}\in S\}$ is represented by the filter

$\uparrow\bigsqcap\{\sigma\to\tau \mid f_{\uparrow\sigma\uparrow\tau}\in S\}$ and by the completeness of the lattice follows that $\mathscr{F}_{\mathrm{Z(T)}}\to(\mathscr{F}_{\mathrm{Z(T)}})_{\mathrm{R}}=(\mathscr{F}_{\mathrm{Z(T)}}\to\mathscr{F}_{\mathrm{Z(T)}})_{\mathrm{CZ}}$.

4. Some particular Z-type theories

It is possible to define some type theories, well known in the literature, in terms of Z-type theories. Usually a type theory is defined by a pair $\langle \mathbf{T}, \leqslant\rangle$, where the set $\mathbf{T}$ of types is introduced as a formal language, defined by a grammar. This grammar describes the constructors by means of which types are formed, starting from a countable set of type variables. The partial order relation $\leqslant$ is obtained by defining $\sigma=\tau$ iff $\sigma\leqslant\tau$ and $\tau\leqslant\sigma$ on a preorder relation $\leqslant$, given by means of a set of clauses. In all type theories here considered, a particular type constructor can be interpreted as the infimum operator for a suitable subset of types. For these type theories one can easily give an answer to the question of the existence of filter models. Moreover, owing to Theorem 2.12, this model, if any, can be made the basis of a completeness proof.

4.1. Intersection type theory

The intersection type theory has been studied in [2]. The set of types $\mathbf{T}_\wedge$ is defined by the following grammar:

$$\sigma := \varphi \mid \mathsf{T} \mid \sigma\to\sigma \mid \sigma\wedge\sigma$$

where φ ranges over a countable set of type variables. In the seminal paper of $\wedge$-types, the type constant ω is used instead of T.

The preorder relation $\leqslant_\wedge$ is the minimal reflexive and transitive relation such that:
- $\sigma\leqslant_\wedge\mathsf{T}$;
- $\mathsf{T}\leqslant_\wedge\mathsf{T}\to\mathsf{T}$;
- $\sigma\leqslant_\wedge\sigma\wedge\sigma$;
- $\sigma\wedge\tau\leqslant_\wedge\sigma \quad \sigma\wedge\tau\leqslant_\wedge\tau$;
- $(\sigma\to\tau)\wedge(\sigma\to\tau')\leqslant_\wedge\sigma\to(\tau\wedge\tau')$;
- if $\sigma\leqslant_\wedge\sigma'$ and $\tau\leqslant_\wedge\tau'$, then $\sigma\wedge\tau\leqslant_\wedge\sigma'\wedge\tau'$;
- if $\sigma\leqslant_\wedge\sigma'$ and $\tau'\leqslant_\wedge\tau$, then $\sigma'\to\tau'\leqslant_\wedge\sigma\to\tau$.

If the subset system $\mathrm{Z}_\wedge$ is defined as the set of finite, non-empty subsets of $\mathbf{T}_\wedge$, it is immediate to verify that $\langle\mathbf{T}_\wedge,\leqslant_\wedge,\mathrm{Z}_\wedge\rangle$ is a $\mathrm{Z}_\wedge$-semilattice, where every $\mathrm{Z}_\wedge$-set $S=\{\sigma_1,\sigma_2,\dots,\sigma_n\}$ has as infimum the type $\sigma_1\wedge\sigma_2\wedge\cdots\wedge\sigma_n$. Condition (1) of Definition 1.6 can be easily proved (Lemma 2.4(ii) in [2]).

The $\mathrm{Z}_\wedge$-type theory is $\cup Z_\wedge$-preserving, hence the set of $\mathrm{Z}_\wedge$-filters: $\mathscr{F}_\wedge$ gives rise to a λ-model. Moreover, the function $\mathrm{z}_\wedge$ that associates with any set S the set of its finite, non-empty subsets is strong, so the application between filters is $\mathrm{z}_\wedge$-continuous and the functions representable over $\langle\mathscr{F}_\wedge,\bullet\rangle$ are the $\mathrm{z}_\wedge$-continuous functions, that is the Scott-continuous ones [2, 5, 6]. As Conditions (2) and (3) are satisfied, one also has that any type interpretation is a simple interpretation.

The conditions on the partial order relation under which a $Z_\wedge$-filter model is obtained, are studied in [21]. In the present paper it is shown that the existence of the filter model can be related to properties of the subset system $Z_\wedge$.

4.2. Polymorphic type theory

Polymorphic types for untyped λ-terms have been introduced [16, 19] by adding to the Curry type system the universal quantification, and to the type assignment system two new rules:

$$(\forall I)^S \quad \frac{B \vdash M : \tau}{B \vdash M : \forall\varphi.\tau} \quad \varphi \text{ does not occur free in } B \upharpoonright M$$

$$(\forall E)^S \quad \frac{B \vdash M : \forall\varphi.\tau}{B \vdash M : \tau}$$

In order to define a subset system for the polymorphic type theory, note that the universal quantification of a type $\tau : \forall\varphi.\tau$ can be seen as the infimum of the set of all substitution instances of the type itself, and that the standard $(\forall I)^s$ rule can be substituted by the infinitary rule:

$$(\forall I) \quad \frac{B \vdash M : \tau[\varphi := \sigma] \quad \text{for all } \sigma \in \mathbf{T}_\forall}{B \vdash M : \forall\varphi.\tau}$$

More formally, let the set $\mathbf{T}_\forall$ be defined by the following grammar:

$$\sigma := \varphi \mid \top \mid \sigma \to \sigma \mid \forall\varphi.\sigma,$$

where φ ranges over a countable set of type variables. Note that $\top$ is added to the standard set of polymorphic types. Let the preorder relation $\leqslant_\forall$ be defined as the minimal, reflexive and transitive realtion such that:

- $\sigma \leqslant_\forall \top$,
- if $\sigma \leqslant_\wedge \sigma'$ and $\tau' \leqslant_\forall \tau$, then $\sigma' \to \tau' \leqslant_\forall \sigma \to \tau$;
- If $\sigma \leqslant_\forall \tau$, then $\forall\boldsymbol{\varphi}.\sigma \leqslant_\forall \forall\boldsymbol{\varphi}.\tau$;
- $\forall\boldsymbol{\varphi}.\sigma \leqslant_\forall \forall\boldsymbol{\psi}.\sigma[\boldsymbol{\varphi} := \boldsymbol{\tau}]$, $\quad\boldsymbol{\psi}$ not free in $\forall\boldsymbol{\varphi}.\sigma$;
- $\forall\boldsymbol{\varphi}.(\sigma \to \tau) \leqslant_\forall (\forall\boldsymbol{\varphi}.\sigma) \to (\forall\boldsymbol{\varphi}.\tau)$.

($\boldsymbol{\varphi}$ and $\boldsymbol{\tau}$ are used for the tuples $\varphi_1, \varphi_2, \ldots, \varphi_n$ and $\tau_1, \tau_2, \ldots, \tau_m$, for some n and m, respectively).

The subset system $Z_\forall$ on $\mathbf{T}_\forall$ can be defined as follows. S is a $Z_\forall$-set if there is a type $\tau \in \mathbf{T}_\forall$ such that:

(a) $\sigma \in S$ implies σ is a substitution instance of τ, i.e. $\sigma = \tau[\boldsymbol{\varphi} := \boldsymbol{\xi}]$ for some tuple $\boldsymbol{\xi}$ of types.

(b) every substitution instance of τ belongs to S, i.e. for all tuple of types, $\boldsymbol{\xi}$, there is a type $\sigma \in S$ such that $\sigma = \tau[\boldsymbol{\varphi} := \boldsymbol{\xi}]$.

It is easy to prove that $\langle \mathbf{T}_\forall, \leqslant_\forall, Z_\forall \rangle$ is a $Z_\forall$-semilattice in which $\bigsqcap S = \forall\boldsymbol{\varphi}.\tau$, if S is the set of all substitution instances of τ.

Condition (1) of Definition 1.6 can be proved by means of the following facts, provable by induction on the definition of $\leqslant_\forall$:

- $\forall\boldsymbol{\varphi}.(\sigma\to\tau)\leqslant_\forall\xi\neq_\forall \mathsf{T}$ implies $\xi=\forall\boldsymbol{\psi}.(\mu\to\nu)$
- $\forall\boldsymbol{\varphi}.(\sigma\to\tau)\leqslant_\forall\forall\psi.(\mu\to\nu)$ implies that for every tuple of types $\boldsymbol{\alpha}$ there are a tuple of type variables $\boldsymbol{\chi}$ and a tuple of types $\boldsymbol{\beta}$ such that $\mu[\boldsymbol{\psi}:=\boldsymbol{\alpha}]\leqslant_\forall\forall\boldsymbol{\chi}.\sigma[\boldsymbol{\varphi}:=\boldsymbol{\beta}]$ and $\forall\boldsymbol{\chi}.\tau[\boldsymbol{\varphi}:=\boldsymbol{\beta}]\leqslant_\forall \nu[\boldsymbol{\psi}:=\boldsymbol{\alpha}]$.

In this case, however, it is not clear how a suitable subset function could be defined.

If $\mathscr{F}_\forall$ denotes the set of $Z_\forall$-filters, $\langle\mathscr{F}_\forall,\ \mathbf{F},\ \mathbf{G}\rangle$ is a λ-semimodel, actually the one defined in [20], except that the constant $\mathbf{T}$ is added, but, since the $Z_\forall$-type theory is not $\cup$-closed the existence of the filter λ-model $\langle\mathscr{F}_\forall,\ \mathbf{F}_\mathrm{R},\ \mathbf{G}_\mathrm{R}\rangle$ cannot be inferred. Indeed, it is easy to show that $\langle\mathscr{F}_\forall, \bullet, [\![-]\!]_{(-)}\rangle$ does not satisfy Condition (6′) of Definition 2.17(iii).

In fact, if φ and ψ are type variables, $\psi\in[\![xx]\!]_{p[x:=\uparrow\{\varphi\to\psi,\varphi\}]}$, while $\psi\notin[\![\lambda x.xx]\!]\bullet\uparrow\{\varphi\to\psi,\varphi\}$.

One can notice that the infinitary rule ($\forall$I) plays an essential role only if infinite basis are allowed; in fact, if $\vdash^{\forall s}$ indicates the standard type assignment introduced for polymorphic types (possibly enriched by the constant T [15]), it is worth noting that, for finite basis: $B\vdash^{\forall} M:\tau$ iff $B\vdash^{\forall s} M:\tau$, whereas for infinite basis: $B\vdash^{\forall s} M:\tau$ implies $B\vdash^{\forall} M:\tau$, but the converse it is not true. In fact: $\{x:\sigma\to(\tau\to\tau)\mid \text{for all } \tau\in\mathbf{T}_\forall\}\cup\{y:\sigma\}\vdash^{\forall} xy:\forall\varphi.\varphi\to\varphi$, but $\{x:\sigma\to(\tau\to\tau)\mid \text{for all } \tau\in\mathbf{T}_\forall\}\cup\{y:\sigma\}\not\vdash^{\forall s} xy:\forall\varphi.\varphi\to\varphi$.

4.3. Polymorphic intersection type theory

By defining the subset system $\mathrm{Z}_{\forall\wedge}$ as the collection of all finite, non-empty subsets of $\mathrm{Z}_\forall$-sets of types, the $\mathrm{Z}_{\forall\wedge}$-type theory, is obtained [15]. The set $\mathbf{T}_{\forall\wedge}$ is defined by the following grammar:

$$\sigma:=\varphi\mid\mathsf{T}\mid\sigma\to\sigma\mid\sigma\wedge\sigma\mid\forall\varphi.\sigma.$$

The preorder relation $\leqslant_{\forall\wedge}$ is defined as the minimal reflexive and transitive relation such that:

- $\sigma\leqslant_{\forall\wedge}\mathsf{T}$;
- if $\sigma\leqslant_{\forall\wedge}\sigma'$ and $\tau'\leqslant_{\forall\wedge}\tau$, then $\sigma'\to\tau'\leqslant_{\forall\wedge}\sigma\to\tau$;
- if $\sigma\leqslant_{\forall\wedge}\tau$, then $\forall\varphi.\sigma\leqslant_{\forall\wedge}\forall\varphi.\tau$;
- $\forall\boldsymbol{\varphi}.\sigma\leqslant_{\forall\wedge}\forall\boldsymbol{\psi}.\sigma[\boldsymbol{\varphi}:=\tau]$, $\boldsymbol{\psi}$ not free in $\forall\boldsymbol{\varphi}.\sigma$;
- $\forall\boldsymbol{\varphi}.(\sigma\to\tau)\leqslant_{\forall\wedge}(\forall\boldsymbol{\varphi}.\sigma)\to(\forall\boldsymbol{\varphi}.\tau)$;
- $\sigma\wedge\tau\leqslant_{\forall\wedge}\sigma \qquad \sigma\wedge\tau\leqslant_{\forall\wedge}\tau$;
- $\sigma\leqslant_{\forall\wedge}\sigma\wedge\sigma$;
- if $\sigma\leqslant_{\forall\wedge}\sigma'$ and $\tau\leqslant_{\forall\wedge}\tau'$, then $\sigma\wedge\tau\leqslant_{\forall\wedge}\sigma'\wedge\tau'$;
- $(\sigma\to\tau)\wedge(\sigma-\tau')\leqslant_{\forall\wedge}\sigma\to(\tau\wedge\tau')$.

It is easy to prove that $\langle\mathbf{T}_{\forall\wedge},\ \leqslant_{\forall\wedge},\ \mathrm{Z}_{\forall\wedge}\rangle$ is a $\mathrm{Z}_{\forall\wedge}$-semilattice in which $\bigsqcap S=\forall\boldsymbol{\varphi}_1.\sigma_1\wedge\forall\boldsymbol{\varphi}_2.\sigma_2\wedge\cdots\wedge\forall\boldsymbol{\varphi}_n.\sigma_n$ and that Condition (1) of Definition 1.6 holds.

The $\mathrm{Z}_{\forall\wedge}$-type theory is U-closed but not $\mathrm{Z}_{\forall\wedge}$-preserving.

So $\langle \mathscr{F}_{\forall\wedge}, \mathbf{F}_{\mathrm{R}}, \mathbf{G}_{\mathrm{R}} \rangle$ (where $\mathscr{F}_{\forall\wedge}$ denotes, as usual, the set of $Z_{\forall\wedge}$-filters) cannot be proved to be a λ-model. On the other hand it is again easy to show that Condition (6′) of Definition 2.17(iii) does not hold. For example, consider a function which associates with every type $\tau \in \mathbf{T}_{\forall\wedge}$ a variable φ_τ. Then

$$\psi \in [\![z(xy)]\!]_{\rho[z := \uparrow(\forall\varphi.\varphi\to\varphi)\to\psi;\ x := \uparrow\{\varphi_\tau\to(\tau\to\tau)\};\ y := \uparrow\{\varphi_\tau\}]},$$

whereas

$$\psi \notin [\![\lambda x.z(xy)]\!]_{\rho[z := \uparrow(\forall\varphi.\varphi\to\varphi)\to\psi;\ y := \uparrow\{\varphi_\tau\}]} \bullet \{\varphi_\tau \to (\tau\to\tau)\}.$$

As for $Z_\forall$-type theory one can compare the type assignment $\vdash^{\forall\wedge}$ with the type assignment $\vdash^{\forall\wedge s}$, in which the infinitary rule $(\forall\mathrm{I})$ is substituted by the standard rule $(\forall\mathrm{I})^{\mathrm{s}}$. Again one has: $B \vdash^{\forall\wedge} M : \tau$ iff $B \vdash^{\forall\wedge\mathrm{s}} M : \tau$ for finite basis $B \vdash^{\forall\wedge\mathrm{s}} M : \tau$ implies $B \vdash^{\forall\wedge} M : \tau$ for infinite basis. In [15] a filter λ-model for $\forall\wedge$-types has been introduced, using a definition of filter different from the present one. This model is not useful to prove completeness of $\vdash^{\forall\wedge\mathrm{s}}$, that, on the contrary, can be easily proved using a standard term model technique. The authors conjecture that, for the $\forall\wedge$-type assignment, there is no filter λ-model which can be made the basis of a completeness proof, although the conjecture is not precise.

4.4. Infinite interesection type theory

The infinite countable extension of intersection types has been proposed in [17]. Among the open problems there is that one of the existence of new filter models, related to infinite intersection. The subset system Z_ω of countable, non-empty subsets of the set of types, allows one to give a positive answer to this question, provided that the type constant T is added. One way to define the set of types $\mathbf{T}_\omega$ is that of generalizing the type constructor $\wedge$ to a constructor $\bigwedge$, that provides the infimum of the countable sets.

So $\mathbf{T}_\omega$ can be defined by the following grammar:

$$\sigma := \varphi \mid \mathsf{T} \mid \sigma\to\sigma \mid \bigwedge\{\sigma \mid \sigma\in S\},$$

where S is a countable set of types.

The subset system Z_ω is U-closed and Z_ω-preserving, so any preorder relation $\leqslant_\omega$ satisfying Condition (1) of Definition 1.6 provides a λ-model. However, it is quite natural to investigate the preorder relation $\leqslant_\omega$ obtained as a natural extension of $\leqslant_\wedge$ relation. Hence, consider $\leqslant_\omega$ as the minimal reflexive and transitive relation such that:

- $\sigma \leqslant_\omega \mathsf{T}$;
- $\sigma \leqslant_\omega \bigwedge\{\sigma\}$;
- $\bigwedge\{\sigma \mid \sigma\in S\} \leqslant_\omega \sigma$, for all $\sigma \in S$;
- $\bigwedge\{\sigma\to\tau \mid \sigma\to\tau\in S\} \leqslant_\omega \bigwedge\{\sigma \mid \sigma\to\tau\in S\} \to \bigwedge\{\tau \mid \sigma\to\tau\in S\}$;
- if for every $\xi \in R$, there is $\sigma\in S$, such that $\sigma \leqslant_\omega \xi$, then $\bigwedge\{\sigma \mid \sigma\in S\} \leqslant_\omega \bigwedge\{\xi \mid \xi\in R\}$;
- if $\sigma \leqslant_\omega \sigma'$ and $\tau' \leqslant_\omega \tau$, then $\sigma'\to\tau' \leqslant_\omega \sigma\to\tau$.

Condition (1) of Definition 1.6 can be easily proved by induction on the definition of $\leqslant_\omega$.

In this case one can define the subset function z_ω as the function that associates with any set S the set of its non-empty, countable subsets. The function z_ω is strong, so the application between Z_ω-filters is z_ω-continuous and the representable functions over $\langle \mathscr{F}_\omega, \bullet \rangle$ are exactly the z_ω-continuous functions. Obviously the set of continuous functions is a proper subset of the set of z_ω-continuous functions, so one could ask for a characterization of filters representing the continuous functions.

Although $\mathscr{M}_\omega = \langle \mathscr{F}_\omega, \mathbf{F}_{\mathrm{R}}, \mathbf{G}_{\mathrm{R}} \rangle$ is not a continuous λ-model in the sense of [1], it seems to be close to this class of models because the z_ω-continuity of the application can be seen as a weaker notion of the continuity [11]. The continuous λ-models enjoy some nice properties such as the approximation property (i.e. $[\![M]\!]_\rho = \sup\{[\![A]\!]_\rho \mid A \underset{\sim}{\sqsubset} M\}$) and the fact that all fixed-point operators in Λ are interpreted in the least fixed-point operator of the model. It is an open question whether similar results hold for $\mathscr{M}_\omega$. In particular, the authors conjecture that, since Tarsky's theorem holds in all complete lattices, $\mathscr{M}_\omega$ has the fixed-point property. On the other hand, an approximation theorem seems difficult to prove; in fact, $[\![\mathbf{Y}]\!]^{\mathscr{M}\omega}$ can be seen as the countable sup of the interpretations of its approximants, but it is not clear how to obtain $[\![\lambda x.x\mathbf{Y}]\!]^{\mathscr{M}\omega}$ from the interpretations of its approximants.

5. Conclusions

This paper characterizes the conditions under which a type system allows the construction of a filter λ-model, in a natural way. The definition of filter here considered is the one reflecting the structure of the semilattice of the considered types. So the filter λ-model, if any, can be used to prove completeness of the type assignment.

In this particular setting, Condition (1) of Definition 1.6 is sufficient to take account of β-reduction, in the sense that, for all environment ρ, if $M \rightarrow_\beta^* N$, then $[\![M]\!]_\rho \subseteq [\![N]\!]_\rho$, while to reflect the behaviour of β-conversion a stronger condition is needed: the way to collect a set of types (Z-set) in a type that subsumes the set (in other words, the definition of the Z-semilattice) cannot be whatever. The $\cup$Z-preserving property of the subset system Z is sufficient to guarantee the existence of a λ-model.

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