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On a stochastic logistic equation with impulsive perturbations

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ABSTRACT

A stochastic logistic model with impulsive perturbations is proposed and investigated. First, we give a new definition of a solution of an impulsive stochastic differential equation (ISDE), which is more convenient for use than the existing one. Using this definition, we show that our model has a global and positive solution and obtain its explicit expression. Then we establish the sufficient conditions for extinction, non-persistence in the mean, weak persistence, persistence in the mean and stochastic permanence of the solution. The critical value between weak persistence and extinction is obtained. In addition, the limit of the average in time of the sample path of the solution is estimated by two constants. Afterwards, the lower-growth rate and the upper-growth rate of the solution are estimated. Finally, sufficient conditions for global attractivity are established.

1. Introduction

The investigation of logistic equation has long been and will continue to be one of the dominant themes in both ecology and mathematical ecology due to its importance. Since population dynamics in the real world is inevitably affected by environmental noise which is an important component in an ecosystem, several authors (see e.g. [1–7]) have investigated the following stochastic logistic equation

$$dx(t) = x(t)(r(t) - a(t)x(t))dt + \sigma(t)x(t)dB(t), \tag{1}$$

where $x(t)$ is the population size and $B(t)$ is a standard Brownian motion. Many important results of solutions of Eq. (1) have been obtained.

On the other hand, the theory of impulsive differential equation appears as a natural description of several kinds of real processes subject to certain perturbations whose duration is negligible in comparison with the duration of the process. Processes of this type are often studied in various fields of science and technology: population dynamics, ecology, biological systems, physics, pharmacokinetics, optimal control, etc.; see e.g., the monographs [8,9]. Various population dynamical systems of impulsive differential equations have been proposed and investigated extensively. Many important and interesting results on the dynamical behaviors for such systems have been found; see e.g., [10–17] and the references therein. Recently, stability of stochastic differential equation (SDE) with impulsive effects has been done by Sakthivel and Luo [18], Zhao et al. [19], Li and Sun [20], Li et al. [21] and Li et al. [22]. However, so far as our knowledge is concerned, very little amount of work on the stochastic population dynamics with impulsive effects has been done.

In this paper, we will study the following stochastic logistic system with impulsive perturbations:

$$\begin{cases} dx(t) = x(t)(r(t) - a(t)x(t))dt + \sigma(t)x(t)dB(t), & t \neq t_k,\ k \in N \\ x(t_k^+) - x(t_k) = b_k x(t_k), & k \in N \end{cases} \tag{2}$$

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where N denotes the set of positive integers, $0 < t_1 < t_2 < \cdots$, $\lim_{k\to+\infty} t_k = +\infty$, $r(t)$, $a(t)$ and $\sigma(t)$ are continuous bounded functions on $R_+ := [0, +\infty)$. The following additional restrictions on (2) are natural for biological meanings:

$$\inf_{t\in R_+} a(t) > 0, \qquad 1 + b_k > 0, \quad k \in N.$$

When $b_k > 0$, the perturbation stands for planting of the species, while $b_k < 0$ stands for harvesting. The main aims of this work are to investigate how impulses affect on the existence of positive solutions, permanence, persistence, extinction and global attractivity of Eq. (2). Our results show that the impulse does not affect all of these properties if the impulsive perturbations are bounded. However, if the impulsive perturbations are unbounded, some properties could be changed significantly. The important contributions of this paper is therefore clear.

To proceed, we need some appropriate definitions of persistence. Based on these definitions, we shall establish the persistence and extinction results for Eq. (2). Ma and his co-workers proposed the concepts of weak persistence [23], non-persistence in the mean [24] and persistence in the mean [24] for some deterministic models. Especially, Wang and Ma [25] pointed out the fact that there is only threshold between weak persistence and extinction of populations for general non-autonomous population models.

Definition 1.
- $x(t)$ is said to be extinctive if $\lim_{t\to+\infty} x(t) = 0$.
- $x(t)$ is said to be nonpersistent in the mean if $\lim_{t\to+\infty} t^{-1}\int_0^t x(s)ds = 0$.
- $x(t)$ is said to be weakly persistent if $\limsup_{t\to+\infty} x(t) > 0$.
- $x(t)$ is said to be persistent in the mean if $\liminf_{t\to+\infty} t^{-1}\int_0^t x(s)ds > 0$.
- $x(t)$ is said to be stochastically permanent if for every $\varepsilon \in (0, 1)$, there are constants $\beta > 0$, $\delta > 0$ such that

$$\liminf_{t\to+\infty} \mathcal{P}\{x(t) \geq \beta\} \geq 1 - \varepsilon, \qquad \liminf_{t\to+\infty} \mathcal{P}\{x(t) \leq \delta\} \geq 1 - \varepsilon.$$

The rest of the paper is arranged as follows. In Section 2, we give a new definition of solution of ISDE, which is more convenient for use than the existing definition. Then we show that Eq. (2) has a global and positive solution for any positive initial condition and give its explicit expression. In Section 3, sufficient conditions for extinction, non-persistence in the mean, weak persistence, persistence in the mean and stochastic permanence of the population represented by Eq. (2) are established. The critical value between weak persistence and extinction is obtained. Moreover, the limit of the average in time of the sample path of the solution is estimated by two constants. In Section 4, the lower-growth rate and the upper-growth rate of the solutions are estimated. In Section 5, we investigate the global attractivity of Eq. (2). In the last section, we give the conclusions and illustrate our main results through some examples and figures.

2. Global positive solution

Throughout this paper, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions. Let $B(t)$ denote a standard Brownian motion defined on this probability space. Moreover, we always assume that a product equals unity if the number of factors is zero.

Definition 2. Consider the following ISDE:

$$\begin{cases} dX(t) = F(t, X(t))dt + G(t, X(t))dB(t), & t \neq t_k,\ k \in N \\ X(t_k^+) - X(t_k) = B_k X(t_k), & k \in N \end{cases} \tag{3}$$

with initial condition $X(0)$. A stochastic process $X(t) = (X_1(t), \ldots, X_n(t))^T$, $t \in R_+$, is said to be a solution of ISDE (3) if

(i) $X(t)$ is $\mathcal{F}_t$-adapted and is continuous on $(0, t_1)$ and each interval $(t_k, t_{k+1}) \subset R_+$, $k \in N$; $F(t, X(t)) \in \mathcal{L}^1(R_+; R^n)$, $G(t, X(t)) \in \mathcal{L}^2(R_+; R^n)$, where $\mathcal{L}^k(R_+; R^n)$ is all R^n valued measurable $\{\mathcal{F}_t\}$-adapted processes $f(t)$ satisfying $\int_0^T |f(t)|^k dt < \infty$ a.s. (almost surely) for every $T > 0$;
(ii) for each t_k, $k \in N$, $X(t_k^+) = \lim_{t\to t_k^+} X(t)$ and $X(t_k^-) = \lim_{t\to t_k^-} X(t)$ exist and $X(t_k) = X(t_k^-)$ with probability one;
(iii) for almost all $t \in [0, t_1]$, $X(t)$ obeys the integral equation

$$X(t) = X(0) + \int_0^t F(s, X(s))ds + \int_0^t G(s, X(s))dB(s). \tag{4}$$

And for almost all $t \in (t_k, t_{k+1}]$, $k \in N$, $X(t)$ obeys the integral equation

$$X(t) = X(t_k^+) + \int_{t_k}^t F(s, X(s))ds + \int_{t_k}^t G(s, X(s))dB(s). \tag{5}$$

Moreover, $X(t)$ satisfies the impulsive conditions at each $t = t_k$, $k \in N$ with probability one.

Remark 1. One of the definitions of the solution of an ISDE is as follows.

Definition 3 (*See e.g. [19–22]*). A function $\tilde{X}(t) = (\tilde{X}_1(t), \ldots, \tilde{X}_n(t))^T$, $t \in R_+$, is said to be a solution of ISDE (3) if

(a) $\tilde{X}(t)$ is **absolutely continuous** on $(0, t_1)$ and $(t_k, t_{k+1}) \subset R_+,\ k \in N$;
(b) for each $t_k,\ k \in N, \tilde{X}(t_k^+) = \lim_{t\to t_k^+} \tilde{X}(t)$ and $\tilde{X}(t_k^-) = \lim_{t\to t_k^-} \tilde{X}(t)$ exist and $\tilde{X}(t_k) = \tilde{X}(t_k^-)$;
(c) $\tilde{X}(t)$ obeys (3) for almost every $t \in R_+ \setminus \{t_k\}$ and satisfies the impulsive conditions at each $t = t_k,\ k \in N$.

This is an important definition. However, if we use this definition, many important ISDEs will not have non-zero solutions. For example, consider the famous Black–Scholes model with impulsive effects:

$$\begin{cases} dY(t) = rY(t)dt + \sigma Y(t)dB(t), & t \neq t_k,\ k \in N;\ Y(0) = Y_0 \\ Y(t_k^+) - Y(t_k) = b_k Y(t_k), & k \in N. \end{cases} \tag{6}$$

It is well-known that $Y(t) = Y_0 \exp\{(r - 0.5\sigma^2)t + \sigma B(t)\}$ is the unique non-zero function that satisfies Eq. (6) on interval $(0, t_1)$, which is continuous on $(0, t_1)$ but is not **absolutely continuous** on $(0, t_1)$ (because $B(t)$ is not absolutely continuous). That is to say if we adopt Definition 3, Eq. (6) will not have a non-zero solution. In the following, we can find that if we use Definition 3, system (2) will not have a non-zero solution either.

Remark 2. Now let us see how we obtain Definition 2. First of all, note that if the impulsive conditions are dropped (i.e. $B_k = 0$), then ISDE (3) becomes the following SDE:

$$dX(t) = F(t, X(t))dt + G(t, X(t))dB(t).$$

Consequently, definition of a solution of SDE should be a special case of definition of a solution of ISDE. According to the classical definition of a solution of SDE (see e.g. [26]), condition (i), Eqs. (4) and (5) should be satisfied. Second, since there are impulsive perturbations in Eq. (3), then condition (ii) and impulsive conditions in (iii) should be satisfied. According to the above two facts, we propose Definition 2.

Theorem 1. *For any initial value $x(0) = x_0 > 0$, there exists a unique positive solution $x(t)$ to Eq. (2) a.s., which is global and represented by*

$$x(t) = \frac{\prod_{0<t_k<t} (1 + b_k) \exp\left\{\int_0^t [r(s) - 0.5\sigma^2(s)]ds + \int_0^t \sigma(s)dB(s)\right\}}{1/x_0 + \int_0^t \prod_{0<t_k<s} (1 + b_k)a(s) \exp\left\{\int_0^s [r(\tau) - 0.5\sigma^2(\tau)]d\tau + \int_0^s \sigma(\tau)dB(\tau)\right\} ds}.$$

Proof. Let

$$\begin{aligned} z(t) = {} & \exp\left\{-\int_0^t [r(s) - 0.5\sigma^2(s)]ds - \int_0^t \sigma(s)dB(s)\right\} \\ & \times \left[1/x_0 + \int_0^t \prod_{0<t_k<s} (1 + b_k)a(s) \exp\left\{\int_0^s [r(\tau) - 0.5\sigma^2(\tau)]d\tau + \int_0^s \sigma(\tau)dB(\tau)\right\} ds\right]. \end{aligned}$$

Then making use of Itô's formula, we obtain that $z(t)$ satisfies the equation

$$dz(t) = z(t)(\sigma^2(t) - r(t))dt - z(t)\sigma(t)dB(t) + \prod_{0<t_k<t} (1 + b_k)a(t)dt.$$

Define $y(t) = 1/z(t)$; then it follows from Itô's formula that

$$\begin{aligned} dy(t) &= -\frac{dz(t)}{z^2(t)} + \frac{(dz(t))^2}{z^3(t)} \\ &= -y(t)(\sigma^2(t) - r(t))dt + y(t)\sigma^2(t)dt - y^2(t) \prod_{0<t_k<t} (1 + b_k)a(t)dt + y(t)\sigma(t)dB(t) \\ &= y(t)\left(r(t) - \prod_{0<t_k<t} (1 + b_k)a(t)y(t)\right) dt + \sigma(t)y(t)dB(t). \end{aligned} \tag{7}$$

Let

$$\begin{aligned} x(t) &= \prod_{0<t_k<t} (1 + b_k)y(t) \\ &= \frac{\prod_{0<t_k<t} (1 + b_k) \exp\left\{\int_0^t [r(s) - 0.5\sigma^2(s)]ds + \int_0^t \sigma(s)dB(s)\right\}}{1/x_0 + \int_0^t \prod_{0<t_k<s} (1 + b_k)a(s) \exp\left\{\int_0^s [r(\tau) - 0.5\sigma^2(\tau)]d\tau + \int_0^s \sigma(\tau)dB(\tau)\right\} ds}; \end{aligned}$$

then it is easy to see that $x(t)$ is continuous on each interval $(t_k, t_{k+1}) \subset R_+,\ k \in N$ and for any $t \neq t_k$,

$$\begin{aligned} dx(t) &= d\left[\prod_{0<t_k<t}(1+b_k)y(t)\right] = \prod_{0<t_k<t}(1+b_k)dy(t) \\ &= \prod_{0<t_k<t}(1+b_k)y(t)\left(r(t) - \prod_{0<t_k<t}(1+b_k)a(t)y(t)\right)dt + \prod_{0<t_k<t}(1+b_k)\sigma(t)y(t)dB(t) \\ &= x(t)(r(t) - a(t)x(t))dt + \sigma(t)x(t)dB(t). \end{aligned}$$

On the other hand, for each $k \in N$ and $t_k \in [0, +\infty)$,

$$x(t_k^+) = \lim_{t\to t_k^+}\prod_{0<t_j<t}(1+b_j)y(t) = \prod_{0<t_j\le t_k}(1+b_j)y(t_k^+) = (1+b_k)\prod_{0<t_j<t_k}(1+b_j)y(t_k) = (1+b_k)x(t_k).$$

At the same time,

$$x(t_k^-) = \lim_{t\to t_k^-}\prod_{0<t_j<t}(1+b_j)y(t) = \prod_{0<t_j<t_k}(1+b_j)y(t_k^-) = \prod_{0<t_j<t_k}(1+b_j)y(t_k) = x(t_k).$$

Now let us prove the uniqueness of the solution. For $t \in (0, t_1]$, system (2) becomes the following classical equation:

$$dx(t) = x(t)(r(t) - a(t)x(t))dt + \sigma(t)x(t)dB(t), \quad t \in (0, t_1]. \tag{8}$$

Since the coefficients of Eq. (8) are local Lipschitz continuous, by the theory of SDE (see e.g. Theorem 3.15 in [26], p. 91), the solution of Eq. (8) is unique. For $t \in (t_k, t_{k+1}]$, $k \in N$, system (2) becomes:

$$dx(t) = x(t)(r(t) - a(t)x(t))dt + \sigma(t)x(t)dB(t), \quad t \in (t_k, t_{k+1}],\ k \in N. \tag{9}$$

Note that the coefficients of Eq. (9) are also local Lipschitz continuous; then the solution of Eq. (9) is also unique. Consequently, the solution of system (2) is unique. □

3. Persistence and extinction

Theorem 1 shows that Eq. (2) has a positive solution for any positive initial value. This nice property provides us with a great opportunity to discuss in more detail how the solution varies in R_+. Now, let us study when the population represented by Eq. (2) goes to extinction and when it does not. Define $\check{f} = \inf_{t\in R_+} f(t)$, $\hat{f} = \sup_{t\in R_+} f(t)$.

Theorem 2. *Suppose that $x(t)$ is a solution of Eq. (2); then*

$$\limsup_{t\to+\infty} t^{-1}\ln x(t) \le \limsup_{t\to+\infty} t^{-1}\left[\sum_{0<t_k<t}\ln(1+b_k) + \int_0^t b(s)ds\right] =: b^*, \quad a.s.,$$

where $b(t) = r(t) - 0.5\sigma^2(t)$. Particularly, if $b^* < 0$, then $\lim_{t\to+\infty} x(t) = 0$ a.s.

Proof. Applying Itô's formula to Eq. (7), we have

$$\begin{aligned} d\ln y(t) &= \frac{dy(t)}{y(t)} - \frac{(dy(t))^2}{2y^2(t)} \\ &= \left[r(t) - a(t)\prod_{0<t_k<t}(1+b_k)y(t) - 0.5\sigma^2(t)\right]dt + \sigma(t)dB(t) \\ &= [b(t) - a(t)x(t)]dt + \sigma(t)dB(t). \end{aligned}$$

Integrating both sides from 0 to t, one can see that

$$\ln y(t) - \ln y(0) = \int_0^t b(s)ds - \int_0^t a(s)x(s)ds + M_1(t), \tag{10}$$

where $M_1(t) = \int_0^t \sigma(s)dB(s)$. Note that $M_1(t)$ is a local martingale, whose quadratic variation is $\langle M_1(t), M_1(t)\rangle = \int_0^t \sigma^2(s)ds \le \hat{\sigma}^2 t$. Making use of the strong law of large numbers for local martingales (see e.g. [26] on page 16) leads to

$$\lim_{t\to+\infty} M_1(t)/t = 0 \quad \text{a.s.} \tag{11}$$

On the other hand, it follows from (10) that

$$\sum_{0<t_k<t}\ln(1+b_k)+\ln y(t)-\ln y(0)=\sum_{0<t_k<t}\ln(1+b_k)+\int_0^t b(s)ds-\int_0^t a(s)x(s)ds+M_1(t).$$

In other words, we have shown that

$$\ln x(t)-\ln x(0)=\sum_{0<t_k<t}\ln(1+b_k)+\int_0^t b(s)ds-\int_0^t a(s)x(s)ds+M_1(t). \tag{12}$$

Therefore

$$\ln x(t)-\ln x(0)\le\sum_{0<t_k<t}\ln(1+b_k)+\int_0^t b(s)ds+M_1(t).$$

Then the desired assertion follows from (11) immediately. □

Theorem 3. *The solution of Eq.* (2) *obeys*

$$\limsup_{t\to+\infty} t^{-1}\int_0^t x(s)ds\le b^*/\check{a}\quad a.s.$$

Particularly, if $b^*=0$, *then the population represented by Eq.* (2) *is non-persistent in the mean a.s., i.e.* $\lim_{t\to+\infty} t^{-1}\int_0^t x(s)ds=0$ *a.s.*

Proof. For arbitrarily fixed $\varepsilon>0$, there is a constant T_1 such that

$$\ln x(0)/t\le\varepsilon/3,\qquad t^{-1}\left[\sum_{0<t_k<t}\ln(1+b_k)+\int_0^t b(s)ds\right]\le b^*+\varepsilon/3,\qquad M_1(t)/t\le\varepsilon/3$$

for $t\ge T$. Substituting this inequality into (12) yields

$$\begin{aligned}\ln x(t)&=\ln x(0)+\sum_{0<t_k<t}\ln(1+b_k)+\int_0^t b(s)ds-\int_0^t a(s)x(s)ds+M_1(t)\\&\le\lambda t-\check{a}\int_0^t x(s)ds\end{aligned}$$

for all $t\ge T$ almost surely, where $\lambda=b^*+\varepsilon$. Denote $h(t)=\int_0^t x(s)ds$. Consequently, $\exp(\check{a}h(t))(dh/dt)\le\exp\{\lambda t\},\ t\ge T$. Integrating this inequality from T to t, we have

$$\check{a}^{-1}[\exp\{\check{a}h(t)\}-\exp\{\check{a}h(T)\}]\le\lambda^{-1}[\exp\{\lambda t\}-\exp\{\lambda T\}].$$

Rewriting this inequality we can observe

$$\exp\{\check{a}h(t)\}\le\exp\{\check{a}h(T)\}+\check{a}\lambda^{-1}\exp\{\lambda t\}-\check{a}\lambda^{-1}\exp\{\lambda T\}.$$

Taking logarithm of both sides, one can see that

$$h(t)\le\check{a}^{-1}\ln\{\check{a}\lambda^{-1}\exp\{\lambda t\}+\exp\{\check{a}h(T)\}-\check{a}\lambda^{-1}\exp\{\lambda T\}\}.$$

In other words, we have shown that

$$\limsup_{t\to+\infty} t^{-1}\int_0^t x(s)ds\le\check{a}^{-1}\limsup_{t\to+\infty}\{t^{-1}\ln\{\check{a}\lambda^{-1}\exp\{\lambda t\}+\exp\{\check{a}h(T)\}-\check{a}\lambda^{-1}\exp\{\lambda T\}\}\}.$$

Making use of the L'Hospital's rule, we obtain

$$\limsup_{t\to+\infty} t^{-1}\int_0^t x(s)ds\le\check{a}^{-1}\limsup_{t\to+\infty}\{t^{-1}\ln[\check{a}\lambda^{-1}\exp(\lambda t)]\}=\lambda/\check{a}.$$

The required assertion follows from the arbitrariness of ε. □

Theorem 4. *If* $b^*>0$, *then the population represented by Eq.* (2) *is weakly persistent a.s., i.e.* $\limsup_{t\to+\infty} x(t)>0$ *a.s.*

Proof. If this assertion is not true, then $\mathcal{P}(S) > 0$, where S is the set $S = \{\limsup_{t\to+\infty} x(t) = 0\}$. It follows from (12) that

$$t^{-1}[\ln x(t) - \ln x(0)] = t^{-1}\left[\sum_{0<t_k<t} \ln(1+b_k) + \int_0^t b(s)ds\right] - t^{-1}\int_0^t a(s)x(s)ds + M_1(t)/t. \quad (13)$$

On the other hand, for $\forall\omega \in S$, we have $\lim_{t\to+\infty} x(t,\omega) = 0$. Thus it follows from the boundedness of $a(t)$ that

$$\limsup_{t\to+\infty} t^{-1}[\ln x(t,\omega) - \ln x(0)] \le 0, \qquad \lim_{t\to+\infty} t^{-1}\int_0^t a(s)x(s,\omega)ds = 0.$$

Substituting these inequalities into (13) and making use of (11), one can obtain the contradiction $0 \ge \limsup_{t\to+\infty} t^{-1} \ln x(t,\omega) = b^* > 0$. $\square$

Remark 3. Theorems 2–4 have an interesting biological interpretation. Observe that the extinction and persistence of species $x(t)$ depend only on b^*. If $b^* > 0$, the population $x(t)$ is weakly persistent. If $b^* < 0$, the population $x(t)$ goes to extinction.

Now, we strengthen the conditions to give some other results.

Theorem 5. *Denote $b_* = \liminf_{t\to+\infty} t^{-1}[\sum_{0<t_k<t} \ln(1+b_k) + \int_0^t b(s)ds]$. Then the solution of Eq. (2) satisfies*

$$\liminf_{t\to+\infty} t^{-1}\int_0^t x(s)ds \ge b_*/\hat{a}, \quad a.s.$$

Particularly, if $b_* > 0$, then the population represented by Eq. (2) is persistent in the mean a.s.

Proof. Without loss of generality, in the proof we suppose that $b_* > 0$. For $\forall\varepsilon > 0$, there exists a T such that

$$t^{-1}\left[\sum_{0<t_k<t} \ln(1+b_k) + \int_0^t b(s)ds\right] \ge b_* - \varepsilon/3, \qquad M_1(t)/t \ge -\varepsilon/3, \qquad \ln x(0)/t \ge -\varepsilon/3$$

for all $t > T$. Substituting these inequalities into Eq. (11) and noting that $t^{-1}\int_0^t a(s)x(s)ds \le \hat{a}t^{-1}\int_0^t x(s)ds$, we obtain

$$\ln x(t) \ge \nu t - \hat{a}\int_0^t x(s)ds; \quad t > T,$$

where $\nu = b_* - \varepsilon$. Let $g(t) = \int_0^t x(s)ds$; then we get

$$\ln(dg/dt) \ge \nu t - \hat{a}g(t); \quad t > T.$$

Consequently

$$\exp\{\hat{a}g(t)\}(dg/dt) \ge \exp\{\nu t\}, \quad t > T.$$

Integrating this inequality from T to t leads to

$$\hat{a}^{-1}[\exp\{\hat{a}g(t)\} - \exp\{\hat{a}g(T)\}] \ge \nu^{-1}[\exp\{\nu t\} - \exp\{\nu T\}].$$

Rewriting this inequality, we have

$$\exp\{\hat{a}g(t)\} \ge \exp\{\hat{a}g(T)\} + \hat{a}\nu^{-1}\exp\{\nu t\} - \hat{a}\nu^{-1}\exp\{\nu T\}.$$

Taking logarithm of both sides gives

$$g(t) \ge \hat{a}^{-1}\ln\{\hat{a}\nu^{-1}\exp\{\nu t\} + \exp\{\hat{a}g(T)\} - \hat{a}\nu^{-1}\exp\{\nu T\}\}.$$

That is to say

$$\liminf_{t\to+\infty} t^{-1}\int_0^t x(s)ds \ge \liminf_{t\to+\infty} \hat{a}^{-1}\{t^{-1}\ln\{\hat{a}\nu^{-1}\exp\{\nu t\} + \exp\{\hat{a}g(T)\} - \hat{a}\nu^{-1}\exp\{\nu T\}\}\}.$$

In view of the L'Hospital's rule, we can observe that

$$\liminf_{t\to+\infty} t^{-1}\int_0^t x(s)ds \ge \liminf_{t\to+\infty} \hat{a}^{-1}\{t^{-1}\ln[\hat{a}\nu^{-1}\exp(\nu t)]\} = \nu/\hat{a}.$$

The desired assertion follows from the arbitrariness of ε. $\square$

Now let us turn to studying the stochastic permanence of Eq. (2).

Assumption 1. There are two positive constants m and M such that $m \le \prod_{0<t_k<t}(1+b_k) \le M$ for all $t > 0$.

Remark 4. Assumption 1 is easy to be satisfied. For example, if $b_k = \exp\{(-1)^{k+1}/k^2\} - 1$, then $\exp\{0.75\} < \prod_{0<t_k<t}(1+b_k) < e$ for all $t > t_2$. Thus $1 \le \prod_{0<t_k<t}(1+b_k) \le e$ for all $t > 0$.

Theorem 6. *Under Assumption 1. If $\check{b} = \min_{t\ge 0}[r(t) - 0.5\sigma^2(t)] > 0$, then $x(t)$ is stochastically permanent.*

Proof. First of all, let us prove that for given $0 < \varepsilon < 1$, there is a positive constant β such that $\liminf_{t\to+\infty} \mathcal{P}\{x(t) \ge \beta\} \ge 1-\varepsilon$. Define $V_1(y) = 1/y$ for $y > 0$. Using Itô's formula to Eq. (7) results in

$$dV_1(y) = -V_1(y)\left[r(t) - \prod_{0<t_k<t}(1+b_k)a(t)y\right]dt + V_1(y)\sigma^2(t)dt - V_1(y)\sigma(t)dB(t).$$

Since $\check{b} > 0$, we can choose a positive constant θ such that $\check{b} > 0.5\theta\hat{\sigma^2}$. Define $V_2(y) = (1+V_1(y))^\theta$. Applying Itô's formula again results in

$$\begin{aligned}
dV_2(y) &= \theta(1+V_1(y))^{\theta-1}dV_1(y) + 0.5\theta(\theta-1)(1+V_1(y))^{\theta-2}(dV_1(y))^2\\
&= \theta(1+V_1(y))^{\theta-2}\left\{-(1+V_1(y))V_1(y)\left[r(t) - a(t)\prod_{0<t_k<t}(1+b_k)y\right]\right.\\
&\quad\left. +\ (1+V_1(y))V_1(y)\sigma^2(t) + 0.5(\theta-1)V_1^2(y)\sigma^2(t)\right\}dt - \theta(1+V_1(y))^{\theta-1}V_1(y)\sigma(t)dB(t)\\
&= \theta(1+V_1(y))^{\theta-2}\left\{-V_1^2(y)[r(t) - 0.5\sigma^2(t) - 0.5\theta\sigma^2(t)]\right.\\
&\quad\left. +\ V_1(y)\left[-r(t) + \sigma^2(t) + a(t)\prod_{0<t_k<t}(1+b_k)\right] + a(t)\prod_{0<t_k<t}(1+b_k)\right\}dt\\
&\quad -\theta(1+V_1(y))^{\theta-1}V_1(y)\sigma(t)dB(t)\\
&\le \theta(1+V_1(y))^{\theta-2}\{-V_1^2(y)[\check{b} - 0.5\theta\hat{\sigma^2}] + V_1(y)[\hat{\sigma^2} + \hat{a}M] + \hat{a}M\}dt\\
&\quad -\theta(1+V_1(y))^{\theta-1}V_1(y)\sigma(t)dB(t).
\end{aligned} \tag{14}$$

Now, choose κ sufficiently small to satisfy

$$0 < \frac{\kappa}{\theta} < \check{b} - 0.5\theta\hat{\sigma^2}.$$

Define $V_3(y) = \exp\{\kappa t\}V_2(y)$. In view of Itô's formula,

$$\begin{aligned}
dV_3(y) &= \kappa\exp\{\kappa t\}V_2(y)dt + \exp\{\kappa t\}dV_2(y)\\
&\le \theta\exp\{\kappa t\}(1+V_1(y))^{\theta-2}\{\kappa(1+V_1(y))^2/\theta - [\check{b} - 0.5\theta\hat{\sigma^2}]V_1^2(y) + V_1(y)[\hat{\sigma^2} + \hat{a}M] + \hat{a}M\}dt\\
&\quad -\theta\exp\{\kappa t\}(1+V_1(y))^{\theta-1}V_1(y)\sigma(t)dB(t)\\
&= \theta\exp\{\kappa t\}(1+V_1(y))^{\theta-2}\{-[\check{b} - 0.5\theta\hat{\sigma^2} - \kappa/\theta]V_1^2(y) + V_1(y)[\hat{\sigma^2} + \hat{a}M + 2\kappa/\theta] + \hat{a}M + \kappa/\theta\}dt\\
&\quad -\theta\exp\{\kappa t\}(1+V_1(y))^{\theta-1}V_1(y)\sigma(t)dB(t)\\
&=: \exp\{\kappa t\}H(y)dt - \theta\exp\{\kappa t\}(1+V_1(y))^{\theta-1}V_1(y)\sigma(t)dB(t),
\end{aligned}$$

where

$$H(y) = \theta(1+V_1(y))^{\theta-2}\{-[\check{b} - 0.5\theta\hat{\sigma^2} - \kappa/\theta]V_1^2(y) + V_1(y)[\hat{\sigma^2} + \hat{a}M + 2\kappa/\theta] + \hat{a}M + \kappa/\theta\}.$$

Now let us prove $H(y)$ is upper bounded in $y > 0$. Set $K_1 = \check{b} - 0.5\theta\hat{\sigma^2} - \kappa/\theta$, $K_2 = \hat{\sigma^2} + \hat{a}M + 2\kappa/\theta$ and $K_3 = \hat{a}M + \kappa/\theta$. Then $K_1 > 0, K_2 > 0, K_3 > 0$ and

$$H(y) = \theta\left(1+\frac{1}{y}\right)^{\theta-2}\left\{-\frac{K_1}{y^2} + \frac{K_2}{y} + K_3\right\} =: \theta\left(1+\frac{1}{y}\right)^{\theta-2}\tilde{H}(y).$$

Case (i). If $\frac{1}{y} \ge \frac{K_2+\sqrt{K_2^2+4K_1K_3}}{2K_1} =: K_4$, then $\tilde{H}(y) \le 0$; thus $H(y) \le 0$.

Case (ii). If $0 < \frac{1}{y} \le K_4$, then $\tilde{H}(y) \le \frac{4K_1K_3+K_2^2}{4K_1}$. As for $\theta(1+\frac{1}{y})^{\theta-2}$, if $\theta \ge 2$, then $\theta(1+\frac{1}{y})^{\theta-2} \le \theta(1+K_4)^{\theta-2}$. If $\theta < 2$, then $\theta(1+\frac{1}{y})^{\theta-2} \le \theta$. Set $K_5 = \max\{\theta(1+K_4)^{\theta-2}, \theta\}$ and $H_1 = K_5\frac{4K_1K_3+K_2^2}{4K_1}$. Then we have shown that $H(y)$ is upper bounded for $y > 0$, namely $\sup_{y>0} H(y) \le H_1$. Consequently,

$$dV_3(y) \le H_1 \exp\{\kappa t\}dt - \theta \exp\{\kappa t\}(1+V_1(y))^{\theta-1}V_1(y)\sigma(t)dB(t).$$

Integrating and then taking expectations, one can get that

$$E[V_3(y(t))] = E[\exp\{\kappa t\}(1+V_1(y(t)))^{\theta}] \le (1+V_1(y(0)))^{\theta} + \frac{H_1}{\kappa}\exp\{\kappa t\}.$$

Consequently,

$$\limsup_{t\to+\infty} E[V_1^{\theta}(y(t))] \le \limsup_{t\to+\infty} E[(1+V_1(y(t)))^{\theta}] \le \frac{H_1}{\kappa}. \tag{15}$$

In other words, we have shown that $\limsup_{t\to+\infty} E[1/y^{\theta}(t)] \le H_1/\kappa =: H_2$. Then

$$\limsup_{t\to+\infty} E[1/x^{\theta}(t)] = \limsup_{t\to+\infty}\left[\prod_{0<t_k<t}(1+b_k)\right]^{-\theta} E[1/y^{\theta}(t)] \le m^{-\theta}H_2 := H_3.$$

Thus for any $\varepsilon > 0$, set $\beta = \varepsilon^{\frac{1}{\theta}}/H_3^{\frac{1}{\theta}}$, by Chebyshev's inequality, we have

$$\mathcal{P}\{x(t) < \beta\} = \mathcal{P}\{x^{-\theta}(t) > \beta^{-\theta}\} \le \frac{E[x^{-\theta}(t)]}{\beta^{-\theta}} = \beta^{\theta}E[x^{-\theta}(t)].$$

In other words, $\limsup_{t\to+\infty} \mathcal{P}\{x(t) < \beta\} \le \beta^{\theta}H_3 = \varepsilon$. Consequently

$$\liminf_{t\to+\infty} \mathcal{P}\{x(t) \ge \beta\} \ge 1-\varepsilon.$$

Next we show that for arbitrary fixed $\varepsilon > 0$, there exists a positive constant δ such that $\liminf_{t\to+\infty} \mathcal{P}\{x(t) \le \delta\} \ge 1-\varepsilon$. For arbitrarily given $q > 0$, applying Itô's formula to Eq. (7), one can observe that

$$\begin{aligned} dy^q(t) &= qy^q(t)\left[r(t) - a(t)\prod_{0<t_k<t}(1+b_k)y(t) + 0.5(q-1)\sigma^2(t)\right]dt + q\sigma(t)y^q(t)dB(t) \\ &\le qy^q(t)[r(t) - a(t)my(t) + 0.5(q-1)\sigma^2(t)]dt + q\sigma(t)y^q(t)dB(t). \end{aligned}$$

Integrating from 0 to t and taking expectations, we obtain

$$E(y^q(t)) - E(y^q(0)) \le q\int_0^t E\{y^q(s)[r(s) - a(s)my(s) + 0.5(q-1)\sigma^2(t)]\}ds.$$

Hence $\frac{dE(y^q(t))}{dt} \le qE(y^q(t))[r(t) + 0.5(q-1)\sigma^2(t)] - ma(t)qE(y^{q+1}(t))$. Note that for $0 < r < p < +\infty$, we have the following Hölder's inequality

$$(E(y^p))^{1/p} \ge (E(y^r))^{1/r}.$$

Choose $p = q+1$ and $r = q$; then we get

$$E(y^{q+1}) \ge (E(y^q))^{(q+1)/q}.$$

Consequently

$$\frac{dE(y^q(t))}{dt} \le qE(y^q(t))[r(t) + 0.5(q-1)\sigma^2(t)] - qma(t)E(y^q(t))^{\frac{q+1}{q}}.$$

Denote $n(t) = E(x^q(t))$; we have

$$\frac{dn}{dt} \le qn(t)[r(t) + 0.5(q-1)\sigma^2(t) - ma(t)n^{1/q}(t)] \le qn(t)[\hat{r} + 0.5q\hat{\sigma^2} - m\check{a}n^{1/q}(t)].$$

Using the standard comparison theorem, we obtain that

$$\limsup_{t\to+\infty} n(t) = \limsup_{t\to+\infty} E(y^q(t)) \le \left[\frac{\hat{r} + 0.5q\hat{\sigma^2}}{m\check{a}}\right]^q =: G_1(q). \tag{16}$$

Consequently,

$$\limsup_{t\to+\infty} E(x^q(t)) = \limsup_{t\to+\infty}\left[\prod_{0<t_k<t}(1+b_k)\right]^q E(y^q(t)) \le \left[M\frac{\hat{r}+0.5q\hat{\sigma^2}}{m\check{a}}\right]^q =: L(q).$$

Then the required assertion follows from Chebyshev's inequality. □

4. Asymptotic pathwise estimation

In Section 3, we have studied the persistence and extinction of Eq. (2). Now let us further examine how this solution pathwisely moves in R_+.

Theorem 7. *Under Assumption 1, the solution of Eq. (2) satisfies*

$$\limsup_{t\to+\infty}\frac{\ln x(t)}{\ln t} \le 1. \tag{17}$$

If moreover, $\check{b} > 0$, *then the solution of Eq. (2) obeys*

$$\liminf_{t\to+\infty}\frac{\ln x(t)}{\ln t} \ge -\frac{\hat{\sigma^2}}{2\check{b}}. \tag{18}$$

Proof. Applying Itô's formula to Eq. (7) results in

$$\begin{aligned} d(\exp(t)\ln y(t)) &= \exp(t)\ln y(t)dt + \exp(t)d\ln y(t) \\ &= \exp(t)\left[\ln y(t) + b(t) - a(t)\prod_{0<t_k<t}(1+b_k)y(t)\right]dt + \exp(t)\sigma(t)dB(t). \end{aligned}$$

Thus, we have already shown that

$$\exp(t)\ln y(t) - \ln y(0) = \int_0^t \exp(s)\left[\ln y(s) + b(s) - a(s)\prod_{0<t_k<s}(1+b_k)y(s)\right]ds + M_2(t), \tag{19}$$

where $M_2(t) = \int_0^t \exp(s)\sigma(s)dB(s)$. The quadratic forms of $M_2(t)$ is

$$\langle M_2(t), M_2(t)\rangle = \int_0^t \exp(2s)\sigma^2(s)ds.$$

In view of the exponential martingale inequality (see e.g. [26], p. 44),

$$\mathcal{P}\left\{\sup_{0\le t\le\gamma k}[M_2(t) - 0.5\exp(-\gamma k)\langle M_2(t), M_2(t)\rangle] > \rho\exp(\gamma k)\ln k\right\} \le k^{-\rho},$$

where $\rho > 1$ and $\gamma > 0$ are arbitrary. By virtue of Borel–Cantelli lemma (see e.g. [26], p. 7), for almost all $\omega \in \Omega$, there exists $k_0(\omega)$ such that for every $k \ge k_0(\omega)$,

$$M_2(t) \le 0.5\exp(-\gamma k)\langle M_2(t), M_2(t)\rangle + \rho\exp(\gamma k)\ln k, \quad 0 \le t \le \gamma k.$$

In other words,

$$M_2(t) \le 0.5\exp(-\gamma k)\int_0^t \exp(2s)\sigma^2(s)ds + \rho\exp(\gamma k)\ln k$$

for $0 \le t \le \gamma k$. Substituting this inequality into (19) results in

$$\begin{aligned} \exp(t)\ln y(t) - \ln y(0) &\le \int_0^t \exp(s)\left[\ln y(s) + b(s) - a(s)\prod_{0<t_k<s}(1+b_k)y(s)\right]ds \\ &\quad + 0.5\exp(-\gamma k)\int_0^t \exp(2s)\sigma^2(s)ds + \rho\exp(\gamma k)\ln k \\ &\le \int_0^t \exp(s)[\ln y(s) + \hat{b} - \check{a}my(s) + 0.5\hat{\sigma^2}]ds + \rho\exp(\gamma k)\ln k, \end{aligned}$$

where in the last inequality, we have used the fact that $s \le \gamma k$. Note that for all $y > 0$, there exists a positive constant C_1 such that $\ln y + \hat{b} - \check{a}my + 0.5\widehat{\sigma^2} \le C_1$. In other words, for any $0 \le t \le \gamma k$, we have

$$\exp(t)\ln y(t) - \ln y(0) \le C_1[\exp(t) - 1] + \rho\exp(\gamma k)\ln k.$$

That is to say

$$\ln y(t) \le \exp(-t)\ln y(0) + C_1[1 - \exp(-t)] + \rho\exp(-t)\exp(\gamma k)\ln k.$$

If $\gamma(k-1) \le t \le \gamma k$ and $k \ge k_0(\omega)$, we have

$$\ln y(t)/\ln t \le \exp(-t)\ln y(0)/\ln t + C_1[1 - \exp(-t)]/\ln t + \rho\exp(-\gamma(k-1))\exp(\gamma k)\ln k/\ln t.$$

That is to say $\limsup_{t\to+\infty}\frac{\ln y(t)}{\ln t} \le \rho\exp(\gamma)$. Letting $\rho \to 1$ and $\gamma \to 0$ leads to $\limsup_{t\to+\infty}\frac{\ln y(t)}{\ln t} \le 1$. Moreover, it follows from Assumption 1 that

$$\lim_{t\to+\infty}\frac{\prod_{0<t_k<t}(1+b_k)}{\ln t} = 0. \tag{20}$$

Then we obtain

$$\limsup_{t\to+\infty}\frac{\ln x(t)}{\ln t} = \limsup_{t\to+\infty}\frac{\sum_{0<t_k<t}\ln(1+b_k) + \ln y(t)}{\ln t} = \limsup_{t\to+\infty}\frac{\ln y(t)}{\ln t} \le 1.$$

Now let us prove (17). By (15), there exists a constant $C_2 > 0$ such that

$$E[(1 + V_1(y(t)))^\theta] \le C_2, \quad t \ge 0. \tag{21}$$

At the same time, it follows from (14) that

$$\begin{aligned} dV_2(y) &\le \theta(1+V_1(y))^{\theta-2}\{-V_1^2(y)[\check{b} - 0.5\theta\widehat{\sigma^2}] + V_1(y)[\widehat{\sigma^2} + \hat{a}M] + \hat{a}M\}dt - \theta(1+V_1(y))^{\theta-1}V_1(y)\sigma(t)dB(t) \\ &\le \theta C_3(1+V_1(y))^\theta - \theta(1+V_1(y))^{\theta-1}V_1(y)\sigma(t)dB(t) \end{aligned} \tag{22}$$

where $C_3 = \max\{|\check{b} - 0.5\theta\widehat{\sigma^2}|, 0.5[\widehat{\sigma^2} + \hat{a}M], \hat{a}M\}$. Let $\upsilon > 0$ be sufficiently small for

$$\theta\left[C_3\upsilon + 6\upsilon^{0.5}\sqrt{\widehat{\sigma^2}}\right] < 0.5. \tag{23}$$

Let $k = 1, 2, \ldots,$ making use of (22) gives that

$$\begin{aligned} E\left(\sup_{(k-1)\upsilon\le t\le k\upsilon}(1+V_1(y(t)))^\theta\right) &\le E(1+V_1(y((k-1)\upsilon)))^\theta + E\left(\sup_{(k-1)\upsilon\le t\le k\upsilon}\left|\int_{(k-1)\upsilon}^t \theta C_3(1+V_1(y(s)))^\theta ds\right|\right) \\ &\quad + E\left(\sup_{(k-1)\upsilon\le t\le k\upsilon}\left|\int_{(k-1)\upsilon}^t \theta(1+V_1(y(s)))^{\theta-1}V_1(y(s))\sigma(s)dB(s)\right|\right). \end{aligned} \tag{24}$$

Compute that

$$\begin{aligned} E\left(\sup_{(k-1)\upsilon\le t\le k\upsilon}\left|\int_{(k-1)\upsilon}^t \theta C_3(1+V_1(y(s)))^\theta ds\right|\right) &\le E\left(\int_{(k-1)\upsilon}^{k\upsilon}\left|\theta C_3(1+V_1(y(s)))^\theta\right|ds\right) \\ &\le \theta C_3\upsilon E\left(\sup_{(k-1)\upsilon\le t\le k\upsilon}(1+V_1(y(t)))^\theta\right). \end{aligned} \tag{25}$$

At the same time, by Burkholder–Davis–Gundy inequality,

$$\begin{aligned} &E\left(\sup_{(k-1)\upsilon\le t\le k\upsilon}\left|\int_{(k-1)\upsilon}^t \theta(1+V_1(y(s)))^{\theta-1}V_1(y(s))\sigma(s)dB(s)\right|\right) \\ &\le 6E\left(\int_{(k-1)\upsilon}^{k\upsilon}\theta^2(1+V_1(y(s)))^{2\theta-2}V_1^2(y(s))\sigma^2(s)ds\right)^{0.5} \\ &\le 6\theta\sqrt{\widehat{\sigma^2}}E\left(\int_{(k-1)\upsilon}^{k\upsilon}(1+V_1(y(s)))^{2\theta}ds\right)^{0.5} \\ &\le 6\theta\upsilon^{0.5}\sqrt{\widehat{\sigma^2}}E\left(\sup_{(k-1)\upsilon\le t\le k\upsilon}(1+V_1(y(t)))^\theta\right). \end{aligned} \tag{26}$$

Substituting (25) and (26) into (24) results in

$$E\left(\sup_{(k-1)\upsilon\le t\le k\upsilon}(1+V_1(x(t)))^{\theta}\right)\le E(1+V_1(x((k-1)\upsilon)))^{\theta}$$
$$+\theta\left[C_3\upsilon+6\upsilon^{0.5}\sqrt{\hat{\sigma^2}}\right]E\left(\sup_{(k-1)\upsilon\le t\le k\upsilon}(1+V_1(x(t)))^{\theta}\right).$$

It then follows from (21) and (23) that

$$E\left(\sup_{(k-1)\upsilon\le t\le k\upsilon}(1+V_1(y(t)))^{\theta}\right)\le 2C_2.$$

Let $\varepsilon>0$ be arbitrary. Then by the Chebyshev inequality, we obtain

$$\mathcal{P}\left\{\omega:\sup_{(k-1)\upsilon\le t\le k\upsilon}(1+V_1(y(t)))^{\theta}>(k\upsilon)^{1+\varepsilon}\right\}\le\frac{2C_2}{(k\upsilon)^{1+\varepsilon}},\quad k=1,2,\ldots.$$

An application of Borel–Cantelli lemma leads to that for almost all $\omega\in\Omega$, there is a random integer $k_0=k_0(\omega)$ such that for $k\ge k_0$ and $(k-1)\upsilon\le t\le k\upsilon$,

$$\frac{\ln(1+V_1(y(t)))^{\theta}}{\ln t}\le\frac{(1+\varepsilon)\ln(k\upsilon)}{\ln((k-1)\upsilon)}.$$

In other words, we have shown that

$$\limsup_{t\to+\infty}\frac{\ln(1+V_1(y(t)))^{\theta}}{\ln t}\le 1.$$

Recalling the definition of $V_1(y)$, we have $\limsup_{t\to+\infty}\frac{\ln y^{-\theta}(t)}{\ln t}\le 1$. Consequently $\liminf_{t\to+\infty}\frac{\ln y(t)}{\ln t}\ge -1/\theta$. But this holds for any θ that obeys $\check{b}>0.5\theta\hat{\sigma^2}$, therefore

$$\liminf_{t\to+\infty}\frac{\ln y(t)}{\ln t}\ge-\frac{\hat{\sigma^2}}{2\check{b}}.$$

Then the required assertion (17) follows from (20) immediately. □

Theorem 7 indicates that for any $\varepsilon>0$, there is a positive random variable T_ε such that $t^{-(0.5\hat{\sigma^2}/\check{b}+\varepsilon)}\le x(t)\le t^{1+\varepsilon}$ for $t\ge T_\varepsilon$ with probability one. In other words, with probability one, the solution will not grow faster than $t^{1+\varepsilon}$ and will not decay faster than $t^{-(0.5\hat{\sigma^2}/\check{b}+\varepsilon)}$.

5. Global attractivity

Now let us study the global attractivity of Eq. (2).

Definition 4. Let $x_1(t), x_2(t)$ be two arbitrary solutions of Eq. (2) with initial values $x_1(0), x_2(0)>0$ respectively. If $\lim_{t\to+\infty}|x_1(t)-x_2(t)|=0$ *a.s.*, then we say Eq. (2) is globally attractive.

To begin with, we prepare some useful lemmas.

Lemma 8 (*See e.g. [27]*)**.** *Suppose that an n-dimensional stochastic process $X(t)$ on $t\ge 0$ satisfies the condition*

$$E|X(t)-X(s)|^{\alpha}\le c|t-s|^{1+\beta},\quad 0\le s,\ t<\infty,$$

for some positive constants α, β and c. Then there exists a continuous modification $\tilde{X}(t)$ of $X(t)$ which has the property that for every $\vartheta\in(0,\beta/\alpha)$ there is a positive random variable $h(\omega)$ such that

$$\mathcal{P}\left\{\omega:\sup_{0<|t-s|<h(\omega),\ 0\le s,\ t<\infty}\frac{|\tilde{X}(t,\omega)-X(t,\omega)|}{|t-s|^{\vartheta}}\le\frac{2}{1-2^{-\vartheta}}\right\}=1.$$

In other words, almost every sample path of $\tilde{X}(t)$ is locally but uniformly Hölder continuous with exponent ϑ.

Lemma 9. *Let $y(t)$ be a solution of (7) for any initial values $y(0)=y_0>0$. If Assumption 1 holds, then almost every sample path of $y(t)$ is uniformly continuous for $t\ge 0$.*

Proof. It follows from (16) that there is a $T > 0$, such that $E(y^q(t)) \leq 1.5G_1(q)$ for all $t \geq T$. At the same time, by the continuity of $E(y^q(t))$, it is clear that there is a $G_2(q) > 0$ such that $E(y^q(t)) \leq G_2(q)$ for $t \leq T$. Denote $G(q) = \max\{1.5G_1(q), G_2(q)\}$; then we have for all $t \geq 0$,

$$E(y^q(t)) \leq G(q).$$

On the other hand, Eq. (7) is equivalent to the following integral equation

$$y(t) = y_0 + \int_0^t y(s)\left[r(s) - a(s)\prod_{0<t_k<s}(1+b_k)y(s)\right]ds + \int_0^t \sigma(s)y(s)dB(s).$$

At the same time, it is easy to see that

$$\begin{aligned} E\left|y(t)\left(r(t) - a(t)\prod_{0<t_k<t}(1+b_k)y(t)\right)\right|^q &= E\left[|y(t)|^q\left|r(t) - a(t)\prod_{0<t_k<t}(1+b_k)y(t)\right|^q\right] \\ &\leq 0.5E|y(t)|^{2q} + 0.5E\left|r(t) - a(t)\prod_{0<t_k<t}(1+b_k)y(t)\right|^{2q} \\ &\leq 0.5G(2q) + 2^{2q-2}(|\hat{r}|^{2q} + |\hat{a}M|^{2p}E|y(t)|^{2q}) \\ &= 0.5G(2q) + 2^{2q-2}(|\hat{r}|^{2p} + |\hat{a}M|^{2q}G(2q)) =: K_1(q). \end{aligned}$$

Making use of the moment inequality for stochastic integrals (see e.g. [26], Theorem 2.11 on p. 69) gives that for $0 \leq t_1 \leq t_2$ and $q > 2$,

$$\begin{aligned} E\left|\int_{t_1}^{t_2}\sigma(s)y(s)dB(s)\right|^q &\leq (\hat{\sigma^2})^q\left[\frac{q(q-1)}{2}\right]^{q/2}(t_2-t_1)^{(q-2)/2}\int_{t_1}^{t_2}E|y(s)|^q ds \\ &\leq (\hat{\sigma^2})^q\left[\frac{q(q-1)}{2}\right]^{q/2}(t_2-t_1)^{q/2}G(q). \end{aligned}$$

Then for $0 < t_1 < t_2 < \infty,\ t_2 - t_1 \leq 1,\ 1/q + 1/p = 1$, one can derive that

$$\begin{aligned} E(|y(t_2) - y(t_1)|^q) &= E\left|\int_{t_1}^{t_2} y(s)\left[r(s) - a(s)\prod_{0<t_k<s}(1+b_k)y(s)\right]ds + \int_{t_1}^{t_2}\sigma(s)x(s)dB(s)\right|^q \\ &\leq 2^{q-1}E\left|\int_{t_1}^{t_2} y(s)\left[r(s) - a(s)\prod_{0<t_k<s}(1+b_k)y(s)\right]ds\right|^q + 2^{q-1}E\left|\int_{t_1}^{t_2}\sigma(s)x(s)dB(s)\right|^q \\ &\leq 2^{q-1}(t_2-t_1)^{q/p}\int_{t_1}^{t_2}E\left|y(s)\left[r(s) - a(s)\prod_{0<t_k<s}(1+b_k)y(s)\right]\right|^q ds \\ &\quad + 2^{q-1}(\hat{\sigma^2})^q\left[\frac{q(q-1)}{2}\right]^{q/2}(t_2-t_1)^{q/2}G(p) \\ &= 2^{q-1}(t_2-t_1)^{q/p+1}K_1(q) + 2^{q-1}(\hat{\sigma^2})^q\left[\frac{q(q-1)}{2}\right]^{q/2}(t_2-t_1)^{q/2}G(q) \\ &\leq 2^{q-1}(t_2-t_1)^{q/2}\left[(t_2-t_1)^{q/2} + \left[\frac{q(q-1)}{2}\right]^{q/2}\right]K_2(q) \\ &\leq 2^{q-1}(t_2-t_1)^{q/2}\left[1 + \left[\frac{q(q-1)}{2}\right]^{q/2}\right]K_2(q), \end{aligned}$$

where $K_2(q) = \max\{K_1(q), (\hat{\sigma_1^2})^pG(q)\}$. Then it follows from Lemma 8 that almost every sample path of $y(t)$ is locally but uniformly Hölder-continuous with exponent ϑ for every $\vartheta \in (0, \frac{q-2}{2q})$ and therefore almost every sample path of $y(t)$ is uniformly continuous on $t \geq 0$. □

Lemma 10 (*See e.g. [28]*). *Let f be a non-negative function defined on R_+ such that f is integrable on R_+ and is uniformly continuous on R_+. Then $\lim_{t\to+\infty} f(t) = 0$.*

Now, we are in the position to give our main result of this section.

Theorem 11. *If Assumption 1 holds, then Eq. (2) is globally attractive.*

Proof. Let $x_1(t)$ and $x_2(t)$ be two arbitrary solutions of Eq. (2) with initial values $x_1(0), x_2(0) > 0$ respectively. Suppose that the solution of Eq.

$$dy(t) = y(t)\left(r(t) - \prod_{0<t_k<t}(1+b_k)a(t)y(t)\right)dt + \sigma(t)y(t)dB(t); \qquad y(0) = x_1(0)$$

is $y_1(t)$ and the solution of Eq.

$$dy(t) = y(t)\left(r(t) - \prod_{0<t_k<t}(1+b_k)a(t)y(t)\right)dt + \sigma(t)y(t)dB(t); \qquad y(0) = x_2(0)$$

is $y_2(t)$. Then we have $x_1(t) = \prod_{0<t_k<t}(1+b_k)y_1(t),\ x_2(t) = \prod_{0<t_k<t}(1+b_k)y_2(t)$. Define $\bar{V}(t) = |\ln y_1(t) - \ln y_2(t)|$. Then $\bar{V}(t)$ is continuous and positive on $t \geq 0$. A calculation of the right differential $d^+\bar{V}(t)$ of $\bar{V}(t)$, and making use of Itô's formula, we obtain

$$\begin{aligned} d^+\bar{V}(t) &= sgn(y_1(t) - y_2(t))d(\ln y_1(t) - \ln y_2(t)) \\ &= sgn(y_1(t) - y_2(t))\left[-\prod_{0<t_k<t}(1+b_k)a(t)(y_1(t) - y_2(t))\right]dt \\ &= -a(t)\prod_{0<t_k<t}(1+b_k)|y_1(t) - y_2(t)|dt \\ &\leq -m\check{a}|y_1(t) - y_2(t)|dt. \end{aligned}$$

Integrating both sides and then taking the expectation, we can see that

$$\bar{V}(t) \leq \bar{V}(0) - \check{a}m\int_0^t |y_1(s) - y_2(s)|ds.$$

Consequently,

$$\bar{V}(t) + \check{a}m\int_0^t |y_1(s) - y_2(s)|ds \leq \bar{V}(0) < \infty.$$

Then it follows from $\bar{V}(t) \geq 0$ that $|y_1(t) - y_2(t)| \in L^1[0, \infty)$. Then it follows from Lemmas 9 and 10 that $\lim_{t\to+\infty}|y_1(t) - y_2(t)| = 0$. Thus

$$\lim_{t\to+\infty}|x_1(t) - x_2(t)| = \lim_{t\to+\infty}\prod_{0<t_k<t}(1+b_k)|y_1(t) - y_2(t)| \leq M\lim_{t\to+\infty}|y_1(t) - y_2(t)| = 0.$$

This completes the proof. □

To close this section, we prepare a corollary which will be used later.

Corollary 12. *Consider Eq. (1), then*

($\mathscr{A}$) (i) *If* $\bar{b}^* = \limsup_{t\to+\infty} t^{-1}\int_0^t b(s)ds < 0$, *then* $x(t)$ *represented by Eq. (1) goes to extinction a.s.;*
(ii) *If* $\bar{b}^* = 0$, *then* $x(t)$ *is non-persistent in the mean a.s.;*
(iii) *If* $\bar{b}^* > 0$, *then* $x(t)$ *is weakly persistent a.s.;*
(iv) *If* $\bar{b}_* = \liminf_{t\to+\infty} t^{-1}\int_0^t b(s)ds > 0$, *then* $x(t)$ *is persistent in the mean a.s.;*
(v) *If* $\check{b} = \min_{t\geq 0}[r(t) - 0.5\sigma^2(t)] > 0$, *then* $x(t)$ *is stochastically permanent.*

($\mathscr{B}$) *Eq. (1) is globally attractive.*

6. Concluding remarks and examples and numerical simulations

In this paper, a stochastic logistic equation with impulsive perturbations is proposed and studied. We first give a new definition of a solution of ISDE, which is more convenient for use than the existing one (see Remark 1). Using this new definition, we show that our model has a global and positive solution for any positive initial condition and obtain its explicit expression. Then we establish the sufficient conditions for extinction, non-persistence in the mean, weak persistence, persistence in the mean and stochastic permanence of the solution. The critical value between weak persistence and extinction is obtained. In addition, the limit of the average in time of the sample path of the solution is estimated by two constants. Afterwards, the lower-growth rate and the upper-growth rate of the solution are estimated. Finally, we investigate the global attractivity. More precisely, if $\check{a} > 0$,

(A) Eq. (2) has an explicit solution of the form:

$$x(t) = \frac{\prod_{0\le t_k<t}(1+b_k)\exp\left\{\int_0^t[r(s)-0.5\sigma^2(s)]ds+\int_0^t\sigma(s)dB(s)\right\}}{1/x_0+\int_0^t\prod_{0<t_k<s}(1+b_k)a(s)\exp\left\{\int_0^s[r(\tau)-0.5\sigma^2(\tau)]d\tau+\int_0^s\sigma(\tau)dB(\tau)\right\}ds}.$$

(B) (a) If $b^* = \limsup_{t\to+\infty} t^{-1}[\sum_{0<t_k<t}\ln(1+b_k)+\int_0^t b(s)ds]$, then $x(t)$ goes to extinction a.s.;
(b) If $b^* = 0$, then $x(t)$ is non-persistent in the mean a.s.;
(c) If $b^* > 0$, then $x(t)$ is weakly persistent a.s.;
(d) If $b_* = \liminf_{t\to+\infty} t^{-1}[\sum_{0<t_k<t}\ln(1+b_k)+\int_0^t b(s)ds] > 0$, then $x(t)$ is persistent in the mean a.s.;
(e) If $\check{b} = \min_{t\ge0}[r(t)-0.5\sigma^2(t)] > 0$ and moreover, Assumption 1 holds, then $x(t)$ is stochastically permanent.

(C) The solution $x(t)$ obeys

$$b_*/\hat{a} \le \liminf_{t\to+\infty} t^{-1}\int_0^t x(s)ds \le \limsup_{t\to+\infty} t^{-1}\int_0^t x(s)ds \le b^*/\check{a} \quad \text{a.s.}$$

(D) Under Assumption 1, the solution of Eq. (2) satisfies

$$\limsup_{t\to+\infty}\frac{\ln x(t)}{\ln t} \le 1 \quad \text{a.s.}$$

If moreover, $\check{b} > 0$, then

$$\liminf_{t\to+\infty}\frac{\ln x(t)}{\ln t} \ge -0.5\hat{\sigma^2}/\check{b} \quad \text{a.s.}$$

(E) If Assumption 1 holds, then Eq. (2) is globally attractive.

The present paper is the first attempt, so far as our knowledge is concerned, to investigate the stochastic population systems with impulsive perturbations.

Note that if the impulsive perturbations are bounded (i.e. Assumption 1 holds), then $b^* = \bar{b}^*$ and by comparing our Theorems 1–7 and 11 with Corollary 12 we can find that the impulse does not affect the properties including extinction, persistence, stochastic permanence, global attractivity. However, if the impulsive perturbations are unbounded, some properties including persistence and extinction could be changed significantly. To see this more clearly, let us consider the following examples.

Example 1. Consider the following model

$$\begin{cases} dx(t) = x(t)(r(t)-a(t)x(t))dt+\sigma(t)x(t)dB(t), & t\ne t_k,\ k\in N\\ x(t_k^+)-x(t_k) = b_k x(t_k), & k\in N \end{cases} \tag{27}$$

where $t_k = 10k$, $r(t) = 0.4+0.1\sin t$, $a(t) = 1-\cos 2t$, $\sigma(t) = \sqrt{0.802}+0.2\sin 2t$. First of all, set $b_k = 0$ for all $k \in N$ (i.e., there are no impulsive perturbations), then we have $\bar{b}^* = b^* = -0.001 < 0$. Then it follows from Corollary 12 that the population $x(t)$ goes to extinction; see Fig. 1(a). However, if we take the impulsive perturbations into account by setting $b_k = \exp\{0.5\}-1$ for all $k \in N$, then $b^* = 0.499 > 0$. By Theorem 4, we can obtain that the population $x(t)$ is weakly persistent; see Fig. 1(b).

Example 2. Consider Eq. (27) again, where $t_k = 10k$, $r(t) = 0.4+0.1\sin 2t$, $a(t) = 1-\cos t$, $\sigma(t) = \sqrt{0.78}+0.2\sin t$. First of all, set $b_k = 0$ for $k \in N$, then we have $\bar{b}^* = b^* = 0.01 > 0$. Making use of Corollary 12, one can see that the population $x(t)$ is weakly persistent; see Fig. 2(a). However, if we set $b_k = \exp\{-1\}-1$ for $k \in N$, then $b^* = -0.99 < 0$. In view of Theorem 2, we can get that the population $x(t)$ goes to extinction; see Fig. 2(b).

Example 3. Consider Eq. (27) again, where $t_k = k$, $a(t) = 1-\cos t$, $b_k = \exp\{(-1)^{k+1}/k^2\}-1$, then $\exp\{0.75\} < \prod_{k=1}^{\infty}(1+b_k) < e$. First, suppose that $r(t) = 0.4+0.1\sin 2t$, $\sigma(t) = \sqrt{0.7}+0.2\sin t$. Then it follows from Theorem 6 that the population represented by Eq. (27) is stochastically permanent; Fig. 3 confirms this. Now suppose that $r(t) = 0.45+0.2\sin 2t$, $\sigma(t) = \sqrt{0.8}+0.1\sin 2t$, $x_1(0) = 0.08$ and $x_2(t) = 0.02$. An application of Theorem 11, we obtain that Eq. (27) is globally attractive; see Fig. 4.

This paper devotes to studying Eq. (2) which is basic and important. Our results are presented for an one dimensional system, part methods developed here are also applicable to Lotka–Volterra systems with two or more species, and we leave this for future work.

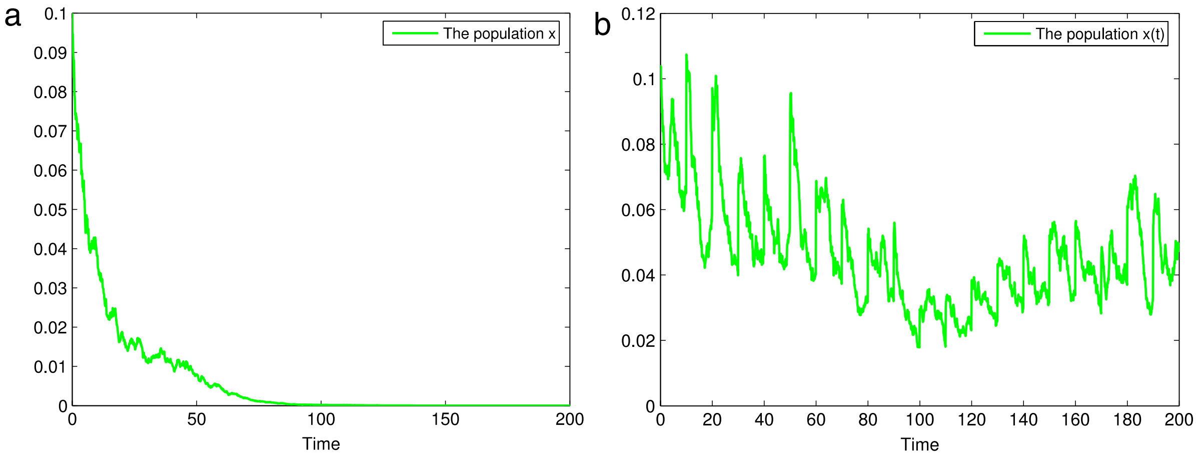

Fig. 1. Eq. (27) for $t_k = 10k$, $r(t) = 0.4 + 0.1\sin t$, $a(t) = 1 - \cos 2t$, $\sigma(t) = \sqrt{0.802} + 0.2\sin 2t$, $x(0) = 0.1$, step size $\Delta t = 0.001$. The horizontal axis in this and following figures represents the time t. (a): $b_k = 0$, $k \in N$; (b): $b_k = \exp\{0.5\} - 1$, $k \in N$.

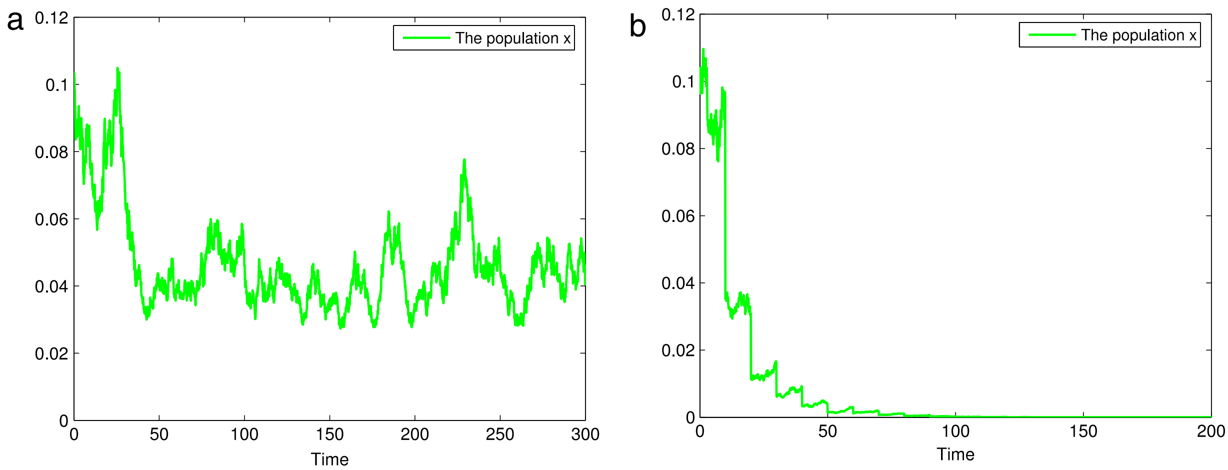

Fig. 2. Eq. (27) for $t_k = 10k$, $r(t) = 0.4 + 0.1\sin 2t$, $a(t) = 1 - \cos t$, $\sigma(t) = \sqrt{0.78} + 0.2\sin t.$, $x(0) = 0.1$, $\Delta t = 0.001$. (a): $b_k = 0$, $k \in N$; (b): $b_k = \exp\{-1\} - 1$, $k \in N$.

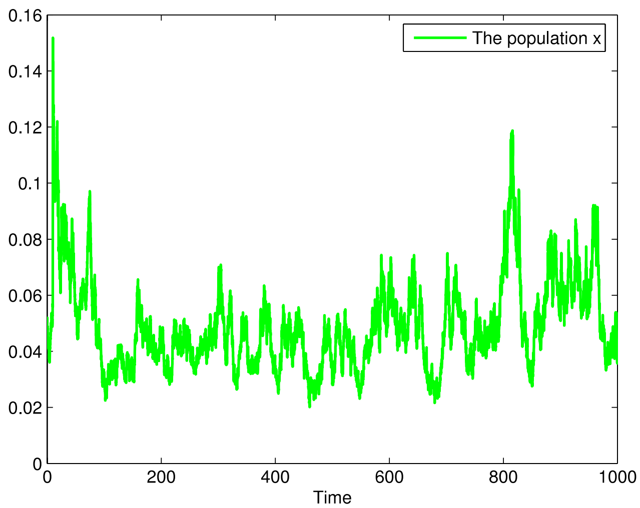

Fig. 3. Eq. (27) for $t_k = k$, $a(t) = 1 - \cos t$, $b_k = \exp\{(-1)^{k+1}/k^2\} - 1$, $r(t) = 0.4 + 0.1\sin 2t$, $\sigma(t) = \sqrt{0.7} + 0.2\sin t$, $x(0) = 0.05$, $\Delta t = 0.001$.

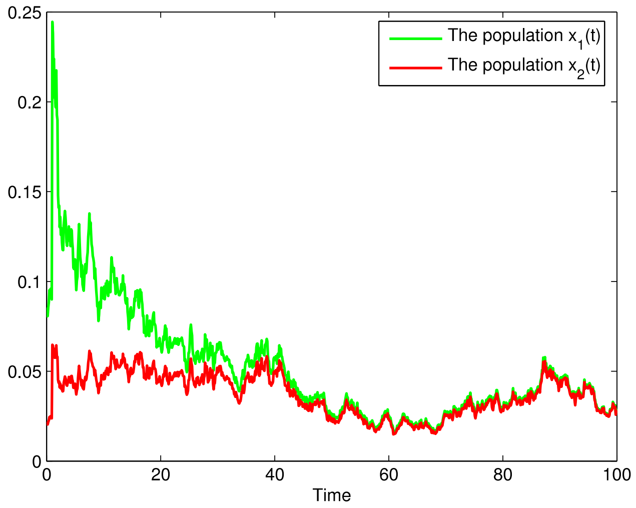

Fig. 4. Eq. (27) for $t_k = k$, $a(t) = 1 - \cos t$, $b_k = \exp\{(-1)^{k+1}/k^2\} - 1$, $r(t) = 0.45 + 0.2\sin 2t$, $\sigma(t) = \sqrt{0.8} + 0.1\sin 2t$, $x_1(0) = 0.08$, $x_2(t) = 0.02$, $\Delta t = 0.001$.

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