Blaschke products and the rank of backward shift invariant subspaces over the bidisk

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Abstract

Let $H^2(D^2)$ be the Hardy space over the bidisk. For sequences of Blaschke products $\{\varphi_n(z): -\infty < n < \infty\}$ and $\{\psi_n(w): -\infty < n < \infty\}$ satisfying some additional conditions, we may define a Rudin type invariant subspace $M$. We shall determine the rank of $H^2(D^2) \ominus M$ for the pair of operators $T_z^*$ and $T_w^*$.

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1. Introduction

Let $H^2 = H^2(D^2)$ be the Hardy space over the bidisk $D^2$. We denote by $z$, $w$ variables in $D^2$. Let $H^2(z)$ and $H^2(w)$ be the one variable Hardy spaces with variables $z$ and $w$, respectively. Then $H^2$ coincides with the tensor product $H^2(z) \otimes H^2(w)$. Let $T_z$ and $T_w$ be the multiplication operators on $H^2$ by $z$ and $w$, respectively. A closed subspace $M$ of $H^2$ is said to be invariant if $T_zM \subset M$ and $T_wM \subset M$. For an invariant subspace $M$, let $N = H^2 \ominus M$. Then $T_z^*N \subset N$ and $T_w^*N \subset N$, so $N$ is said to be a backward shift invariant subspace of $H^2$. The structure of
invariant and backward shift invariant subspaces has been studied from various points of view [4, 6, 9–11, 17–19].

For a subset $E$ of $M$, we denote by $[E]_T$ the smallest invariant subspace of $H^2$ containing $E$ for the pair of operators $T_*$ and $T_w$. Then $E \subset [E]_T \subset M$. When $[E]_T = M$, $E$ is said to be a generating set of $M$, and if $[f]_T = M$, then $f$ is called a generator of $M$. We denote by $\#E$ the number of elements in $E$. The purpose of this paper is to determine rank $\{4, 6, 9–11, 17–19\}$. The number of elements in $E$ has been studied from various points of view.

Ahern and Clark [1] (see also Chen and Guo’s book [4]). In Theorem 3.1, we shall determine rank to know the values of these indices. But in many cases, it is difficult to determine rank $M$ and we denote it by rank$_T M$. Similarly we may define rank$_T N$ the rank of $N$ for the pair of operators $T_*$ and $T_w$. Since rank$_T M$ and rank$_T N$ are important indices for invariant subspaces $M$ of $H^2$, to study the structure of invariant subspaces we need to know the values of these indices. But in many cases, it is difficult to determine rank$_T M$ and rank$_T N$. In [8], the authors determined rank$_T M_\phi$ for special type of invariant subspaces $M_\phi$ (defined below). The purpose of this paper is to determine rank$_T (H^2 \ominus M_\phi)$.

Let $L^2_a(\mathbb{D})$ be the Bergman space over $\mathbb{D}$ and $B$ be the Bergman shift. In 1996, Aleman, Richter and Sundberg [2] gave a big progress studying invariant subspaces of $L^2_a(\mathbb{D})$ for $B$. They proved that rank$_B M = \dim(M \ominus BM)$ for every invariant subspace $M$ of $L^2_a(\mathbb{D})$. This result reveals the inside structure of invariant subspaces in the Bergman space and is a fundamental theorem in the function theory in $L^2_a$. Authors think it is significant to determine the rank of invariant subspaces for the basic operators.

For a sequence $\{\alpha_n\}_{n \geq 1}$ in $\mathbb{D}$ satisfying $\sum_{n=1}^{\infty} (1 - |\alpha_n|) < \infty$, the function

$$\varphi(z) = \prod_{n=1}^{\infty} \frac{-\alpha_n}{|\alpha_n|} \frac{z - \alpha_n}{1 - \overline{\alpha_n} z}, \quad z \in \mathbb{D}$$

is called the Blaschke product with zeros $\{\alpha_n\}_{n \geq 1}$, where we consider $-\overline{\alpha_n}/|\alpha_n| = 1$ if $\alpha_n = 0$. Then $\varphi(z)$ is an inner function, that is, $|\varphi(z)| = 1$ a.e. on $\partial \mathbb{D}$ (see [7] for the study of $H^2(z)$).

We consider that a unimodular constant is a Blaschke product. For Blaschke products $\varphi(z)$ and $\psi(z)$, we write $\varphi(z) < \psi(z)$ if $\psi(z)/\varphi(z)$ is a Blaschke product.

Let $\{\varphi_n(z)\}_{n \geq 0}$ be a sequence of Blaschke products such that $\varphi_n(z)/\varphi_{n+1}(z)$ is a nonconstant Blaschke product for every $n \geq 0$ and $\{\varphi_n(z)\}_{n \geq 0}$ has no nonconstant common Blaschke factors. Let

$$M_\varphi = \bigvee_{n=0}^{\infty} \varphi_n(z) w^n H^2,$$

where $\bigvee_{n=0}^{\infty} E_n$ denotes the closed linear span of $\bigcup_{n=0}^{\infty} E_n$. Then $M_\varphi$ is an invariant subspace of $H^2$. This type of invariant subspaces were first studied by Rudin [13, p. 72]. He gave an example of $M_\varphi$ satisfying rank$_T M_\varphi = \infty$. In [8], the authors determined rank$_T M_\varphi$ completely. The number of rank$_T M_\varphi$ is deeply concerned with the orders of zeros of $\varphi_n$ for $n = 0, 1, 2, \ldots$. See also [5, 14, 15] for the study of $M_\varphi$.

Let $N_\varphi = H^2 \ominus M_\varphi$. In this paper, we shall study rank$_T N_\varphi$. As we see later, determining rank$_T N_\varphi$ is much easier than determining rank$_T M_\varphi$. First, we study backward shift invariant subspaces $N$ satisfying dim $N < \infty$. In this case, the structures of $M$ and $N$ were studied by Ahern and Clark [1] (see also Chen and Guo’s book [4]). In Theorem 3.1, we shall determine rank$_T N$. The ideas in this paper are in the proof of Theorem 3.1.

Let $\{\varphi_n(z)\}_{n=\infty}^{\infty}$ and $\{\psi_n(w)\}_{n=\infty}^{\infty}$ be sequences of Blaschke products satisfying that
(1) \( \varphi_{n+1}(z) < \varphi_n(z) \) and \( \psi_n(w) < \psi_{n+1}(w) \) for every \( n \)

and

(2) each \( \{ \varphi_n(z) \}_{n=-\infty}^{\infty} \) and \( \{ \psi_n(w) \}_{n=-\infty}^{\infty} \) has no nonconstant common Blaschke factors.

Let

\[
M = \bigvee_{n=-\infty}^{\infty} \varphi_n(z) \psi_{n-1}(w) H^2.
\]

Then \( M \) is an invariant subspace of \( H^2 \) and a generalized space of \( M \varphi \).

Let \( N = H^2 \ominus M \). Then \( N \) is a backward shift invariant subspace of \( H^2 \) and

\[
N = \bigvee_{n=-\infty}^{\infty} \left( H^2(z) \ominus \varphi_n(z) H^2(z) \right) \otimes \left( H^2(w) \ominus \psi_n(w) H^2(w) \right).
\]

In Theorem 4.1, as an application of Section 3 we shall determine \( \text{rank}_T N \).

It is interesting to study the structure of invariant subspaces \( M \) of \( H^2 \) generated by some functions of the forms \( \varphi(z) \psi(w) \) for the Blaschke products \( \varphi \) and \( \psi \). The ideas used in this paper may give us some light upon determining \( \text{rank}_T M \) and \( \text{rank}_T N \) in the further study.

2. Lemmas

For an inner function \( \varphi(z) \), it is known that \( T_z^* \varphi(z) \) is a generator of \( H^2(z) \ominus \varphi(z) H^2(z) \) for \( T_z^* \) (see [3,12]). For a function \( \psi(z) \in H^\infty(z) \), let \( T_{\psi} \) be the Toeplitz operator on \( H^2(z) \). The following is an immediate consequence of Proposition III.4.7 in [16].

**Lemma 2.1.** Let \( \varphi(z), \psi(z) \) be inner functions having no nonconstant common inner factors. If \( f(z) \) is a generator of \( H^2(z) \ominus \varphi(z) H^2(z) \) for \( T_z^* \), then \( T_{\psi}^* f(z) \) is a generator of \( H^2(z) \ominus \varphi(z) H^2(z) \) for \( T_z^* \).

**Lemma 2.2.** Let \( \varphi(z), \psi(z) \) be inner functions and \( q(z) \) be the greatest common inner factor of \( \varphi(z) \) and \( \psi(z) \). Put \( \varphi_1(z) = \varphi(z)/q(z) \). If \( f(z) \) is a generator of \( H^2(z) \ominus \varphi(z) H^2(z) \) for \( T_z^* \), then \( T_{\psi}^* f(z) \) is a generator of \( H^2(z) \ominus \varphi_1(z) H^2(z) \) for \( T_z^* \).

**Proof.** It is easy to see that \( T_q^* f(z) \) is a generator of \( H^2(z) \ominus \varphi_1(z) H^2(z) \) for \( T_z^* \). By Lemma 2.1, \( T_{\psi}^* f(z) = T_{\psi_1}^* T_q^* f(z) \) is a generator of \( H^2(z) \ominus \varphi_1(z) H^2(z) \) for \( T_z^* \). \( \square \)

Let fix a point \((\alpha, \beta)\) in \( \mathbb{D}^2 \). Let \( b_\alpha(z) \) and \( b_\beta(w) \) be simple Blaschke products with zeros \( \alpha \) and \( \beta \) respectively, i.e.,

\[
b_\alpha(z) = \frac{-\overline{\alpha}}{|\alpha|} \frac{z - \alpha}{1 - \overline{\alpha} z} \quad \text{and} \quad b_\beta(w) = \frac{-\overline{\beta}}{|\beta|} \frac{w - \beta}{1 - \overline{\beta} w}, \quad z, w \in \mathbb{D}.
\]
For a closed subspace $E$ of $H^2$, we denote by $P_E$ the orthogonal projection from $H^2$ onto $E$.

The following is a list of well-known facts.

**Lemma 2.3.** For each positive integer $s$, we have the following:

(i) $H^2(z) \ominus b_\alpha(z)^s H^2(z) = \sum_{\sigma=0}^{s-1} \oplus \mathbb{C} \cdot b_\alpha(z)^\sigma T_z^* b_\alpha(z)$.

(ii) If $f(z)$ is a generator of $H^2(z) \ominus b_\alpha(z)^s H^2(z)$ for $T_z^*$, then $f(z) \not\perp b_\alpha(z)^{s-1} T_z^* b_\alpha(z)$.

(iii) Let $h(z) = b_\alpha(z)^{s-1} T_z^* b_\alpha(z)$. Then $P_{\mathbb{C} h} T_z^* h(z) = \mathbb{C} h(z)$.

Let $m$ and $s_1, s_2, \ldots, s_m, t_1, t_2, \ldots, t_m$ be positive integers such that

$$1 \leq s_m < s_{m-1} < \cdots < s_1 \quad \text{and} \quad 1 \leq t_1 < t_2 < \cdots < t_m.$$  

Put $t_0 = 0$. Let

$$N = \bigoplus_{n=1}^m (H^2(z) \ominus b_\alpha(z)^{s_n} H^2(z)) \otimes (H^2(w) \ominus b_\beta(w)^{t_n} H^2(w)).$$

Then $N$ is a finite dimensional backward shift invariant subspace of $H^2$. By Lemma 2.3(i), we have the following.

**Lemma 2.4.**

$$N = \bigoplus_{n=1}^m \left( \bigoplus_{t=t_0}^{t_n-1} \left( \bigoplus_{s=s_n}^{s_n-1} \mathbb{C} \cdot b_\alpha(z)^\sigma T_z^* b_\alpha(z) b_\beta(w)^\ell T_w^* b_\beta(w) \right) \right).$$

By the above lemma, we have

$$\dim N = \sum_{n=1}^m s_n (t_n - t_{n-1}).$$

**Lemma 2.5.** For each $1 \leq n \leq m$, let $f_n(z)$ and $g_n(w)$ be generators of $H^2(z) \ominus b_\alpha(z)^{s_n} H^2(z)$ for $T_z^*$ and $H^2(w) \ominus b_\beta(w)^{t_n} H^2(w)$ for $T_w^*$, respectively. Then we have the following:

(i) $[f_1(z)g_1(w), f_2(z)g_2(w), \ldots, f_m(z)g_m(w)]_{T^*} = N$.

(ii) $\text{rank}_{T^*} N = m$.

**Proof.** (i) It is not difficult to see that

$$[f_n(z)g_n(w)]_{T^*} = (H^2(z) \ominus b_\alpha(z)^{s_n} H^2(z)) \otimes (H^2(w) \ominus b_\beta(w)^{t_n} H^2(w))$$

for $1 \leq n \leq m$. Hence we get (i).
(ii) Let
\[ E = \sum_{n=1}^{m} \oplus \mathbb{C} \cdot b_{\alpha}(z)^{s_n-1} T_{z}^{s_n} b_{\alpha}(w) T_{w}^{t_n-1} b_{\beta}(w). \]

By Lemmas 2.3(i) and 2.4, \( E \subset N \) and \( N \ominus E \) is a backward shift invariant subspace. By Lemma 2.3(iii), \( P_{E} T_{z}^{\ast} = \overline{\alpha I_{E}} \) and \( P_{E} T_{w}^{\ast} = \overline{\beta I_{E}} \), where \( I_{E} \) is the identity operator on \( E \). By a standard argument, we have \( m \leq \text{rank} T_{z}^{\ast} N \). By (i), we have \( \text{rank} T_{z}^{\ast} N \leq m \), so we get (ii). \( \square \)

Let \( \varphi(z) \) and \( \psi(w) \) be Blaschke products with zeros \( \{\alpha_n\}_{n \geq 1} \) and \( \{\beta_n\}_{n \geq 1} \) satisfying \( \alpha_n \neq \alpha_j \) and \( \beta_n \neq \beta_j \) for every \( n \neq j \), respectively, and

\[ \varphi(z) = \prod_{n=1}^{\infty} b_{\alpha_n}(z)^{s_n} \quad \text{and} \quad \psi(w) = \prod_{n=1}^{\infty} b_{\beta_n}(w)^{t_n}. \]

Then one easily see the following.

**Lemma 2.6.**

(i) \( H^{2}(z) \ominus \varphi(z) H^{2}(z) = \bigvee_{n=1}^{\infty} (H^{2}(z) \ominus b_{\alpha_n}(z)^{s_n} H^{2}(z)). \)

(ii) \( (H^{2}(z) \ominus \varphi(z) H^{2}(z)) \otimes (H^{2}(w) \ominus \psi(w) H^{2}(w)) \)

\[ = \bigvee_{n,\ell=1}^{\infty} (H^{2}(z) \ominus b_{\alpha_n}(z)^{s_n} H^{2}(z)) \otimes (H^{2}(w) \ominus b_{\beta_{\ell}}(w)^{t_{\ell}} H^{2}(w)). \]

**3. Finite dimensional backward shift invariant subspaces**

Let \( M \) be an invariant subspace of \( H^{2} \) of finite codimension. Let \( N = H^{2} \ominus M \). In [1], Ahern and Clark studied \( M \) and \( N \). For \( f \in H^{2} \), we put \( Z(f) = \{ \lambda \in \mathbb{D}^{2} : f(\lambda) = 0 \} \). By [1], \( \bigcap_{f \in M} Z(f) \) is a finite set. Let

\[ \bigcap_{f \in M} Z(f) = \{ \lambda_1, \lambda_2, \ldots, \lambda_k \}, \quad \lambda_j \neq \lambda_\ell \ (j \neq \ell). \]

For each \( 1 \leq j \leq k \), we write \( \lambda_j = (\alpha_j, \beta_j) \in \mathbb{D}^{2} \). Also by [1], there are positive integers \( m_j; s_{j,1}, s_{j,2}, \ldots, s_{j,m_j}; t_{j,1}, t_{j,2}, \ldots, t_{j,m_j} \) satisfying

\[ 1 \leq s_{j,m_j} < s_{j,m_j-1} < \cdots < s_{j,1} \quad \text{and} \quad 1 \leq t_{j,1} < t_{j,2} < \cdots < t_{j,m_j} \]

such that

\[ N = \sum_{j=1}^{k} N_j, \]
where
\[ N_j = \sum_{n=1}^{m_j} \left( H^2(z) \ominus b_{\alpha_j}(z)^{s_j,n} H^2(z) \right) \otimes \left( H^2(w) \ominus b_{\beta_j}(w)^{t_j,n} H^2(w) \right). \]

We note that \( \text{dim } N = \sum_{j=1}^{k} \text{dim } N_j \).

**Theorem 3.1.** Let \( N \) be a finite dimensional backward shift invariant subspace of \( H^2 \). Under the above notations, we have \( \text{rank } T^* N = \max_{1 \leq j \leq k} m_j \).

**Proof.** Let
\[ m_0 = \max_{1 \leq j \leq k} m_j \]
and \( 1 \leq j_0 \leq k \). We put
\[ \varphi(z) = \prod_{j : \alpha_j \neq \alpha_{j_0}} b_{\alpha_j}(z)^{s_j,1} \quad \text{and} \quad \psi(w) = \prod_{j : \beta_j \neq \beta_{j_0}} b_{\beta_j}(w)^{t_j,1}. \]

Let \( j \neq j_0 \). Since \( \lambda_j \neq \lambda_{j_0} \), either \( \alpha_j \neq \alpha_{j_0} \) or \( \beta_j \neq \beta_{j_0} \). If \( \alpha_j \neq \alpha_{j_0} \), then \( b_{\alpha_j}(z)^{s_j,1} < \varphi(z) \).

Since \( s_j,n \leq s_j,1 \), we have
\[ T^*_{\varphi(z)} \left( H^2(z) \ominus b_{\alpha_j}(z)^{s_j,n} H^2(z) \right) = \{0\}, \quad 1 \leq n \leq m_j. \]

Similarly, if \( \beta_j \neq \beta_{j_0} \) then
\[ T^*_{\psi(w)} \left( H^2(w) \ominus b_{\beta_j}(w)^{t_j,n} H^2(w) \right) = \{0\}, \quad 1 \leq n \leq m_j. \]

Hence for \( j \neq j_0 \), we have
\[
T^*_{\varphi(z)} T^*_{\psi(w)} N_j = \sum_{n=1}^{m_j} \left( T^*_{\varphi(z)} \left( H^2(z) \ominus b_{\alpha_j}(z)^{s_j,n} H^2(z) \right) \right) \\
\otimes \left( T^*_{\psi(w)} \left( H^2(w) \ominus b_{\beta_j}(w)^{t_j,n} H^2(w) \right) \right) = \{0\}.
\]

By Lemma 2.1, \( T^*_{\varphi(z)} T^*_{\psi(w)} N_{j_0} = N_{j_0} \), so \( T^*_{\varphi(z)} T^*_{\psi(w)} N = N_{j_0} \). Hence \( \text{rank } T^* N_{j_0} \leq \text{rank } T^* N \). By Lemma 2.5, we get \( m_0 \leq \text{rank } T^* N \).

For each \( 1 \leq j \leq k \), let \( f_{j,n}(z) \) and \( g_{j,n}(w) \), \( 1 \leq n \leq m_j \), be generators of \( H^2(z) \ominus b_{\alpha_j}(z)^{s_j,n} H^2(z) \) for \( T^*_z \) and \( H^2(w) \ominus b_{\beta_j}(w)^{t_j,n} H^2(w) \) for \( T^*_w \), respectively. Let
\[ F_{j,n} = \begin{cases} f_{j,n}(z)g_{j,n}(w), & 1 \leq n \leq m_j, \\ 0, & m_j + 1 \leq n \leq m_0 \end{cases} \]
and
\[ G_n = \sum_{j=1}^{k} F_{j,n}, \quad 1 \leq n \leq m_0. \]

Put \( \Omega = [G_1, G_2, \ldots, G_{m_0}]^* \). Then \( \Omega \subset N \). We shall show that \( \Omega = N \). For \( 1 \leq n \leq m_{j_0} \), we have
\[ \Omega \ni T^*_{\phi(z)} f_{j_0,n}(z) T^*_{\psi(w)} g_{j_0,n}(w) = (T^*_{\phi(z)} f_{j_0,n}(z)) (T^*_{\psi(w)} g_{j_0,n}(w)). \]

By Lemma 2.1, \( T^*_{\phi(z)} f_{j_0,n}(z) \) and \( T^*_{\psi(w)} g_{j_0,n}(w) \) are generators of \( H^2(z) \ominus b_{\alpha_{j_0}} \varphi_n(z) H^2(z) \) for \( T^*_z \) and \( H^2(w) \ominus b_{\beta_{j_0}} \psi_n(w) H^2(w) \) for \( T^*_w \), respectively. Therefore we have
\[ (H^2(z) \ominus b_{\alpha_{j_0}} \varphi_n(z) H^2(z)) \otimes (H^2(w) \ominus b_{\beta_{j_0}} \psi_n(w) H^2(w)) \subset \Omega \]
for every \( 1 \leq n \leq m_{j_0} \). Thus \( N_{j_0} \subset \Omega \) for every \( 1 \leq j_0 \leq k \), so we get \( N \subset \Omega \). Hence \( N = \Omega \). This shows that \( \text{rank}_{T^*} N \leq m_0 \). By the first paragraph of the proof, we get \( \text{rank}_{T^*} N = m_0 \). \( \square \)

4. Applications

Recall that \( \{\varphi_n(z)\}_{n=-\infty}^{\infty} \) and \( \{\psi_n(w)\}_{n=-\infty}^{\infty} \) are sequences of Blaschke products satisfying conditions (1) and (2). Let
\[ \mathcal{M} = \bigvee_{n=-\infty}^{\infty} \varphi_n(z) \psi_{n-1}(w) H^2 = \bigcap_{n=-\infty}^{\infty} (\varphi_n(z) H^2 + \psi_n(w) H^2) \]
and \( \mathcal{N} = H^2 \ominus \mathcal{M} \). We note that
\[ (H^2(z) \ominus \varphi_n(z) H^2(z)) \otimes (H^2(w) \ominus \psi_n(w) H^2(w)) = H^2 \ominus (\varphi_n(z) H^2 + \psi_n(w) H^2). \]

The above type of subspaces were studied in [9,19]. We have
\[ \mathcal{N} = \bigvee_{n=-\infty}^{\infty} (H^2(z) \ominus \varphi_n(z) H^2(z)) \otimes (H^2(w) \ominus \psi_n(w) H^2(w)). \]

We shall determine \( \text{rank}_{T^*} \mathcal{N} \). Let
\[ Z = \{ (\alpha, \beta) \in \mathbb{D}^2: \varphi_n(\alpha) = \psi_n(\beta) \text{ for some } n \}. \]

Then \( Z \) is at most a countable set. Put
\[ Z = \{ (\alpha_j, \beta_j): j \geq 1 \}, \quad (\alpha_j, \beta_j) \neq (\alpha_i, \beta_i) \quad (j \neq i). \]
For each $j \geq 1$, let

$$Z_j = \{ n : \varphi_n(\alpha_j) = \psi_n(\beta_j) = 0, \ -\infty < n < \infty \}.$$ 

By conditions (\alpha) and (\beta), $Z_j$ is a finite set. For each $n \in Z_j$, let

$$k_{j,n} = \text{ord}(\varphi_n(z), \alpha_j) \quad \text{and} \quad \ell_{j,n} = \text{ord}(\psi_n(w), \beta_j),$$

where $\text{ord}(\varphi_n(z), \alpha_j)$ denotes zero’s order of $\varphi_n(z)$ at $z = \alpha_j$. If $n_1, n_2 \in Z_j$ and $n_1 < n_2$, then $k_{j,n_2} \leq k_{j,n_1}$ and $\ell_{j,n_1} \leq \ell_{j,n_2}$. Let

$$N_j = \sum_{n \in Z_j} \left( H^2(z) \ominus b_{\alpha_j}(z)^{k_{j,n}} H^2(z) \right) \otimes \left( H^2(w) \ominus b_{\beta_j}(w)^{\ell_{j,n}} H^2(w) \right).$$

Then $N_j \subset \mathcal{N}$, $N_j$ is a finite dimensional backward shift invariant subspace, and by Lemma 2.6 we have

$$\mathcal{N} = \bigvee_{j=1}^{\infty} N_j.$$

**Theorem 4.1.** Under the above notations, we have

$$\text{rank}_{T^*} \mathcal{N} = \sup_{j \geq 1} \text{rank}_{T^*} N_j.$$

**Proof.** To write down the number of $\text{rank}_{T^*} N_j$, we need to rewrite $N_j$. As in Section 3, there are positive integers $m_j, s_{j,1}, s_{j,2}, \ldots, s_{j,m_j}; t_{j,1}, t_{j,2}, \ldots, t_{j,m_j}$ such that

$$1 \leq s_{j,m_j} < s_{j,m_j-1} < \cdots < s_{j,1}, \quad 1 \leq t_{j,1} < t_{j,2} < \cdots < t_{j,m_j},$$

and

$$N_j = \sum_{n=1}^{m_j} \left( H^2(z) \ominus b_{\alpha_j}(z)^{s_{j,n}} H^2(z) \right) \otimes \left( H^2(w) \ominus b_{\beta_j}(w)^{t_{j,n}} H^2(w) \right).$$

By Theorem 3.1, we have $\text{rank}_{T^*} N_j = m_j$. Let

$$m_0 = \sup_{j \geq 1} m_j.$$

We shall show that $\text{rank}_{T^*} \mathcal{N} = m_0$.

For each $n \in Z_j$, let

$$\tilde{\varphi}_n(z) = \frac{\varphi_n(z)}{b_{\alpha_j}(z)^{k_{j,n}}} \quad \text{and} \quad \tilde{\psi}_n(w) = \frac{\psi_n(w)}{b_{\beta_j}(w)^{\ell_{j,n}}}.$$ 

By Lemma 2.2, we have
\[ T_{\overline{\psi}_n(z)}^* T_{\overline{\psi}_n(w)}^* \left( (H^2(z) \ominus \varphi_n(z)H^2(z)) \otimes (H^2(w) \ominus \psi_n(w)H^2(w)) \right) = (H^2(z) \ominus b_{\alpha_j}(z)^{k_j,n}H^2(z)) \otimes (H^2(w) \ominus b_{\beta_j}(w)^{\ell_j,n}H^2(w)). \]

Let \( p_j = \max\{n: n \in Z_j\} \) and \( q_j = \min\{n: n \in Z_j\} \). By conditions \((\#1)\) and \((\#2)\), \( \varphi_n(\alpha_j) \neq 0 \) for every \( n > p_j \), \( \psi_n(\beta_j) \neq 0 \) for every \( n < q_j \) and \( Z_j = \{n: q_j \leq n \leq p_j\} \). Let

\[ \Phi_j(z) = \frac{\varphi_{q_j}(z)}{b_{\alpha_j}(z)^{s_{j,1}}} \quad \text{and} \quad \Psi_j(w) = \frac{\psi_{p_j}(w)}{b_{\beta_j}(w)^{t_{j,m_j}}}. \]

For \( n \in Z_j \), we have

\[ s_{j,1} = \text{ord}(\varphi_{q_j}(z), \alpha_j) = k_{j,q_j} \geq k_{j,n} \]

and

\[ t_{j,m_j} = \text{ord}(\psi_{p_j}(w), \beta_j) = \ell_{j,p_j} \geq \ell_{j,n}. \]

Then \( \Phi_j(z) \) and \( \Psi_j(w) \) are Blaschke products, and we have \( \Phi_j(\alpha_j) \neq 0 \) and \( \Psi_j(\beta_j) \neq 0 \). Also for every \( n \in Z_j \), we have

\[ \tilde{\varphi}_n(z) = \frac{\varphi_n(z)}{b_{\alpha_j}(z)^{s_{j,n}}} \times \frac{\varphi_{q_j}(z)}{b_{\alpha_j}(z)^{s_{j,1}}} = \Phi_j(z) \]

and similarly \( \tilde{\psi}_n(w) \prec \Psi_j(w) \). By the last paragraph and Lemma 2.2, we get

\[ \sum_{n \in Z_j} T_{\Phi_j(z)}^* T_{\Psi_j(w)}^* \left( (H^2(z) \ominus \varphi_n(z)H^2(z)) \otimes (H^2(w) \ominus \psi_n(w)H^2(w)) \right) = N_j. \]

For \( n > p_j \), we have \( \varphi_n(z) \prec \Phi_j(z) \), so

\[ T_{\Phi_j(z)}^* T_{\Psi_j(w)}^* \left( (H^2(z) \ominus \varphi_n(z)H^2(z)) \otimes (H^2(w) \ominus \psi_n(w)H^2(w)) \right) = \{0\}. \]

Also for \( n < q_j \), we have \( \psi_n(w) \prec \Psi_j(w) \), so

\[ T_{\Phi_j(z)}^* T_{\Psi_j(w)}^* \left( (H^2(z) \ominus \varphi_n(z)H^2(z)) \otimes (H^2(w) \ominus \psi_n(w)H^2(w)) \right) = \{0\}. \]

Hence we get

\[ T_{\Phi_j(z)}^* T_{\Psi_j(w)}^* \mathcal{N} = N_j. \]

This shows that \( m_j = \text{rank} T^* \mathcal{N} \leq \text{rank} T^* \mathcal{N} \). Thus we get \( m_0 \leq \text{rank} T^* \mathcal{N} \). When \( m_0 = \infty \), we finish the proof.

Suppose that \( m_0 < \infty \). For \( 1 \leq n \leq m_j \), let \( f_{j,n}(z) \) and \( g_{j,n}(w) \) be generators of \( H^2(z) \ominus b_{\alpha_j}(z)^{s_{j,n}}H^2(z) \) for \( T_{z}^* \) and \( H^2(w) \ominus b_{\beta_j}(w)^{\ell_{j,n}}H^2(w) \) for \( T_{w}^* \), respectively. We may assume that \( \|f_{j,n}(z)\| = \|g_{j,n}(w)\| = 1 \) for every \( n \). We note that \( f_{j,n}(z)g_{j,n}(w) \) is a generator of
\[(H^2(z) \ominus b_{\alpha_{j}}(z)^{i_{j}} H^2(z)) \otimes (H^2(w) \ominus b_{\beta_{j}}(w)^{i_{j}} H^2(w))\]

for $T^*_z$ and $T^*_w$. For $1 \leq n \leq m_0$, let

\[F_{j,n} = \begin{cases} f_{j,n}(z)g_{j,n}(w), & 1 \leq n \leq m_j, \\ 0, & m_j + 1 \leq n \leq m_0 \end{cases}\]

and

\[G_n = \sum_{\ell=1}^{\infty} \frac{F_{\ell,n}}{\ell^2}, \quad 1 \leq n \leq m_0.\]

Then $G_n \in \mathcal{N}$ for every $1 \leq n \leq m_0$. Let $\Omega = [G_1, G_2, \ldots, G_{m_0}]_{T^*}$. For $1 \leq n \leq m_j$, we have

\[
\Omega \ni T^*_c(\Phi_j(z))T^*_c(\Psi_j(w))G_{j,n} = T^*_c(\Phi_j(z)f_{j,n}(z))T^*_c(\Psi_j(w)g_{j,n}(w)) \frac{j^2}{j^2}.
\]

By Lemma 2.2, $T^*_c(\Phi_j(z)f_{j,n}(z))T^*_c(\Psi_j(w)g_{j,n}(w))$ is a generator of

\[(H^2(z) \ominus b_{\alpha_{j}}(z)^{i_{j}} H^2(z)) \otimes (H^2(w) \ominus b_{\beta_{j}}(w)^{i_{j}} H^2(w))\]

for $T^*_z$ and $T^*_w$. Hence $\mathcal{N}_j \subset \Omega$ for every $j \geq 1$. Therefore we get $\Omega = \mathcal{N}$, so $\text{rank}_{T^*} \mathcal{N} \leq m_0$. Thus we get the assertion. \(\square\)

We put

\[\zeta_n(z) = \frac{\varphi_n(z)}{\varphi_{n+1}(z)} \quad \text{and} \quad \xi_n(w) = \frac{\psi_n(w)}{\psi_{n-1}(w)}, \quad -\infty < n < \infty.\]

By the proof of Theorem 4.1, one may prove that

\[\#\{n: \zeta_n(\alpha_j) = \xi_n(\beta_j) = 0, \quad -\infty < n < \infty\} = m_j.\]

Hence we may rewrite Theorem 4.1 as the following way.

**Theorem 4.2.**

\[\text{rank}_{T^*} \mathcal{N} = \sup_{j \geq 1} \#\{n: \zeta_n(\alpha_j) = \xi_n(\beta_j) = 0, \quad -\infty < n < \infty\}.\]
Let \( \{ \varphi_n(z) \}_{n \geq 0} \) and \( \{ \psi_n(w) \}_{n \geq 0} \) be sequences of Blaschke products satisfying conditions (\#1) and (\#2). Moreover we assume that \( \varphi_n(z) = \varphi_0(z) \), \( \psi_n(w) = \psi_0(w) = 1 \) for every \( n < 0 \), and both \( \varphi_0(z) \) and \( \psi_1(w) \) are nonconstant. In this case, we have

\[
\mathcal{M} = \bigvee_{n=0}^{\infty} \varphi_n(z) \psi_{n-1}(w) H^2.
\]

Let \( \mathcal{N} = H^2 \ominus \mathcal{M} \). Then \( \mathcal{N} \) is a backward shift invariant subspace and

\[
\mathcal{N} = \bigvee_{n=0}^{\infty} \left( H^2(z) \ominus \varphi_n(z) H^2(z) \right) \otimes \left( H^2(w) \ominus \psi_n(w) H^2(w) \right).
\]

In the same way as the proof of Theorem 4.1, we may define \( Z = \{ (\alpha_j, \beta_j) \in \mathbb{D}^2 : j \geq 1 \}, \mathcal{Z}_j, \mathcal{N}_j, m_0, \{ \zeta_n(z) \}_{n \geq 0}, \{ \xi_n(z) \}_{n \geq 0} \), and we get the following.

**Theorem 4.3.**

\[
\text{rank}_{T^*} \mathcal{N} = \sup_{j \geq 1} \# \{ n : \zeta_n(\alpha_j) = \xi_n(\beta_j) = 0, n \geq 0 \}.
\]

When \( \psi_n(w) = w^n \) for \( n \geq 0 \), we have \( \mathcal{M} = M_\varphi \) and \( \mathcal{N} = \{ w \in \mathbb{D} : \varphi_0(\alpha) = 0 \} = \{ \alpha_j : j \geq 1 \} \). Then \( \mathcal{Z} = \{ (\alpha_j, 0) : j \geq 1 \} \). As a corollary of Theorem 4.3, we get the following.

**Corollary 4.4.**

\[
\text{rank}_{T^*} \mathcal{N} = \sup_{j \geq 1} \# \{ n : \zeta_n(\alpha_j) = 0, n \geq 0 \}.
\]

**References**