Phantom maps and spaces of the same \( n \)-type for all \( n \)

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Abstract


This paper develops the connection between the set of phantom maps from \( X \) to \( Y \) and the set of homotopy types \( W \) having the same \( n \)-type as \( Z \), for all \( n \), where \( Z = X \times \Omega Y \) or \( Y \vee \Sigma X \).

Using recent work making calculations of the space of phantom maps possible, we can give explicit constructions of various \( W \)'s.

1. Introduction

In this paper, we explore a connection between \( \text{Ph}(X, Y) \), the set of homotopy classes of phantom maps from \( X \) to \( Y \), and \( \text{SNT}(Z) \), the set of homotopy types of spaces \( W \) such that \( W \) has the same \( n \)-type as \( Z \) for all \( n \). Here \( Z \) may be taken to be either \( X \times \Omega Y \) or \( Y \vee \Sigma X \).

The idea that such a connection exists is not novel; it goes back to Gray [1], who showed that for suitably chosen \( X, Y \) (for instance, \( X = K(Z, 2) \), \( Y = S^3 \)), \( \text{SNT}(\Omega \Sigma(Y \vee \Sigma X)) \) is nontrivial, that is, has more than one element. We elaborate Gray's idea, taking advantage of the technology developed in [10]—and further studied in [6-8]—which has made computation of \( \text{Ph}(X, Y) \) accessible. Thus, we are able to construct explicitly nontrivial elements in \( \text{SNT}(Z) \) for
various $Z$ by performing standard homotopy-theoretical constructions with phantom maps.

In a series of papers [3–5], McGibbon and Moller systematically study $\text{SNT}(Z)$, proving both ‘vanishing’ and ‘nonvanishing’ theorems. Their point of departure is a theorem of Wilkerson [9] identifying $\text{SNT}(Z)$ with a certain $\lim^1$-set. The approach taken in this paper, which does not use any $\lim^1$ arguments, leads only to nonvanishing results and complements and enhances the work of McGibbon and Moller.

The paper is organized as follows. In Section 2, we review some salient results on $\text{Ph}(X, Y)$ from [6–8, 10]. For use in Section 3, we also study the relationship between $\text{Ph}(X, Y)$ and the collection $\{\text{Ph}(X, Y_{(p)})\}$, where $Y_{(p)}$ denotes, as usual, the $p$-localization of $Y$. In Section 3, we define and examine functions $F : \text{Ph}(X, Y) \rightarrow \text{SNT}(X \times \Omega Y)$, $C : \text{Ph}(X, Y) \rightarrow \text{SNT}(Y \vee \Sigma X)$ and establish the nontriviality of these functions under appropriate hypotheses. Finally, in Section 4, we specialize the theorems of Section 3 to obtain explicit examples of spaces $Z$—for example, $\Omega^k(K(Z, n) \times S^n) (k \geq 2, n - k \geq 2)$ and $\Sigma^k(S^n \vee K(Z, n)) (k \geq 2, n \geq 2)$—for which $\text{SNT}(Z)$ is nontrivial, indeed uncountable.

2. Phantom maps and localization

Throughout, $X$ and $Y$ will be taken to be path-connected CW-spaces such that either $Y$ is a grouplike space or $X$ is a cogroup; in this way, the set $\text{Ph}(X, Y)$ admits a natural group structure. In order to have available the results of [6, 7, 10], we also require $X$ and $Y$ to satisfy modest connectivity and finiteness conditions; it is sufficient to have $X$ and $Y$ 1-connected, of finite type, and, in the case that $X$ is a cogroup, $\pi_* X$ a finite group.

Let $X \xrightarrow{\tau} X_{(0)}$ be a rationalization map, and consider the fibration-cofibration

$$
X \xrightarrow{\tau} X \rightarrow X_{(0)}
$$

with connecting map $X_{(0)} \rightarrow \Sigma X_{\tau}$. The main structure theorem on $\text{Ph}(X, Y)$ may be stated as follows.

**Theorem 2.1** [6, 7, 10]. Under the stated conditions on $X$ and $Y$, the group $\text{Ph}(X, Y)$ is isomorphic to the quotient group $[X_{(0)}, Y] / q^* [\Sigma X_{\tau}, Y]$ and the group $[X_{(0)}, Y]$ is isomorphic to the product group

$$
\prod \text{Ext}(H_{n-1} X_{(0)}, \pi_n Y).
$$

If, in addition, $X$ is a finite Postnikov space, or an iterated suspension of such, and $Y$ has the homotopy type of a finite cell complex, or an iterated loop of such, then

$$
\text{Ph}(X, Y) \cong [X_{(0)}, Y] \cong [X, Y].
$$

\[ \Box \]
Remark. Since the argument in [10, Theorem B(b)] that $\text{Ph}(X, Y) = r^*[X_{(0)}, Y] \cong [X_{(0)}, Y]/q^*[\Sigma Y_r, Y]$ has a minor flaw, we sketch a proof: For any phantom map $X \xrightarrow{\phi} Y$, the composite $X \xrightarrow{\phi} Y \xrightarrow{c} Y$ is trivial, where $Y \xrightarrow{c} \hat{Y}$ is profinite completion. Hence $\phi$ lifts to $X \xrightarrow{\phi'} Y_{\hat{p}}$, $Y_{\hat{p}}$ the homotopy-fibre of $c$. But $Y_{\hat{p}}$ is a rational space (one readily checks that each homotopy group $\pi_n Y_{\hat{p}}$ is a direct sum of groups of the form $\mathbb{Z}/2^m \cong \text{Ext}(\mathbb{Q}, \mathbb{Z})$) so $\phi'$ factors as $X \xrightarrow{\phi''} X_{(0)} \xrightarrow{\phi''} Y_{\hat{p}}$. Thus $\text{Ph}(X, Y) \subset r^*[X_{(0)}, Y]$. The opposite inclusion $\text{Ph}(X, Y) \supset r^*[X_{(0)}, Y]$ is correctly proved in [10, Theorem B(b)].

There is a verbatim version of Theorem 2.1 with $Y_{(p)}$, in place of $Y$, where $p$ is an arbitrary prime and $Y_{(p)}$ is the $p$-localization of $Y$. It will be useful to understand the homomorphisms

$$\text{Ph}(X, Y) \xrightarrow{e_p} \text{Ph}(X, Y_{(p)})$$

induced by $p$-localization $Y \xrightarrow{e_p} Y_{(p)}$. Setting $\hat{Y} = \prod Y_{(p)}$ (the ‘local expansion’ of $Y$) and $Y \xrightarrow{e'} \hat{Y}$ the map with components $e'_p$, we state the following:

**Theorem 2.2.** The homomorphism

$$\text{Ph}(X, Y) \xrightarrow{e'} \text{Ph}(X, \hat{Y})$$

is an epimorphism. If $X$ is (an iterated suspension of) a finite Postnikov space and $Y$ is (an iterated loop space of) a finite cell complex, and if $\text{Ph}(X, Y) \neq 0$, then each of the groups in the short exact sequence\footnote{We remind the reader that the symbols $\rightarrow$ and $\twoheadrightarrow$ stand for mono and epi respectively.}

$$\ker \hat{e} \twoheadrightarrow \text{Ph}(X, Y) \twoheadrightarrow \text{Ph}(X, \hat{Y})$$

is a rational vector space of uncountable dimension over $\mathbb{Q}$.

**Remark.** Note that by [2, Theorem II.5.3], the kernel of $[X, Y] \xrightarrow{e'} [X, \hat{Y}]$ consists of phantom maps (just as the kernel of $[X, Y] \xrightarrow{e'} [X, \hat{Y}]$ consists of (all!) phantom maps). We refer to the elements of $\ker \hat{e}$ as ‘special phantom maps’.

The proof of Theorem 2.2 depends on the following lemma:

**Lemma 2.3.** The canonical homomorphism $\mathbb{Z} \rightarrow \hat{\mathbb{Z}}$ induces a short exact sequence

$$\text{Hom}(\mathbb{Q}, \hat{\mathbb{Z}}) \twoheadrightarrow \text{Ext}(\mathbb{Q}, \mathbb{Z}) \rightarrow \text{Ext}(\mathbb{Q}, \hat{\mathbb{Z}})$$

of rational vector spaces of uncountable dimension over $\mathbb{Q}$.
Proof. We simply apply $\text{Hom}(\mathbb{Q}, -)$ to the short exact sequence

$$\mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z},$$

taking into account that $\mathbb{Q}/\mathbb{Z}$ is a rational vector space of uncountable dimension over $\mathbb{Q}$ [2, Theorem 1.3.7].

Proof of Theorem 2.2. Using Theorem 2.1, we identify the homomorphism $\text{Ph}(X, Y) \xrightarrow{\epsilon} \text{Ph}(X, \hat{Y})$ with the evident homomorphism

$$\prod \frac{\text{Ext}(\mathcal{L}_{n-1}X_{(0)}, \pi_n Y)}{q^n[\Sigma X, Y]} \to \prod \frac{\text{Ext}(\mathcal{L}_{n-1}X_{(0)}, \pi_n \hat{Y})}{q^n[\Sigma X, \hat{Y}]}$$

induced by $Y \xrightarrow{\epsilon} \hat{Y}$. To prove $\epsilon_+ \text{ epi}$, it suffices to prove

$$\prod \text{Ext}(\mathcal{L}_{n-1}X_{(0)}, \pi_n Y) \to \prod \text{Ext}(\mathcal{L}_{n-1}X_{(0)}, \pi_n \hat{Y})$$

epi, which is an immediate consequence of Lemma 2.3. The second assertion of Theorem 2.2 likewise follows from Lemma 2.3 since, in view of Theorem 2.1,

$$[\Sigma X, Y] = 0 = [\Sigma X, \hat{Y}]$$

under the given conditions on $X$ and $Y$.  

3. Phantom maps and SNT(Z)

If $Z$ is a path-connected CW-space, SNT($Z$) denotes the set of homotopy types of spaces $W$ such that, for all $n$,

$$P_n W = P_n Z,$$

where $P_n$ denotes the $n$th Postnikov approximation. The (homotopy type of the) space $Z$ serves as basepoint for SNT($Z$).

For a map $X \xrightarrow{f} Y$, let $F_f$ denote the homotopy-fiber of $f$ and $C_f$ the mapping cone of $f$. As $X$ and $Y$ are 1-connected, $F_f$ is path-connected and $C_f$ is 1-connected. Standard arguments show that

$$F_\phi \in \text{SNT}(X \times \Omega Y), \quad C_\phi \in \text{SNT}(Y \vee \Sigma X).$$

We state this formally as the following theorem:
Theorem 3.1. There are (pointed) functions

\[ F : \text{Ph}(X, Y) \rightarrow \text{SNT}(X \times \Omega Y) , \]
\[ C : \text{Ph}(X, Y) \rightarrow \text{SNT}(Y \vee \Sigma X) \]

defined by \( F(\phi) = F_\phi \), \( C(\phi) = C_\phi \). \( \square \)

The functions \( F \) and \( C \) are far from being injective. In fact, the automorphism group \( \text{Aut} \ X \) acts on the set \( \text{Ph}(X, Y) \) on the right while the automorphism group \( \text{Aut} \ Y \) acts on the set \( \text{Ph}(X, Y) \) on the left and we observe the following:

Theorem 3.2. If \( \phi \in \text{Ph}(X, Y) \), \( \xi \in \text{Aut} \ X \), \( \eta \in \text{Aut} \ Y \), then

\[ F(\eta \circ \phi \circ \xi) = F(\phi) , \quad C(\eta \circ \phi \circ \xi) = C(\phi) . \]

Thus \( F \) and \( C \) induce functions

\[ \bar{F} : \text{Aut} \ Y \backslash \text{Ph}(X, Y) / \text{Aut} \ X \rightarrow \text{SNT}(X \times \Omega Y) , \]
\[ \bar{C} : \text{Aut} \ Y \backslash \text{Ph}(X, Y) / \text{Aut} \ X \rightarrow \text{SNT}(Y \vee \Sigma X) , \]

where \( \text{Aut} \ Y \backslash \text{Ph}(X, Y) / \text{Aut} \ X \) denotes the set of double cosets arising from the given group actions on \( \text{Ph}(X, Y) \).

Proof. The map of fibrations

\[ F_{\eta \circ \phi \circ \xi} \rightarrow X \xrightarrow[\eta \circ \phi \circ \xi]{F_\phi} Y \]
\[ F_\phi \rightarrow X \xrightarrow[\phi]{\xi} Y \]

may be filled in with a suitable map \( F_{\eta \circ \phi \circ \xi} \rightarrow F_\phi \). As \( \xi \) and \( \eta^{-1} \) are homotopy equivalences, it is plain that \( h \) induces isomorphisms on homotopy groups, hence is itself a homotopy equivalence.

Dually, one finds a map \( C_{\eta \circ \phi \circ \xi} \rightarrow C_\phi \) inducing isomorphisms on homology groups, hence a homotopy equivalence. \( \square \)

The functions \( F \) and \( C \) are also far from being surjective. Our aim is to find conditions on \( X \) and \( Y \) ensuring that \( F \) and \( C \) have nontrivial images.

We begin with \( F \): Let \( X = K(\mathbb{Z}, n) \), \( n \geq 2 \) and let \( Y \) be (an iterated loop space of) a finite cell complex.
Lemma 3.3. If \( K(\mathbb{Z}, n) \xrightarrow{\phi} Y \) is a nontrivial phantom map, then

\[ F_\phi \neq K(\mathbb{Z}, n) \times \Omega Y. \]

Proof. Let \( K(\mathbb{Z}, n) \times \Omega Y \xrightarrow{f} F_{\phi} \) be any map, \( K(\mathbb{Z}, n) \xrightarrow{i} K(\mathbb{Z}, n) \times \Omega Y \) the canonical embedding and consider the diagram

\[
\begin{array}{ccc}
\Omega Y & \xrightarrow{b} & F_{\phi} \\
\downarrow{f} & & \downarrow{i} \\
K(\mathbb{Z}, n) \times \Omega Y & \xrightarrow{i} & K(\mathbb{Z}, n) \xrightarrow{\phi} Y
\end{array}
\]

with the top row the fibration sequence spawned by \( \phi \). We write

\[ d = i \circ f \circ j \] (3.1)

and think of the latter as an integer, using the identification

\[ [K(\mathbb{Z}, n), K(\mathbb{Z}, n)] \cong \mathbb{Z}. \]

We have

\[ \phi \circ d = \delta \circ i \circ f \circ j = 0, \quad \text{since } \phi \circ i = 0. \] (3.2)

By Theorem 2.1, we may write

\[ \phi = \tilde{\phi} \circ r \] (3.3)

with \( K(\mathbb{Q}, n) \xrightarrow{\tilde{\phi}} Y \) uniquely determined by \( \phi \). There is a commutative diagram

\[
\begin{array}{ccc}
K(\mathbb{Z}, n) & \xrightarrow{r} & K(\mathbb{Q}, n) \\
\downarrow{\delta} & & \downarrow{\delta} \\
K(\mathbb{Z}, n) & \xrightarrow{\delta} & K(\mathbb{Q}, n) \xrightarrow{\tilde{\phi}} Y
\end{array}
\]

(3.4)

with \( \delta \) induced by the endomorphism of \( \mathbb{Q} \) given by multiplication by \( d \). Combining (3.2), (3.3) and (3.4), we find that

\[ 0 = \phi \circ d = \tilde{\phi} \circ r \circ d = \tilde{\phi} \circ \delta \circ r, \]

hence, again by Theorem 2.1, that

\[ \tilde{\phi} \circ \delta = 0. \] (3.5)
If $d \neq 0$, then $\delta$ would be a homotopy equivalence. Thus, by (3.5),
\[
\delta = 0
\]
so that, by (3.3),
\[
\phi = 0,
\]
a contradiction. We therefore conclude that
\[
d = 0.
\] (3.6)

Next, from (3.1) and (3.6), we may write
\[
(3.7)
\]
for some map $K(\mathbb{Z}, n) \xrightarrow{g} \Omega Y$. Since, by Theorem 2.1, $g$ is necessarily phantom, it follows that
\[
\pi_n K(\mathbb{Z}, n) \xrightarrow{\gamma} \pi_n \Omega Y
\]
is the 0 map. Finally, using (3.7), we see that
\[
\pi_n (K(\mathbb{Z}, n) \times \Omega Y) \xrightarrow{f} \pi_2 \Phi
\]
maps the factor $\pi_n K(\mathbb{Z}, n)$ trivially, hence that $f$ cannot be a homotopy equivalence. □

Remark. Note that in Lemma 3.3, the grouplike structure on $Y$ does not come into play.

Theorem 3.4. If $\text{Ph}(K(\mathbb{Z}, n), Y) \neq 0$, then the image of
\[
F : \text{Ph}(K(\mathbb{Z}, n), Y) \rightarrow \text{SNT}(K(\mathbb{Z}, n) \times \Omega Y)
\]
is a nontrivial, indeed an uncountable, subset of $\text{SNT}(K(\mathbb{Z}, n) \times \Omega Y)$. If, in addition, $\text{Ph}(K(\mathbb{Z}, n) \wedge K(\mathbb{Z}, n), Y) = 0$, then all the elements in the image of $F$ admit $H$-space structures.

[Explicit examples illustrating Theorem 3.4, and its dual, Theorem 3.4′ below, will be presented in Section 4.]

Proof. If $\phi$ is a nontrivial phantom map, Lemma 3.3 asserts that $F_\phi$ is a nontrivial element in the image of $F$. To establish the uncountability of the image of $F$, we
proceed as follows. According to Theorem 2.2, \( \text{Ph}(K(Z, n), Y) \neq 0 \) implies \( \text{Ph}(K(Z, n), \hat{Y}) \neq 0 \); indeed, the argument in Theorem 2.2 actually shows that each \( \text{Ph}(K(Z, n), Y_{(p)}) \neq 0 \). For each \( p \), let \( \psi(p) \in \text{Ph}(K(Z, n), Y_{(p)}) \) be nontrivial and for each set of primes \( P \), let \( \Psi(P) \in \text{Ph}(X, \hat{Y}) \) be the element whose \( p \)-th component is \( \psi(p) \) for \( p \in P \), 0 for \( p \not\in P \). By Theorem 2.2, \( \exists \Phi(P) \in \text{Ph}(X, Y) \) such that

\[ \dot{e}_e \Phi(P) = \Psi(P). \]

We will show that

\[ F(\Phi(P_1)) = F(\Phi(P_2)) \iff P_1 = P_2, \]

thereby proving the first assertion of Theorem 3.4. Suppose then that \( P_1 \) and \( P_2 \) are different sets of primes and let \( q \in P_1, q \not\in P_2 \). To show that

\[ F_{\Phi(P_1)} \not\simeq F_{\Phi(P_2)}, \]

if suffices to show that

\[ (F_{\Phi(P_1)})_{(q)} \not\simeq (F_{\Phi(P_2)})_{(q)}. \]

Now

\[ (F_{\Phi(P_1)})_{(q)} = (F_{\Psi(q)})_{(q)} \]

while

\[ (F_{\Phi(P_2)})_{(q)} = K(Z_{(q)}, n) \times \Omega Y_{(q)} \]

and the argument in Lemma 3.3 shows that

\[ (F_{\Psi(q)})_{(q)} \not\simeq K(Z_{(q)}, n) \times \Omega Y_{(q)} . \]

Finally, if \( \text{Ph}(K(Z, n) \wedge K(Z, n), Y) = 0 \), it is not difficult to argue [8, Lemma 2.1] that any phantom map \( K(Z, n) \xrightarrow{\psi} Y \) is an \( \mathcal{H} \)-map. But then \( F_{\Phi} \) clearly admits an \( \mathcal{H} \)-space structure and the proof of Theorem 3.4 is completed. \( \Box \)

Remarks. (1) Knowing that SNT\( (K(Z, n) \times \Omega Y) \) is nontrivial, one infers from [3, Corollary 2.1] that SNT\( (K(Z, n) \times \Omega Y) \) is uncountable. However, Theorem 3.4 provides finer and more explicit information about the structure of SNT\( (K(Z, n) \times \Omega Y) \) than is available from [3, Corollary 2.1].
(2) If \( K(\mathbb{Z}, n) \xrightarrow{\phi} Y \) is chosen to be a nontrivial special phantom map, then for all primes \( p \),

\[
(F_\phi)_{(p)} = K(\mathbb{Z}_{(p)}, n) \times \Omega Y_{(p)}
\]

and so \( F_\phi \) is, in the terminology of [5], a ‘clone’ of \( K(\mathbb{Z}, n) \times \Omega Y \).

We now turn to a study of the function \( C \): Let \( Y = S^n \), \( n \geq 2 \) and let \( X \) be an iterated suspension of a finite Postnikov space, say \( X = \Sigma^k W \). We first state a dual version of Lemma 3.3.

**Lemma 3.3’.** If \( X \xrightarrow{\phi} S^n \) is a nontrivial phantom map, then

\[
C_\phi \not\in S^n \vee \Sigma X.
\]

**Proof.** The dual argument to that in Lemma 3.3 leads to a map \( S^n \xrightarrow{d} S^n \) satisfying

\[
d \circ \phi = 0 \quad (3.1')
\]

and it suffices to show, as in the proof of Lemma 3.3, that

\[
d = 0.
\]

To this end, we select an integral approximation \( S^n \xrightarrow{j} L \) [7, 10] as follows. If \( n \) is odd, we set \( K = K(\mathbb{Z}, n) \) and take \( S^n \xrightarrow{j} L \) to a generator. There is a commutative diagram

\[
\begin{array}{ccc}
S^n & \xrightarrow{j} & L \\
\downarrow d & & \downarrow d \\
S^n & \xrightarrow{j} & L
\end{array}
\]

inducing a commutative diagram

\[
\begin{array}{ccc}
[X_{(0)}, S^n] & \xrightarrow{\pi} & [X_{(0)}, L] \\
\downarrow d & & \downarrow d \\
[X_{(0)}, S^n] & \xrightarrow{\pi} & [X_{(0)}, L]
\end{array} \quad (3.2')
\]

and the group structure on \([X_{(0)}, L]\) induced by the cogroup \( X_{(0)} \) agrees with that induced by the grouplike \( L \). But
\[ d_+ \gamma = d_+ \gamma, \quad \gamma \in [X(0), \mathcal{L}], \]

so (3.1') and (3.2') yield

\[ d(j \phi) = 0, \]

whence

\[ d = 0. \]

If \( n \) is even, we take \( L \) to be the homotopy-fiber of

\[ \iota^n: K(\mathbb{Z}, n) \to K(\mathbb{Z}, 2n), \]

\( \iota_n \) denoting the identity map of \( K(\mathbb{Z}, n) \). Writing \( S^n \xrightarrow{\varepsilon} K(\mathbb{Z}, n) \) for a generator, we have, since the composite

\[ S^n \xrightarrow{\varepsilon} K(\mathbb{Z}, n) \xrightarrow{\iota^n} K(\mathbb{Z}, 2n) \]

is trivial, a map \( S^n \xrightarrow{\iota} L \) (unique since \( \pi_n K(\mathbb{Z}, 2n - 1) = 0 \)) lifting the map \( S^n \xrightarrow{\varepsilon} K(\mathbb{Z}, n) \). There is a map \( L \xrightarrow{\ell} L \) fitting into commutative diagrams

\[
\begin{array}{cccc}
S^n & \xrightarrow{\iota} & L & \xrightarrow{\iota^n} K(\mathbb{Z}, n) & \xrightarrow{\varepsilon} K(\mathbb{Z}, 2n) \\
\downarrow{d} & & \downarrow{d} & & \downarrow{d^2} \\
S^n & \xrightarrow{\iota} & L & \xrightarrow{\iota^n} K(\mathbb{Z}, n) & \xrightarrow{\varepsilon} K(\mathbb{Z}, 2n) \\
\end{array}
\]

(with the rows fibration sequences) and

\[
\begin{array}{cccc}
\cdots & \xrightarrow{d^2} & K(\mathbb{Z}, 2n - 1) & \xrightarrow{\iota} L & \xrightarrow{\iota^n} K(\mathbb{Z}, n) & \xrightarrow{\varepsilon} K(\mathbb{Z}, 2n) \\
\downarrow{d^2} & & \downarrow{d} & & \downarrow{d^2} \\
\cdots & \xrightarrow{d^2} & K(\mathbb{Z}, 2n - 1) & \xrightarrow{\iota} L & \xrightarrow{\iota^n} K(\mathbb{Z}, n) & \xrightarrow{\varepsilon} K(\mathbb{Z}, 2n) \\
\end{array}
\]

These, in turn, induce commutative diagrams

\[
\begin{array}{cccc}
[\ldots, K(\mathbb{Z}, 2n - 1)] & \xrightarrow{d^2} & [X(0), K(\mathbb{Z}, n)] & \xrightarrow{\ell} [X(0), K(\mathbb{Z}, n)] \\
\downarrow{d^2} & & \downarrow{d^2} & & \downarrow{d^2} \\
[\ldots, K(\mathbb{Z}, 2n - 1)] & \xrightarrow{d^2} & [X(0), K(\mathbb{Z}, n)] & \xrightarrow{\ell} [X(0), K(\mathbb{Z}, n)] \\
\end{array}
\]

(3.3')

(with the rows exact) and
Arguing as in the case $n$ odd, we see from (3.3') that

$$\ell_* \gamma = 0 \Rightarrow d^2 \gamma = 0, \quad \gamma \in [X^{(0)}, L]$$

and therefore

either $\gamma = 0$ or $d = 0$.

Thus, from (3.1') and (3.4'),

$$d^2(j_* \phi) = 0,$$

whence again

$$d = 0.$$

**Remark.** If we knew, a priori, that $\phi_{(p)}$ were nontrivial for infinitely many primes $p$, we could quickly infer from $d \circ \phi = 0$ that $d = 0$, even without exploiting the cogroup structure on $X$. Indeed, from

$$d \circ \phi = 0,$$

we have

$$d_{(p)} \circ \phi_{(p)} = 0, \quad \text{for all primes } p.$$

If $p \nmid d$, the $S^n_{(p)} \xrightarrow{d_{(p)}} S^n_{(p)}$ is a homotopy equivalence. Thus for all primes $p$ excluding those dividing $d$,

$$\phi_{(p)} = 0,$$

contrary to the assumption.

As Lemma 3.3 leads to Theorem 3.4, so Lemma 3.3' leads to the following theorem:

**Theorem 3.4'.** If $\text{Ph}(X, S^n) \neq 0$, then the image of
C : Ph(X, S”) → SNT(S” ∨ ΣX)

is a nontrivial, indeed an uncountable, subset of SNT(S” ∨ ΣX). If, in addition,

Ph(x, S”) = Σ Ph(Σ^{k-1}W, S^{n-1}),

then all the elements in the image of C admit co-H-space (indeed, suspension) structures. □

4. Examples

Example 4.1. Let \( m \geq 2 \) and let \( n \) be an even integer satisfying \( 2 \leq n \leq 2m - 2 \). Set

\[
X = K(\mathbb{Z}, n), \quad Y = SU(m).
\]

Using Theorem 2.1, we see that

\[
\text{Ph}(K(\mathbb{Z}, n), SU(m)) \neq 0.
\]

Thus, by Theorem 3.4, \( SNT(K(\mathbb{Z}, n) \times \Omega SU(m)) \) is uncountable. If, in addition, \( n \) satisfies \( n \geq m \), Theorem 2.1 allows us to conclude that

\[
\text{Ph}(K(\mathbb{Z}, n) \wedge K(\mathbb{Z}, n), SU(m)) = 0.
\]

Hence, if \( m \leq n \leq 2m - 2 \), it follows from Theorem 3.4 that \( SNT(K(\mathbb{Z}, n) \times \Omega SU(m)) \) has uncountably many elements admitting H-space structures. [The simplest instance of Example 4.1 occurs when \( m = n = 2 \). In this case, a nontrivial phantom map \( K(\mathbb{Z}, 2) \) → \( S^3 \) gives rise to an H-space \( \Gamma_\phi \) which defines a nontrivial element of \( SNT(K(\mathbb{Z}, 2) \times \Omega S^3) \). This type of example was sought and achieved by Gray [1] in a slightly different way; see Section 1.]

Example 4.2. Let \( k \geq 2 \), \( n - k \geq 2 \) and set

\[
X = K(\mathbb{Z}, n - k), \quad Y = \Omega^{k-1}S^n.
\]

As in Example 4.1,

\[
\text{Ph}(K(\mathbb{Z}, n - k), \Omega^{k-1}S^n) \neq 0,
\]

\[
\text{Ph}(K(\mathbb{Z}, n - k) \wedge K(\mathbb{Z}, n - k), \Omega^{k-1}S^n) = 0
\]
so that \( \text{SNT}(\Omega^k(K(\mathbb{Z}, n) \times S^n)) \) has uncountably many elements admitting \( H \)-space structures. [According to [3, Example F], \( \text{SNT}(\Omega^k(K(\mathbb{Z}, n) \times S^n)) \) is uncountable for \( k \leq 1 \) and \( n \) odd, \( n \geq 3 \); also, by [3, Example G], \( \text{SNT}(\Omega(K(\mathbb{Z}, n) \times S^n)) \) is uncountable for \( n \) even, \( n \geq 4 \) but \( \text{SNT}(K(\mathbb{Z}, n) \times S^n) \) is trivial for \( n \) even, \( n \geq 4 \).]

Our techniques extend to the case \( k = 1 \) in Example 4.2 even though the target space \( Y = S^n \) is not grouplike (unless \( n = 3 \)). In fact, we still have

\[
\text{Ph}(K(\mathbb{Z}, n-1), S^n) \cong [K(\mathbb{Q}, n-1), S^n] \neq 0
\]

so that Lemma 3.3 implies the nontriviality of \( \text{SNT}(\Omega(K(\mathbb{Z}, n) \times S^n)) \). However, we do not know whether the elements in the image of

\[
F : \text{Ph}(K(\mathbb{Z}, n-1), S^n) \rightarrow \text{SNT}(\Omega(K(\mathbb{Z}, n) \times S^n))
\]

admit \( H \)-space structures.

The next two examples likewise involve target spaces \( Y \) which are not (necessarily) grouplike.

**Example 4.3.** Let \( m \geq 2 \) and let \( n \) be an odd integer satisfying \( 3 \leq n \leq 2m - 1 \). Set

\[
X = K(\mathbb{Z}, n), \quad Y = B\text{SU}(m).
\]

Using (a version of) Theorem 2.1, we see that

\[
\text{Ph}(K(\mathbb{Z}, n), B\text{SU}(m)) \neq 0.
\]

In fact,

\[
[X_{(i)} , Y] = [X_{(i)} , \Omega Y] = [X_{(i)} , \text{SU}(m)] = 0,
\]

so that

\[
\text{Ph}(X, Y) \cong [X_{(i)}, Y] = [K(\mathbb{Q}, n), B\text{SU}(m)] \neq 0.
\]

Thus, by Lemma 3.3, \( \text{SNT}(K(\mathbb{Z}, n) \times \text{SU}(m)) \) is nontrivial.

**Example 4.4.** Let \( n \) be even, \( n \geq 2 \), and set

\[
X = K(\mathbb{Z}, n), \quad Y = S^{2n+1}.
\]

Once again,
Ph(K(\mathbb{Z}, n), S^{2n+1}) \neq 0

and so SNT(K(\mathbb{Z}, n) \times \Omega S^{2n+1}) is nontrivial. [Note that since \( \Omega \text{Ph}(K(\mathbb{Z}, n), S^{2n+1}) = 0 \), the space \( F_\phi \) satisfies

\[
\Omega F_\phi = \Omega(K(\mathbb{Z}, n) \times \Omega S^{2n+1}),
\]

that is, the space \( \Omega F_\phi \) defines the trivial element of \( \text{SNT}(\Omega(K(\mathbb{Z}, n) \times \Omega S^{2n+1})) \). The same sort of thing is true in those instances of Example 4.1 corresponding to \( n = 2 \) and those instances of Example 4.2 corresponding to \( n - k = 2 \). By way of contrast, if \( n \) is odd, then it may be verified that

\[
\Omega : \text{Ph}(K(\mathbb{Z}, n), Y) \rightarrow \text{Ph}(K(\mathbb{Z}, n - 1), \Omega Y)
\]

has trivial kernel provided \( Y \) is (an iterated loop space of) a finite cell complex.]

The final two examples are dual versions of Examples 4.2 and 4.4.

**Example 4.2'.** Let \( k \geq 2, n \geq 2 \) and set

\[
X = \Sigma^{k-1} K(\mathbb{Z}, n), \quad Y = S^{n+k}.
\]

Using Theorem 2.1, we see that

\[
\text{Ph}(\Sigma^{k-1} K(\mathbb{Z}, n), S^{n+k}) \neq 0.
\]

Thus, by Theorem 3.4', \( \text{SNT}(\Sigma^k (S^n \vee K(\mathbb{Z}, n))) \) is uncountable. Moreover, the suspension map

\[
\Sigma : \text{Ph}(\Sigma^{k-2} K(\mathbb{Z}, n), S^{n+k}) \rightarrow (\text{Ph}(\Sigma^{k-1} K(\mathbb{Z}, n), S^{n+k})
\]

is checked to be bijective (cf. [7, Example 4.1]). Hence, it follows from Theorem 3.4' that \( \text{SNT}(\Sigma^k (S^n \vee K(\mathbb{Z}, n))) \) has uncountably many elements admitting co-H-space structures. [According to [4, Example C], \( \text{SNT}(\Sigma^k (S^n \vee K(\mathbb{Z}, n))) \) is uncountable for \( k \geq 1, n \geq 2 \).]

**Example 4.4'.** Let \( n \) be even, \( n \geq 4 \), and set

\[
X = K(\mathbb{Z}, 2n - 2), \quad Y = S^n.
\]

As usual,

\[
\text{Ph}(K(\mathbb{Z}, 2n - 2), S^n) \neq 0.
\]
We cannot appeal directly to Theorem 3.4’ or even Lemma 3.3’ since $K(\mathbb{Z}, 2n - 2)$ has no cogroup structure but we must resort to the method of proof in Lemma 3.3’ to infer that $\Sigma S(n' \vee \Sigma K(\mathbb{Z}, 2n - 2))$ is nontrivial. In fact, if a nontrivial phantom map $K(\mathbb{Z}, 2n - 2) \xrightarrow{\phi} S^n$ were to give rise to a mapping cone $C_\phi$ homotopy equivalent to $S^n \vee \Sigma K(\mathbb{Z}, 2n - 2)$, we would be led to the conclusion

$$(\pm 1)^n \phi = 0,$$

an absurdity. [Dually to Example 4.4, the mapping cone $C_\phi$ corresponding to a phantom map $K(\mathbb{Z}, 2n - 2) \xrightarrow{\phi} S^n$ satisfies

$$\Sigma C_\phi = \Sigma (S^n \vee \Sigma K(\mathbb{Z}, 2n - 2))$$

cf. [7, Example 4.2] and [4, Example D]. Also, if $n$ is odd,

$$\Sigma : \text{Ph}(X, S^n) \to \text{Ph}(\Sigma X, S^{n+1})$$

has trivial kernel provided $X$ is (an iterated suspension of) a finite Postnikov space.]

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References

[5] C.A. McGibbon and J.M. Moller, How can you tell two spaces apart when they have the same $n$-type for all $n$?, Preprint.