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Vertices and Sources

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In two important papers [3], [4], J. A. Green has shown how a knowledge of indecomposable modules of p-groups may be used to obtain information about finite groups. The object of this paper is to give two applications of Green's work. First, Theorem B of Hall-Higman [5] is proved without resorting to their Lemmas 2.5.2 and 2.5.3. Second, I give a simple proof that, if a Sylow p-subgroup $\mathfrak{P}$ of $\mathfrak{G}$ is a cyclic, self-centralizing, T.I. set in $\mathfrak{G}$, then each p-rational irreducible character of $\mathfrak{G}$ assumes the value 0, 1 or -1 on p-singular elements of $\mathfrak{G}$. The decomposition matrix for the principal p-block of $\mathfrak{G}$ is also determined. Thus, for example, we recover some results of Brauer on permutation groups of prime degree. In fact, the present paper represents an attempt to understand the fundamental results of Brauer [1] on blocks of defect 1.

The notation is standard. K is an algebraic number field which is a splitting field for every subgroup of the finite group $\mathfrak{G}$, p is a prime, R is the ring of all integers of K, P is a prime ideal of R containing p, R_p is the ring of P-integers of K, R^*, K^*, P^* are the completions of R_p, K and the maximal ideal of R_p, respectively, and $\bar{K} = R^*/P^*$. For each ring S and group $\mathfrak{G}$, $S\mathfrak{G}$ is the group ring of $\mathfrak{G}$ over S. If S is a subring of S' and $\mathfrak{g}$ is a subgroup of $\mathfrak{G}$, we view $S\mathfrak{g}$ as a subring of $S'\mathfrak{G}$.

All modules will be right modules, and if M is an $R^*\mathfrak{G}$-module, we set $\bar{M} = M/MP^*$, so that $\bar{M}$ is a $\bar{K}\mathfrak{G}$-module.

Let $\bar{K}\mathfrak{G} = \epsilon_1\bar{K}\mathfrak{G} \oplus \cdots \oplus \epsilon_m\bar{K}\mathfrak{G}$ be a decomposition of $\bar{K}\mathfrak{G}$ into principal indecomposable submodules, where $\epsilon_1, \ldots, \epsilon_m$ are orthogonal idempotents. Let $e_1, \ldots, e_m$ be orthogonal idempotents of R^* with $\bar{e}_i = \epsilon_i$ ([2], Theorem 77.11), and set $U_i = e_iR^*\mathfrak{G}$. This notation is fixed throughout the paper. Also, we let $\tau^{(i)}$ be the character afforded by $e_iR^*\mathfrak{G}$.

If α, β are characters of $\mathfrak{G}, \alpha \subset \beta$ means that $\beta - \alpha$ is either zero or a character.

THEOREM 1. *Suppose χ is a character of $\mathfrak{G}$, and for some i, $\chi \subset \tau^{(i)}$. Then there is a pure submodule W of U_i such that $M = U_i/W$ affords χ.*

Proof. Let $\tau^{(i)} = \chi + \chi'$. Set $V = K^*U_i$. Since V is completely reducible, we have $V = V_1 \oplus V_2$ where V_1, V_2 afford χ, χ', respectively. Let $W = U_i \cap V_2$.

Then W is a pure submodule of U_i and $V_2 = K^*W$. Thus, W affords χ'. Since U_i/W is torsion free, the theorem follows.

COROLLARY. *$\bar{M}$ is indecomposable.*

Proof. Since $\epsilon_i\bar{K}\mathfrak{G} = \bar{U}_i$ has just one maximal submodule ([*2*], Theorem 54.11), the same holds for $\bar{M}$. This guarantees that $\bar{M}$ is indecomposable.

LEMMA 1. *Suppose M is a torsion-free $R^*\mathfrak{G}$-module such that $\bar{M} = N_1 \oplus P_1$ where P_1 is projective. Then $M = N \oplus P$ where $\bar{N} \simeq N_1$, $\bar{P} \simeq P_1$ and P is projective.*

Proof.

$$\begin{array}{ccc} & & 0 \\ & & \downarrow \\ & & QP^* \\ & & \downarrow \\ & & Q \\ & \overset{\alpha}{\swarrow} & \downarrow \\ M & \xrightarrow{\pi} & P_1 \to 0 \\ & & \downarrow \\ & & 0 \end{array}$$

Since idempotents can be lifted, we are guaranteed the existence of a projective $R^*\mathfrak{G}$-module Q such that the column is exact. If π denotes the composition of the homomorphism $m \to \bar{m}$ with projection onto P_1, then the row is exact. Since Q is projective, we are guaranteed the existence of α so that the diagram commutes. Set $P = \alpha(Q)$. Since M is torsion-free, it follows that α is a monomorphism, and it is straightforward to verify that P is a pure submodule of M. Since projective modules are also injective [*2*], P is a summand of M, $M = N \oplus P$. By the Krull-Schmidt theorem applied to $\bar{M}$, we get $\bar{N} \simeq N_1$. The proof is complete (and well known).

LEMMA 2. *Suppose $\mathfrak{G} = \mathfrak{P}\mathfrak{F}$ is a Frobenius group whose Frobenius kernel $\mathfrak{P}$ is a cyclic p-group of order p^n and $\mathfrak{F}$ is a complement to $\mathfrak{P}$ in $\mathfrak{G}$ of order e. Then each indecomposable $\bar{K}\mathfrak{G}$-module has dimension $\leqslant p^n$ and for each k, $1 \leqslant k \leqslant p^n$ there are exactly e indecomposable nonisomorphic $\bar{K}\mathfrak{G}$-modules of dimension k. The restriction to $\mathfrak{P}$ of each indecomposable $\bar{K}\mathfrak{G}$-module remains indecomposable.*

Proof. Since $\mathfrak{F}$ is Abelian, the e primitive idempotents $\delta_1, \ldots, \delta_e$ of $\bar{K}\mathfrak{F}$ yield that $\bar{K}\mathfrak{G}$ is the direct sum of the indecomposable submodules $\delta_i\bar{K}\mathfrak{G}$,

$1 \leqslant i \leqslant e$. Since $\delta_i \bar{K}\mathfrak{G} = \delta_i \bar{K}\mathfrak{P}$, we have $\dim (\delta_i \bar{K}\mathfrak{G}) = p^n$. Let N be the radical of $\bar{K}\mathfrak{P}$. Then for each j, $\delta_i N^j$ is a submodule of $\delta_i \bar{K}\mathfrak{G}$, and it is easy to see that the $p^n e$-modules $M_{i,j} = \delta_i \bar{K}\mathfrak{G}/\delta_i N^j$ are pairwise nonisomorphic. Thus, it suffices to show that each indecomposable $\bar{K}\mathfrak{G}$-module is isomorphic to some $M_{i,j}$.

Since every indecomposable $\bar{K}\mathfrak{G}$-module is $\mathfrak{P}$-projective by Higman's theorem ([*3*], Theorem 1][1], and since, every indecomposable $\bar{K}\mathfrak{P}$-module is of the shape $\bar{K}\mathfrak{P}/N^j = M_j$ (a trivial but unfortunately crucial step), it suffices to show that $M_j^{\mathfrak{G}} \simeq M_{1,j} \oplus \cdots \oplus M_{e,j}$.[2] But $M_{i,j|\mathfrak{P}} \simeq M_j$, so by Higman's theorem again, we get that $M_{i,j} \mid M_j^{\mathfrak{G}}$. As the $M_{i,j}$ are pairwise non-isomorphic the lemma follows from the Krull-Schmidt theorem.

Lemma 2 is a special case of results of Srinivasan [*6*].

LEMMA 3. *Suppose a Sylow p-subgroup $\mathfrak{P}$ of $\mathfrak{G}$ is a T.I. set. Let $\mathfrak{N} = N(\mathfrak{P})$. Let M be a torsion-free $R^*\mathfrak{G}$-module which is not projective such that $\bar{M}$ is indecomposable. Suppose L is an indecomposable torsion-free $R^*\mathfrak{N}$-module with $M \mid L^{\mathfrak{G}}$, and S is an indecomposable $\bar{K}\mathfrak{N}$-module with $\bar{M} \mid S^{\mathfrak{G}}$. Then $M_{\mathfrak{N}} \simeq L \oplus P$, where P is projective and $\bar{L} \simeq S$.*

Proof. Since $\mathfrak{P}$ is a T.I. set in $\mathfrak{G}$, the Mackey decomposition ([*4*], Eq. (2.12)) implies that $(L^{\mathfrak{G}})_{\mathfrak{N}} \simeq L \oplus P_0$, where P_0 is projective. Similarly, $(S^{\mathfrak{G}})_{\mathfrak{N}} \simeq S \oplus P_0'$ where P_0' is projective. Since neither M nor $\bar{M}$ is projective (M by hypothesis, $\bar{M}$ by Lemma 1), it follows from the Krull-Schmidt theorem that $M_{\mathfrak{N}} \simeq L \oplus P$, $\bar{M}_{\mathfrak{N}} \simeq S \oplus P'$, where P and P' are projective. Since L is indecomposable it follows from Lemma 1 that no component of $\bar{L}$ is projective. Since $\overline{(M_{\mathfrak{N}})} \simeq \bar{M}_{\mathfrak{N}}$, it follows from the indecomposability of S that $\bar{L} \simeq S$.

LEMMA 4. *Assume that the hypotheses of Lemma 2 are satisfied. Let M be an indecomposable torsion free $R^*\mathfrak{G}$-module such that $\bar{M}$ is indecomposable. Let χ be the character of $\mathfrak{P}$ afforded by $M_{\mathfrak{P}}$. Then each linear character of $\mathfrak{P}$ has multiplicity at most 1 in χ.*

Proof. By Lemma 2, $\bar{M}_{\mathfrak{P}}$ is indecomposable. Hence, every submodule of $\bar{M}_{\mathfrak{P}}$ is indecomposable. Hence, every pure submodule of $M_{\mathfrak{P}}$ is indecomposable. Let φ be a linear character of $\mathfrak{P}$ and let

$$M(\varphi) = \{m \in M \mid m(P - \varphi(P))^k = 0 \text{ for some } k \text{ and all } P \text{ in } \mathfrak{P}\}.$$

Then $M(\varphi)$ is a pure submodule of $M_{\mathfrak{P}}$ and by consideration of $K^* \otimes_{R^*} M$, it follows that the rank of $M(\varphi)$ is the multiplicity of φ in χ. Since $P - \varphi(P) = N(P)$ induces a nilpotent endomorphism of $M(\varphi)$ for each P

[1] Or by [*4*], p. 111, Corollary.

[2] We use Green's notation for induction and restriction of modules.

in $\mathfrak{P}$, and since R^* is of characteristic 0, we get that $m(P - \varphi(P)) = 0$ for all m in $M(\varphi)$ and P in $\mathfrak{P}$. Hence, $M(\varphi)$ is the direct sum of submodules of rank 1; so $M(\varphi)$ has rank 0 or 1, as required.

LEMMA 5. *Suppose $\mathfrak{P}$ is a self-centralizing cyclic Sylow p-subgroup of $\mathfrak{G}$ which is a T.I. set in $\mathfrak{G}$. Let χ be a character of $\mathfrak{G}$ and suppose that for some i, $\chi \subset \tau^{(i)}$ ($\tau^{(i)}$ is the character afforded by U_i). For each linear character φ of $\mathfrak{P}$, let $a(\varphi)$ be the multiplicity of φ in $\chi_{|\mathfrak{P}}$, and set $h(\chi) = \max | a(\varphi) - a(\varphi') |$, where φ, φ' range over all linear characters of $\mathfrak{P}$. Then $h(\chi) \leqslant 1$.*

Proof. Let W be a pure submodule of U_i such that $M = U_i/W$ affords χ (Theorem 1). If M is projective, then $h(\chi) = 0$, and we are done. We may assume that M is not projective.

Since $\bar{M}$ is indecomposable (Corollary to Theorem 1), we may apply Lemma 3, where again we are using Higman's theorem to guarantee the existence of L. Thus, $M_{\mathfrak{N}} = L \oplus P$ where P is projective and $\bar{L}$ is indecomposable. Since the character afforded by P vanishes on p-singular elements, this lemma follows from Lemma 4.

For the first application, we turn to that portion of Theorem B for which an alternative treatment is available. Namely, assume that $\mathfrak{G} = \mathfrak{P}\mathfrak{Q}$ where $\mathfrak{Q}$ is a normal extra-special q-group and $\mathfrak{P}$ is a cyclic Sylow p-subgroup of $\mathfrak{G}$ which acts faithfully and irreducibly on $\mathfrak{Q}/Z(\mathfrak{Q})$, and trivially on $Z(\mathfrak{Q})$. Let V be an irreducible $\bar{K}\mathfrak{G}$-module on which $\mathfrak{G}$ is faithfully represented. Let $| \mathfrak{Q} | = q^{2m+1}$, $| \mathfrak{P} | = p^n$. Since $\mathfrak{Q}/Z(\mathfrak{Q})$ is a chief factor of $\mathfrak{G}$ on which $\mathfrak{P}$ acts faithfully, it follows that

$$q^m \equiv -1 \pmod{p^n}. \tag{*}$$

Since it is easy to see that every faithful ordinary irreducible character of $\mathfrak{G}$ has degree q^m, and since every irreducible $\bar{K}\mathfrak{G}$-module is a composition factor of $\bar{M}$ for a suitable torsion-free $R^*\mathfrak{G}$-module M, we may choose such an M so that $\bar{M} \simeq V$. Now $\mathfrak{N} = N(\mathfrak{P}) = \mathfrak{P} \times Z(\mathfrak{Q})$, and since $\mathfrak{P}$ is cyclic, every indecomposable $\mathfrak{N}$-module is uniquely determined by its dimension and a linear character of $Z(\mathfrak{Q})$.

Let $M_{\mathfrak{N}} = L \oplus P$ as in Lemma 3. By (*) and the fact that P is projective, we get $\operatorname{rank} L \equiv -1 \pmod{p^n}$. Since $\bar{L}$ is indecomposable, we have rank $L \leqslant p^n$. Hence, $\operatorname{rank} L = p^n - 1$, obtaining Theorem B.

For the second application, we assume that $\mathfrak{P}$ is a self-centralizing cyclic Sylow p-subgroup of $\mathfrak{G}$ which is a T.I. set. Let $| \mathfrak{P} | = p^n$, $| \mathfrak{N} | = p^n e$, $\mathfrak{N} = \mathfrak{P}\mathfrak{F} = N(\mathfrak{P})$ where $\mathfrak{F}$ is a complement to $\mathfrak{P}$ in $\mathfrak{N}$. To avoid trivialities, we assume in addition that $\mathfrak{P} \subset \mathfrak{N} \subset \mathfrak{G}$. Hence, $e \mid p - 1$ and $\mathfrak{N}$ has exactly $(p^n - 1)/e = a$ nonlinear irreducible characters $\lambda_1, \ldots, \lambda_a$. From the theory of exceptional characters, we have a sign $\epsilon = \pm 1$ and irreducible characters $\chi_1, \ldots, \chi_a$ of $\mathfrak{G}$ such that

$$(\lambda_i - \lambda_j)^* = \epsilon(\chi_i - \chi_j), \qquad 1 \leqslant i, j \leqslant a. \tag{E}$$

(If $a = 1$, omit the above argument, since exceptional characters cannot be defined in this case.)

Next, let $\Gamma = 1_{\mathfrak{P}} - \xi$, where ξ is a linear character of $\mathfrak{P}$ which is a constituent of $\lambda_{1|\mathfrak{P}}$. Then

$$\Gamma^* = 1 - \epsilon\chi_1 + c \sum_{i=1}^{a} \chi_i + \sum_{i=1}^{f} a_i\theta_i, \tag{F}$$

where c, is an integer, $\theta_1, \ldots, \theta_f$ are distinct nonprincipal irreducible characters of $\mathfrak{G}$ different from all χ_i's, and $a_1, \ldots, a_f$ are nonzero integers. Also, taking norms yields

$$e = (c - \epsilon)^2 + (a - 1)\, c^2 + \sum_{i=1} a_i^2. \tag{N}$$

[If $a = 1$, omit all reference to the χ_i's, and define the θ_i's directly by (F).]

By Frobenius reciprocity, we get that

$$h(\theta_i) = | a_i |, \qquad h(\chi_i) = \max\{| \epsilon - c |, | c |\},$$

where $h(\chi)$ is defined by Lemma 5. Since each irreducible character of $\mathfrak{G}$ is a constituent of some $\tau^{(i)}$, Lemma 5 implies that $| a_i | = 1$ and that $c = 0$ or ϵ.

Let $\varphi = \sum_{i=1}^{a} \chi_i$. Since each $\tau^{(j)}$ vanishes on p-singular elements [2], (E) implies that for each $i = 1, \ldots, a$, $\chi_i \subset \tau^{(j)}$ if and only if $\varphi \subset \tau^{(j)}$. Thus, if $c = \epsilon$, it follows that $a = 2$.

Let $\Xi = \{1_{\mathfrak{G}}, \varphi, \theta_1, \ldots, \theta_f\}$. By Lemma 5, if $\zeta \in \Xi$, then there is no j such that $2\zeta \subset \tau^{(j)}$. Thus, the decomposition numbers for the principal p-block of $\mathfrak{G}$ are 0 or 1. If $\zeta, \zeta' \in \Xi$ and $\chi = \zeta + \zeta' \subset \tau^{(j)}$ for some j, Lemma 5 forces $h(\chi) = 0$. Let $M = U_i/W$ be the module given by Theorem 1, and let $M = L \oplus P$ be the decomposition given in Lemma 3. Then $h(\chi) = 0$ implies that L is projective. Hence, M is projective, so that $W = 0$, $\tau^{(i)} = \chi$.

Since the determinant of the Cartan matrix is p^n [2], we may now apply the purely combinatorial argument of Brauer [1] which yields that the linear graph associated with the principal p-block of $\mathfrak{G}$ is a tree.

Also, of course, (N) implies that $\mathfrak{N}$ has no more conjugacy classes than $\mathfrak{G}$, a fact which still seems to defy an elementary proof.

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