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Neumann Problems of Semilinear Elliptic Equations Involving Critical Sobolev Exponents

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In this paper we study the existence of positive solutions to the equation $\Delta u + u^p + f(x, u) = 0$ under the Neumann boundary condition $D_\gamma u + \alpha(x) u = 0$, where $p = (n+2)/(n-2)$, $f(x, u)$ is a lower order perturbation of u^p at infinity. When $\alpha(x) = 0$, we prove the existence of a positive solution provided $\lim_{u \to 0} f(x, u)/u = a(x) \leqslant 0$, $a(x) \not\equiv 0$, and $f(x, u) \geqslant -Au - Bu^q$ for some constants $A, B \geqslant 0$, $q \in (1, n/(n-2))$. For general $\alpha(x)$, we prove the existence under an additional assumption on the boundary $\partial\Omega$. © 1991 Academic Press, Inc.

1. INTRODUCTION

Let Ω be a bounded domain in R^n with C^1 boundary, $n \geqslant 3$. In this paper we are concerned with the problem of existence of a function u satisfying the nonlinear elliptic problem

$$\begin{aligned} -\Delta u &= u^p + f(x, u) && \text{in } \Omega, \\ D_\gamma u + \alpha(x) u &= 0 && \text{on } \partial\Omega, \qquad (1.1) \\ u &> 0 && \text{in } \Omega, \end{aligned}$$

where $p = (n+2)/(n-2)$, $\gamma = (\gamma_1, ..., \gamma_n)$ is the unit outward normal to $\partial\Omega$, $\alpha(x)$ is a nonnegative function, $f(x, u)$ is a lower order perturbation of u^p at infinity, and $f(x, 0) = 0$.

$u \in H(\Omega)$ is a weak solution of (1.1) if

$$\int_\Omega (D_i u \, D_i v - u^p v - f(x, u) v) \, dx + \int_{\partial\Omega} \alpha(x) uv \, ds = 0 \qquad \forall v \in H(\Omega),$$

and $u \geqslant 0$, $u \not\equiv 0$. We verify in Section 2 that the weak solutions of (1.1) are equivalent to the nonzero critical points of the functional

$$J(u) = \int_\Omega \left(\frac{1}{2} |Du|^2 - \frac{1}{p+1} u_+^{p+1} - F(x, u) \right) dx + \frac{1}{2} \int_{\partial\Omega} \alpha(x) u^2 \, ds,$$

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where $F(x, u) = \int_0^t f(x, t)\,dt$, $u_+ = \max(u, 0)$. Since $p+1 = 2n/(n-2)$, the embedding $H(\Omega) \subset L^{p+1}(\Omega)$ is not compact, the functional $J(u)$ does not satisfy the (PS) condition. Hence we cannot apply the standard variational methods directly.

The Neumann problem of semilinear elliptic equations with subcritical growth was studied by Ni, Takagi, and Lin, and many existence results were obtained (see [8–10]). The Dirichlet counterpart of (1.1), namely

$$\begin{aligned} -\Delta u &= u^p + f(x, u) && \text{in } \Omega, \qquad p = \frac{n+2}{n-2} \\ u &= 0 && \text{on } \partial\Omega, \\ u &> 0 && \text{in } \Omega. \end{aligned} \tag{1.2}$$

was studied by Brezis and Nirenberg [3]. Their results show that the existence of solutions of (1.2) depends strongly on the behavior of $f(x, u)$. But Problem (1.1) is different from (1.2). We shall prove that Problem (1.1) possesses a solution for a large class of $f(x, u)$.

This paper is organized as follows. In Section 2, we present a general existence theorem (Theorem 2.1) which is based on a variant of the Mountain Pass Lemma. We prove that $J(u)$ satisfies the $(\mathrm{PS})_c$ condition in a weak sense for $c \in (0, (1/2n)S^{n/2})$. That is, if $(u_j) \subset H(\Omega)$ is a sequence of functions satisfying $J(u_j) \to c \in (0, (1/2n)S^{n/2})$, and $J'(u_j) \to 0$ in $H^{-1}(\Omega)$ as $j \to \infty$, then there exists a subsequence of (u_j) which converges weakly to $u_0 \not\equiv 0$, and u_0 is a critical point of $J(u)$, where S is the best Sobolev embedding constant, i.e.,

$$S = \inf_{u \in H^1_0(\Omega)} \left\{ \int_\Omega |Du|^2\,dx;\ \int_\Omega u^{p+1}\,dx = 1 \right\}.$$

In Section 3, we deal with the problem

$$\begin{aligned} -\Delta u &= u^p + f(x, u) && \text{in } \Omega, \\ D_\gamma u &= 0 && \text{on } \partial\Omega, \\ u &> 0 && \text{in } \Omega. \end{aligned} \tag{1.3}$$

By means of Theorem 2.1, we prove the existence of a nonconstant solution to (1.3) when $f(x, u) = -\lambda u$ for $\lambda > 0$ sufficiently large. In Section 4, we are concerned with the problem

$$\begin{aligned} -\Delta u &= u^p && \text{in } \Omega, \\ D_\gamma u + \alpha(x) u &= 0 && \text{on } \partial\Omega, \\ u &> 0 && \text{in } \Omega. \end{aligned} \tag{1.4}$$

where $\alpha(x) \geqslant 0$, $\alpha(x) \not\equiv 0$ (Indeed, there is no solution of (1.4) if $\alpha(x) \equiv 0$.)

We prove the existence of a solution under an assumption on the boundary $\partial\Omega$. Finally, in Section 5, we discuss the regularity of solutions of (1.1). We also treat equations with variable coefficients briefly.

2. A General Existence Theorem

Let $\Omega \subset R^n$, $n \geqslant 3$, be a bounded domain with C^1 boundary. We assume that $f(x, u)$ is measurable in x and continuous in u and that $\sup\{f(x, u); x \in \Omega, 0 \leqslant u \leqslant M\} < \infty$ for every $M > 0$.

Let $p = (n+2)/(n-2)$, $\alpha(x) \in L^\infty(\Omega)$, $\alpha(x) \geqslant 0$. We are concerned with the problem of the existence of a function u satisfying

$$\begin{aligned} -\Delta u &= u^p + f(x, u) && \text{in } \Omega, \\ D_\gamma u + \alpha(x) u &= 0 && \text{on } \partial\Omega, \\ u &> 0 && \text{in } \Omega, \end{aligned} \tag{2.1}$$

where $\gamma = (\gamma_1, ..., \gamma_n)$ is the unit outward normal to $\partial\Omega$. We assume that there exists $a(x) \in L^\infty(\Omega)$ such that

$$\lim_{u \to 0} f(x, u)/u = a(x) \qquad \text{uniformly for} \quad x \in \Omega, \tag{2.2}$$

$$\lim_{u \to \infty} f(x, u)/u^p = 0 \qquad \text{uniformly for} \quad x \in \Omega. \tag{2.3}$$

Moreover, we assume that the first eigenvalue λ_1 of the following problem is positive:

$$\begin{aligned} -\Delta u - a(x) u &= \lambda u && \text{in } \Omega, \\ D_\gamma u + \alpha(x) u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

That is,

$$\lambda_1 = \inf \left\{ \int_\Omega (|Du|^2 - a(x) u^2)\, dx + \int_{\partial\Omega} \alpha(x) u^2\, ds; \int_\Omega u^2\, dx = 1 \right\} > 0. \tag{2.4}$$

Assumption (2.4) is satisfied if $\alpha(x) \equiv 0$, $a(x) \leqslant 0$, $a(x) \not\equiv 0$; or $a(x) \equiv 0$, $\alpha(x) \geqslant 0$, $\alpha(x) \not\equiv 0$. Hence the norm $\|u\|_H = \|Du\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}$ is equivalent to

$$|u|_H = \left[\int_\Omega (|Du|^2 - a(x) u^2)\, dx + \int_{\partial\Omega} \alpha(x) u^2\, ds \right]^{1/2}.$$

Since the values of $f(x, u)$ for $u<0$ are irrelevant, we may assume

$$f(x, u)=a(x)u \qquad \text{for} \quad u<0,\ x\in\Omega.$$

We claim that the weak solutions of (2.1) are equivalent to the nonzero critical points of the functional

$$J(u)=\int_\Omega\left[\frac{1}{2}|Du|^2-\frac{1}{p+1}u_+^{p+1}-F(x,u)\right]dx+\frac{1}{2}\int_{\partial\Omega}\alpha(x)u^2\,ds, \tag{2.5}$$

where $F(x, u)=\int_0^u f(x, u)\,du$. Indeed, a weak solution of (2.1) is obviously a critical point of $J(u)$. Conversely, if $u\in H(\Omega)$ is a critical point of $J(u)$, then

$$0=\langle J'(u), u_-\rangle=\int_\Omega[|Du_-|^2-a(x)(u_-)^2]\,dx+\int_{\partial\Omega}\alpha(x)(u_-)^2\,ds,$$

where $u_-=\min(u, 0)$. By virtue of (2.4) we see that $u_-\equiv 0$, which implies $u\geqslant 0$. Hence u is a weak solution of (2.1).

Denote

$$c=\inf_{\psi\in\Psi}\sup_{t\in(0,1)}J(\psi(t)), \tag{2.6}$$

where $\Psi=\{\psi\in C([0,1], H(\Omega));\ \psi(0)=0,\ \psi(1)=\psi_0\equiv t_0\}$, the constant t_0 is so large that $J(t\psi_0)\leqslant 0$ for all $t\geqslant 1$. By (2.4), we have

$$\begin{aligned}J(u)&\geqslant C\,\|u\|_H^2-\int_\Omega\left[F(x,u)-a(x)u^2+\frac{1}{p+1}u_+^{p+1}\right]dx\\&\geqslant (C-\varepsilon)\,\|u\|_H^2-C_\varepsilon\int_\Omega u_+^{p+1}\,dx\\&\geqslant (C-\varepsilon)\,\|u\|_H^2-C'_\varepsilon\,\|u\|_H^{p+1}\end{aligned}$$

for some $C>0$ (in the following, we use C to denote various positive constants). Let $\varepsilon=\frac{1}{2}C$; we obtain

$$c=\inf_{\psi\in\Psi}\sup_{t\in(0,1)}J(\psi(t))>0. \tag{2.7}$$

Set

$$S=\inf_{u\in H_0^1(\Omega)}\left\{\int_\Omega|Du|^2\,dx;\ \int_\Omega|u|^{p+1}\,dx=1\right\}, \tag{2.8}$$

which is the best Sobolev constant of the embedding $H_0^1(\Omega) \subset L^{p+1}(\Omega)$, $p = (n+2)/(n-2)$. It is known from [3, 11] that S depends only on n; the infimum in (2.8) is never achieved when Ω is a bounded domain. When $\Omega = R^n$, the infimum in (2.8) is achieved by the function $w(x) = (1 + |x|^2)^{-(n-2)/2}$, or (after rescaling) by any of the functions $w_\varepsilon(x) = C(\varepsilon + |x|^2)^{-(n-2)/2}$.

LEMMA 2.1. *Denote* $\tilde{B} = B_1 \cap \{x_n > h(x')\}$, *where* $B_1 = B(0, 1)$ *is the unit ball in* R^n, $h(x')$ *is a* C^1 *function defined in* $\{x' \in R^{n-1}, |x'| < 1\}$ *with* h, Dh *vanishing at* $0'$. *For any* $u \in H(B_1)$ *with* $\operatorname{supp} u \subset B_1$, *we have*

(i) *If* $h \equiv 0$, *then*

$$\int_{\tilde{B}} |Du|^2 \, dx \geqslant 2^{-2/n} S \left[\int_{\tilde{B}} |u|^{p+1} \, dx \right]^{2/(p+1)}. \tag{2.9}$$

(ii) $\forall \varepsilon > 0$, $\exists \delta > 0$ *depending only on* ε, *such that if* $|Dh| \leqslant \delta$, *then*

$$\int_{\tilde{B}} |Du|^2 \, dx \geqslant (2^{-2/n} S - \varepsilon) \left[\int_{\tilde{B}} |u|^{p+1} \, dx \right]^{2/(p+1)}. \tag{2.10}$$

Proof. (i) Since the values of $u(x)$ for $x_n < 0$ are irrelevant, we may suppose that $u(x)$ is even in x_n. Therefore

$$\begin{aligned} \int_{\tilde{B}} |Du|^2 \, dx &= \frac{1}{2} \int_{B_1} |Du|^2 \, dx \\ &\geqslant \frac{1}{2} S \left[\int_{B_1} |u|^{p+1} \, dx \right]^{2/(p+1)} \\ &= 2^{-2/n} S \left[\int_{\tilde{B}} |u|^{p+1} \, dx \right]^{2/(p+1)}. \end{aligned}$$

(ii) By (2.9) and the coordinate transformation $y' = x'$, $y_n = x_n - h(x')$, which straightens the bottom of $\tilde{B}$, we obtain (2.10) immediately. ∎

Now we give the main existence theorem of this section.

THEOREM 2.1. *Suppose* (2.2)–(2.4) *hold, and*

$$c < \frac{1}{2n} S^{n/2}; \tag{2.11}$$

then there is a solution u *of* (2.1) *which satisfies* $J(u) \leqslant c$.

Proof. By Theorem 2.2 in [3], there exists a sequence $(u_j) \subset H(\Omega)$ such that $J(u) \to c$ and $J'(u_j) \to 0$ in $H^{-1}(\Omega)$ as $j \to \infty$; that is,

$$J(u_j)=\int_\Omega\left[\frac{1}{2}|Du_j|^2-\frac{1}{p+1}(u_j)_+^{p+1}-F(x,u_j)\right]dx$$
$$+\frac{1}{2}\int_{\partial\Omega}\alpha(x)u_j^2\,ds=c+o(1), \tag{2.12}$$

$$\langle J'(u_j),\varphi\rangle=\int_\Omega[Du_j\,D\varphi-(u_j)_+^p\,\varphi-f(x,u_j)\varphi]\,dx$$
$$+\int_{\partial\Omega}\alpha(x)u_j\varphi\,ds=o(\|\varphi\|_H). \tag{2.13}$$

Let $\varphi=u_j$; then

$$\frac{1}{n}\int_\Omega(u_j)_+^{p+1}\,dx=\int_\Omega\left[F(x,u_j)-\frac{1}{2}u_jf(x,u_j)\right]dx+o(1)+o(\|u_j\|_h).$$

Since $f(x,u)=a(x)u$ for $u<0$, we have

$$F(x,u)-\tfrac{1}{2}uf(x,u)=0\qquad\text{for}\quad u<0.$$

Therefore

$$\int_\Omega\left[F(x,u_j)-\frac{1}{2}u_jf(x,u_j)\right]dx\leqslant\frac{1}{2n}\int_\Omega(u_j)_+^{p+1}\,dx+C(1+\|u_j\|_H).$$

Thus

$$\int_\Omega(u_j)_+^{P+1}\,dx\leqslant C(1+\|u_j\|_H).$$

Combining with (2.12) we obtain

$$\frac{1}{2}\int_\Omega[|Du_j|^2-a(x)u^2]\,dx+\frac{1}{2}\int_{\partial\Omega}\alpha(x)u_j^2\,ds\leqslant C(1+\|u_j\|_H),$$

that is, $\|u_j\|_H^2\leqslant C(1+\|u_j\|_H)$ for some different C, which implies $\|u_j\|_H\leqslant C$.

Extract a subsequence, still denoted by u_j, so that

$$\begin{aligned}
&u_j\rightharpoonup u && \text{weakly in } H(\Omega),\\
&u_j\rightharpoonup u && \text{weakly in } (L^{p+1}(\Omega))^*,\\
&u_j\to u && \text{strongly in } L^q(\Omega) \text{ for all } q<p+1,\\
&u_j\to u && \text{strongly in } L^2(\partial\Omega).
\end{aligned}$$

Passing to the limit in (2.13) we see that u is a critical point of J.

We now verify $u \not\equiv 0$. Indeed, if $u \equiv 0$, we have (see [3])

$$\int_\Omega F(x, u_j)\,dx \to 0, \qquad \int_\Omega u_j f(x, u_j)\,dx \to 0 \qquad \text{as} \quad j \to \infty. \tag{2.14}$$

By the compact embedding $H(\Omega) \hookrightarrow L^2(\partial\Omega)$, we also have

$$\int_{\partial\Omega} \alpha(x) u_j^2\,ds \to 0 \qquad \text{as} \quad j \to \infty. \tag{2.15}$$

Let ε be a small positive constant to be determined, and let $(\varphi_\alpha)_{\alpha=1}^N$ be a unit partition on $\bar{\Omega}$ with diam(supp φ_α) $\leqslant \delta$ for each α, where diam(D) is the diameter of the set D. Since $\partial\Omega \in C^1$, from Lemma 2.1 we have

$$\int_\Omega |D(u\varphi_\alpha)|^2\,dx \geqslant (2^{-2/n}S - \varepsilon)\left[\int_\Omega |u\varphi_\alpha|^{p+1}\,dx\right]^{2/(p+1)}$$
$$\forall 1 \leqslant \alpha \leqslant N,\ u \in H(\Omega)$$

provided δ is sufficiently small. Thus

$$\begin{aligned}
&\left[\int_\Omega (u_j)_+^{p+1}\,dx\right]^{2/(p+1)} \\
&\quad\leqslant \|u_j^2\|_{L^{(p+1)/2}(\Omega)} \\
&\quad= \left\|\sum_{\alpha=1}^N \varphi_\alpha u_j^2\right\|_{L^{(p+1)/2}} \leqslant \sum_{\alpha=1}^N \|\varphi_\alpha u_j^2\|_{L^{(p+1)/2}} \\
&\quad\leqslant (2^{-2/n}S - \varepsilon)^{-1} \sum_{\alpha=1}^N \int_\Omega |D(u_j\varphi_\alpha^{1/2})|^2\,dx \\
&\quad\leqslant (2^{-2/n}S - \varepsilon)^{-1}\left[(1+\varepsilon)\int_\Omega |Du_j|^2\,dx + C_\varepsilon \int_\Omega |u_j|^2\,dx\right] \\
&\quad= (2^{-2/n}S - \varepsilon)^{-1}(1+\varepsilon)\int_\Omega |Du_j|^2\,dx + o(1) \qquad \text{as} \quad j \to \infty.
\end{aligned} \tag{2.16}$$

From (2.13), we have

$$\int_\Omega [|Du_j|^2 - (u_j)_+^{p+1} - u_j f(x, u_j)]\,dx + \int_{\partial\Omega} \alpha(x) u_j^2\,ds = o(1). \tag{2.17}$$

Combining (2.12), (2.17), (2.14) and (2.15), we deduce that

$$\int_\Omega |Du_j|^2\,dx \to nc, \qquad \int_\Omega (u_j)_+^{p+1}\,dx \to nc \qquad \text{as} \quad j \to \infty.$$

Passing to the limit in (2.16) we therefore obtain

$$(nc)^{2/(p+1)} \leqslant (2^{-2/n}S-\varepsilon)^{-1}(1+\varepsilon)nc,$$

namely,

$$c \geqslant \frac{1}{n}\left[(2^{-2/n}S-\varepsilon)/(1+\varepsilon)\right]^{n/2}, \tag{2.18}$$

which contradicts (2.11) when $\varepsilon > 0$ is sufficiently small. Thus $u \not\equiv 0$.

Finally we show that $J(u) \leqslant c$. Since $u_j \to u$ weakly in $H(\Omega)$, we have

$$\int_\Omega F(x, u_j)\,dx \to \int_\Omega F(x, u)\,dx,$$

$$\int_\Omega u_j f(x, u_j)\,dx \to \int_\Omega uf(x, u)\,dx$$

as $j \to \infty$. By virtue of the compact embedding $H(\Omega) \subset L^2(\partial\Omega)$, we also have

$$\int_{\partial\Omega} \alpha(x)u_j^2\,dx \to \int_{\partial\Omega} \alpha(x)u^2\,dx \qquad \text{as} \quad j \to \infty.$$

Set $v_j = u_j - u$ (then $v_j \rightharpoonup 0$ in $H(\Omega)$), from [2] we have

$$\int_\Omega (u_j)_+^{p+1}\,dx = \int_\Omega (v_j)_+^{p+1}\,dx + \int_\Omega u^{p+1}\,dx + o(1),$$

and

$$\int_\Omega |Du_j|^2\,dx = \int_\Omega |Du|^2\,dx + \int_\Omega |Dv_j|^2\,dx + o(1).$$

Therefore (2.12) and (2.17) reduce to

$$J(u) + \int_\Omega \left[\frac{1}{2}|Dv_j|^2 - \frac{1}{p+1}(v_j)_+^{p+1}\right]dx = c + o(1)$$

and

$$\int_\Omega \left[|Dv_j|^2 - (v_j)_+^{p+1}\right]dx = o(1),$$

respectively. Consequently,

$$J(u) = c + o(1) - \frac{1}{n}\int_\Omega |Dv_j|^2\,dx,$$

which implies $J(u) \leqslant c$. ∎

Set

$$c^* = \inf_{u \in H(\Omega)} \{\sup_{t>0} J(tu);\ u \geq 0 \text{ and } u \not\equiv 0\}. \tag{2.19}$$

Then $c \leq c^*$ (see, e.g., [9]). Hence the condition (2.11) in Theorem 2.1 can be replaced by

$$c^* < \frac{1}{2n} S^{n/2}. \tag{2.20}$$

With this notation we have

COROLLARY 2.1. *Suppose $a(x) \in L^\infty(\Omega)$ is a nonpositive function, and $a(x) \not\equiv 0$; then there is a $\lambda_0 > 0$ such that the problem*

$$\begin{aligned} -\Delta u &= u^p + \lambda a(x) u && \text{in } \Omega, \\ D_\gamma u &= 0 && \text{on } \partial\Omega, \\ u &> 0 && \text{in } \Omega \end{aligned} \tag{2.21}$$

possesses at least a solution for each $\lambda \in (0, \lambda_0)$.

Proof. Let $v(x) \equiv 1$, we have $\sup_{t>0} J(tv) < (1/2n) S^{n/2}$ if $\lambda > 0$ is sufficiently small, which implies the conclusion of Corollary 2.1, where

$$J(u) = \int_\Omega \left[\frac{1}{2} |Du|^2 - \frac{1}{p+1} u_+^{p+1} - \frac{1}{2} \lambda a(x) u^2 \right] dx. \quad \blacksquare$$

Similarly we have

COROLLARY 2.2. *If $\alpha(x) \geq 0$ is a bounded measurable function, and $\alpha(x) \not\equiv 0$, then there is a solution of the problem*

$$\begin{aligned} -\Delta u &= u^p && \text{in } \Omega, \\ D_\gamma u + \lambda \alpha(x) u &= 0 && \text{on } \partial\Omega, \\ u &> 0 && \text{in } \Omega \end{aligned} \tag{2.22}$$

for $\lambda > 0$ small.

3. EXISTENCE OF SOLUTIONS TO (1.3)

We consider the problem

$$\begin{aligned} -\Delta u &= u^p - \lambda u && \text{in } \Omega, \\ D_\gamma u &= 0 && \text{on } \partial\Omega, \\ u &> 0 && \text{in } \Omega, \end{aligned} \tag{3.1}$$

where $\lambda > 0$, $p = (n+2)/(n-2)$, $\Omega \subset R^n$ is a bounded domain with C^2 boundary, $n \geqslant 3$. Obviously $w_\lambda = \lambda^{1/(p-1)}$ is a constant solution of (3.1).

THEOREM 3.1. *Problem* (3.1) *possesses a nonconstant solution for* $\lambda > 0$ *suitably large.*

Remark 3.1. In the case when $1 < p < (n+2)/(n-2)$, the result of Theorem 3.1 was proved by Ni and Takagi [10].

Proof of Theorem 3.1. The solutions of (3.1) correspond to the nonzero critical points of the functional

$$J(u) = \int_\Omega \left[\frac{1}{2}|Du|^2 - \frac{1}{p+1} u_+^{p+1} + \frac{1}{2}\lambda u^2\right] dx. \tag{3.2}$$

If we have proved

$$c^* = \inf_{u \in H(\Omega)} \{\sup_{t>0} J(tu);\ u \geqslant 0 \text{ and } u \not\equiv 0\} < \frac{1}{2n} S^{n/2}, \tag{3.3}$$

then by Theorem 2.1 we obtain a solution u_λ satisfying

$$J(u_\lambda) \leqslant c \leqslant c^* < \frac{1}{2n} S^{n/2}.$$

On the other hand, a simple computation shows that $J(w_\lambda) = (1/n)\lambda^{n/2}\,\mathrm{mes}(\Omega)$. Hence if $J(w_\lambda) \geqslant (1/2n) S^{n/2}$, namely,

$$\lambda \geqslant S/(2\,\mathrm{mes}(\Omega))^{2/n},$$

then u_λ is a nonconstant solution. We now prove (3.3).

Let $B(\bar{x}, R)$ be a ball containing Ω, and $\partial B(\bar{x}, R) \cap \bar{\Omega} \neq \varnothing$. Choosing $x_0 \in \partial B(\bar{x}, R) \cap \bar{\Omega}$, we have $\alpha_i \geqslant R^{-1}$ for each $1 \leqslant i \leqslant n-1$, where $\alpha_1, ..., \alpha_{n-1}$ are the principal curvatures of $\partial\Omega$ at x_0 (relative to the inner normal). Then with no loss of generality we may suppose that x_0 is the origin and $\Omega \subset \{x_n > 0\}$. Hence the boundary $\partial\Omega$ near the origin is represented by (rotating the $x_1, ..., x_{n-1}$ directions if needed)

$$x_n = h(x') = \frac{1}{2}\sum_{i=1}^{n-1} \alpha_i x_i^2 + o(|x'|^2), \qquad \forall x' = (x_1, ..., x_n) \in D(0, \delta)$$

for some $\delta > 0$, where $D(0, \delta) = B(0, \delta) \cap \{x_n = 0\}$. Set

$$u_\varepsilon(x) = \varepsilon^{(n-2)/4}(\varepsilon + |x|^2)^{-(n-2)/2}.$$

We claim that

$$Y_\varepsilon = \sup_{t>0} J(tu_\varepsilon) < \frac{1}{2n} S^{n/2} \tag{3.4}$$

for $\varepsilon > 0$ sufficiently small (consequently (3.3) follows). Denote

$$K_1(\varepsilon) = \int_\Omega |Du_\varepsilon|^2\, dx, \qquad K_2(\varepsilon) = \int_\Omega |u_\varepsilon|^{p+1}\, dx,$$

and $g(x') = \frac{1}{2} \sum_{i=1}^{n-1} \alpha_i x_i^2$. The proof is divided into two cases.

Case 1, $n \geqslant 4$. We have

$$\begin{aligned} K_1(\varepsilon) &= \int_{R^n_+} |Du_\varepsilon|^2\, dx - \int_{D(0,\delta)} dx' \int_0^{h(x')} |Du_\varepsilon|^2\, dx_n + O(\varepsilon^{(n-2)/2}) \\ &= \tfrac{1}{2} K_1 - \int_{R^{n-1}} dx' \int_0^{g(x')} |Du_\varepsilon|^2\, dx_n \\ &\quad - \int_{D(0,\delta)} dx' \int_{g(x')}^{h(x')} |Du_\varepsilon|^2\, dx_n + O(\varepsilon^{(n-2)/2}), \end{aligned}$$

where $R^n_+ = R^n \cap \{x_n > 0\}$, and

$$K_1 = \int_{R^n} |Du_\varepsilon|^2\, dx = (n-2)^2 \int_{R^n} \frac{|x|^2}{(1+|x|^2)^n}\, dx \tag{3.5}$$

is a constant independent of ε. Observing that

$$\begin{aligned} I(\varepsilon) &= \int_{R^{n-1}} dx' \int_0^{g(x')} |Du_\varepsilon|^2\, dx_n \\ &= (n-2)^2\, \varepsilon^{(n-2)/2} \int_{R^{n-1}} dx' \int_0^{g(x')} \frac{|x|^2}{(\varepsilon+|x|^2)^n}\, dx_n \\ &= (n-2)^2 \int_{R^{n-1}} dy' \int_0^{\sqrt{\varepsilon}\, g(y')} \frac{|y|^2}{(1+|y|^2)^n}\, dy_n, \end{aligned} \tag{3.6}$$

we have

$$\lim_{\varepsilon \to 0} \varepsilon^{-1/2} I(\varepsilon) = (n-2)^2 \int_{R^{n-1}} \frac{|y'|^2\, g(y')}{(1+|y'|^2)^n}\, dy', \tag{3.7}$$

which implies $I(\varepsilon) = O(\varepsilon^{1/2})$. Moreover,

$$I_1(\varepsilon)=\left|\int_{D(0,\delta)} dx' \int_{g(x')}^{h(x')} |Du_\varepsilon|^2\, dx_n\right|$$

$$=\left|(n-2)^2\,\varepsilon^{(n-2)/2}\int_{D(0,\delta)} dx' \int_{g(x')}^{h(x')} \frac{|x|^2}{(\varepsilon+|x|^2)^n}\, dx_n\right|$$

$$\leqslant C(n-2)^2\,\varepsilon^{(n-2)/2}\int_{D(0,\delta)} \frac{|h(x')-g(x')|}{(\varepsilon+|x'|^2)^{n-1}}\, dx',$$

where C depends only on δ and n. Since $h(x')=g(x')+o(|x'|^2)$, it follows that $\forall\sigma>0$, $\exists C(\sigma)>0$ such that $|h(x')-g(x')|\leqslant\sigma\,|x'|^2+C(\sigma)\,|x'|^{5/2}$ for $x'\in D(0,\delta)$. Therefore

$$I_1(\varepsilon)\leqslant C\varepsilon^{(n-2)/2}\int_{D(0,\delta)} \frac{\sigma\,|x'|^2+C(\sigma)\,|x'|^{5/2}}{(\varepsilon+|x'|^2)^{n-1}}\, dx'\leqslant C\varepsilon^{1/2}(\sigma+C(\sigma)\varepsilon^{1/4}),$$

which implies

$$I_1(\varepsilon)=o(\varepsilon^{1/2}) \qquad \text{as} \quad \varepsilon\to 0. \tag{3.8}$$

Thus we obtain

$$K_1(\varepsilon)=\tfrac{1}{2}K_1-I(\varepsilon)+o(\varepsilon^{1/2}). \tag{3.9}$$

On the other hand

$$K_2(\varepsilon)=\int_{R^n_+} u_\varepsilon^{p+1}\, dx-\int_{D(0,\delta)} dx' \int_0^{h(x')} u_\varepsilon^{p+1}\, dx_n+O(\varepsilon^{n/2}),$$

$$=\tfrac{1}{2}K_2-\int_{R^{n-1}} dx' \int_0^{g(x')} u_\varepsilon^{p+1}\, dx_n$$

$$-\int_{D(0,\delta)} dx' \int_{g(x')}^{h(x')} u_\varepsilon^{p+1}\, dx_n+O(\varepsilon^{n/2}),$$

where

$$K_2=\int_{R^n} u_\varepsilon^{p+1}\, dx=\int_{R^n}\frac{1}{(1+|x|^2)^n}\, dx. \tag{3.10}$$

K_1 and K_2 satisfy (see [3])

$$K_1/K_2^{(n-2)/n}=S.$$

Since

$$\begin{aligned} II(\varepsilon) &= \int_{R^{n-1}} dx' \int_0^{g(x')} u_\varepsilon^{p+1}\, dx_n \\ &= \varepsilon^{n/2} \int_{R^{n-1}} dx' \int_0^{g(x')} \frac{1}{(\varepsilon + |x|^2)^n}\, dx_n \\ &= \int_{R^{n-1}} dy' \int_0^{\sqrt{\varepsilon}\, g(y')} \frac{1}{(1+|y|^2)^n}\, dy_n, \end{aligned} \tag{3.11}$$

we have

$$\lim_{\varepsilon \to 0} \varepsilon^{-1/2} II(\varepsilon) = \int_{R^{n-1}} \frac{g(y')}{(1+|y'|^2)^n}\, dy'. \tag{3.12}$$

Thus $II(\varepsilon) = O(\varepsilon^{1/2})$. Similarly to (3.8) we have

$$\left| \int_{K(0,\delta)} dx' \int_{g(x')}^{h(x')} u_\varepsilon^{p+1}\, dx_n \right| = o(\varepsilon^{1/2}) \qquad \text{as} \quad \varepsilon \to 0.$$

Therefore

$$K_2(\varepsilon) = \tfrac{1}{2} K_2 - II(\varepsilon) + o(\varepsilon^{1/2}). \tag{3.13}$$

Moreover (see [3])

$$K_3(\varepsilon) = \int_\Omega u_\varepsilon^2\, dx = \begin{cases} O(\varepsilon^{1/2}) & n = 3, \\ O(|\varepsilon \log \varepsilon|) & n = 4, \\ O(\varepsilon) & n \geqslant 5. \end{cases} \tag{3.14}$$

Let $t_\varepsilon > 0$ be such a constant that

$$\begin{aligned} J(t_\varepsilon u_\varepsilon) = Y_\varepsilon &= \sup_{t>0} J(t u_\varepsilon) \\ &= \sup_{t>0} \left[\frac{1}{2} (K_1(\varepsilon) + \lambda K_3(\varepsilon)) t^2 - \frac{1}{p+1} K_2(\varepsilon) t^{p+1} \right]. \end{aligned}$$

From (3.9), (3.13), and (3.14), there exist positive constants ε_0, K', and K'' such that $K_2(\varepsilon) \geqslant K'$, $K_1(\varepsilon) + K_3(\varepsilon) \leqslant K''$ for $\varepsilon \in (0, \varepsilon_0)$. Hence t_ε are uniformly bounded for $\varepsilon \in (0, \varepsilon_0)$. Note that $K_3(\varepsilon) = o(\varepsilon^{1/2})$ when $n \geqslant 4$. Therefore

$$\begin{aligned} Y_\varepsilon = J(t_\varepsilon u_\varepsilon) &\leqslant \sup_{t>0} \left[\frac{1}{2} K_1(\varepsilon) t^2 - \frac{1}{p+1} K_2(\varepsilon) t^{p+1} \right] + o(\varepsilon^{1/2}) \\ &= \frac{1}{n} [K_1(\varepsilon)/(K_2(\varepsilon))^{(n-2)/n}]^{n/2} + o(\varepsilon^{1/2}). \end{aligned}$$

We claim that

$$K_1(\varepsilon)/(K_2(\varepsilon))^{(n-2)/n} < 2^{-2/n}S + o(\varepsilon^{1/2})$$
$$= \tfrac{1}{2}K_1/(\tfrac{1}{2}K_2)^{(n-2)/n} + o(\varepsilon^{1/2}) \tag{3.15}$$

for ε sufficiently small, which implies (3.4) and thereby (3.3).

Indeed, by (3.9), (3.13), and $II(\varepsilon) = O(\varepsilon^{1/2})$, (3.15) is equivalent to

$$\left(\frac{1}{2}K_1 - I(\varepsilon)\right)\left(\frac{1}{2}K_2\right)^{(n-2)/n}$$
$$< \frac{1}{2}K_1\left(\frac{1}{2}K_2 - II(\varepsilon) + o(\varepsilon^{1/2})\right)^{(n-2)/n} + o(\varepsilon^{1/2})$$
$$= \frac{1}{2}K_1\left[\left(\frac{1}{2}K_2\right)^{(n-2)/n} - \frac{n-2}{n}\left(\frac{1}{2}K_2\right)^{-2/n} II(\varepsilon)\right] + o(\varepsilon^{1/2}),$$

which reduces to

$$I(\varepsilon)/II(\varepsilon) > \frac{n-2}{n}K_1/K_2 + o(1) \qquad \text{as} \quad \varepsilon \to 0, \tag{3.16}$$

namely,

$$\lim_{\varepsilon \to 0} \frac{I(\varepsilon)}{II(\varepsilon)} > \frac{n-2}{n}K_1/K_2. \tag{3.17}$$

From (3.6) and (3.11) we have

$$\lim_{\varepsilon \to 0} \frac{I(\varepsilon)}{II(\varepsilon)} = \lim_{\varepsilon \to 0} \frac{I'(\varepsilon)}{II'(\varepsilon)}$$
$$= (n-2)^2 \int_{R^{n-1}} \frac{|x'|^2\, g(x')}{(1+|x'|^2)^n}\, dx' \Big/ \int_{R^{n-1}} \frac{g(x')}{(1+|x'|^2)^n}\, dx'$$
$$= (n-2)^2 \int_0^\infty \frac{r^{n+2}}{(1+r^2)^n}\, dr \Big/ \int_0^\infty \frac{r^n}{(1+r^2)^n}\, dr.$$

$\forall 2 \leqslant \beta < 2n-1$, integrating by parts we have

$$\int_0^\infty \frac{r^{\beta-2}}{(1+r^2)^{n-1}}\, dr = \frac{2(n-1)}{\beta-1}\int_0^\infty \frac{r^\beta}{(1+r^2)^n}\, dr.$$

Observing that

$$\int_0^\infty \frac{r^\beta}{(1+r^2)^n}\, dr = \int_0^\infty \frac{r^{\beta-2}}{(1+r^2)^{n-1}}\, dr - \int_0^\infty \frac{r^{\beta-2}}{(1+r^2)^n}\, dr,$$

we obtain

$$\int_0^\infty \frac{r^\beta}{(1+r^2)^n}\,dr = \frac{\beta-1}{2n-\beta-1}\int_0^\infty \frac{r^{\beta-2}}{(1+r^2)^n}\,dr. \tag{3.18}$$

Hence

$$\lim_{\varepsilon\to 0}\frac{I(\varepsilon)}{II(\varepsilon)} = (n-2)^2\,\frac{n+1}{n-3}.$$

On the other hand, from (3.5) and (3.10), we have

$$\frac{n-2}{n}\,\frac{K_1}{K_2} = \frac{(n-2)^3}{n}\int_0^\infty \frac{r^{n+1}}{(1+r^2)^n}\,dr \Big/ \int_0^\infty \frac{r^{n-1}}{(1+r^2)^n}\,dr = (n-2)^2.$$

Thus (3.17) follows.

Case 2, $n=3$. Let $0<a\leqslant A<\infty$ such that $a\,|x'|^2 \leqslant h(x') \leqslant A\,|x'|^2$ for $x' \in D(0,\delta)$; we have

$$\begin{aligned} K_1(\varepsilon) &= \int_{R^n_+} |Du_\varepsilon|^2\,dx - \int_{D(0,\delta)} dx' \int_0^{h(x')} |Du_\varepsilon|^2\,dx_n + O(\varepsilon^{(n-2)/2}) \\ &\leqslant \tfrac{1}{2}K_1 - \int_{D(0,\delta)} dx' \int_0^{a|x'|^2} |Du_\varepsilon|^2\,dx_n + O(\varepsilon^{1/2}). \end{aligned}$$

Since

$$\begin{aligned} \int_{D(0,\delta)} dx' \int_0^{a|x'|^2} |Du_\varepsilon|^2\,dx_n &\geqslant C\varepsilon^{(n-2)/2}\int_{D(0,\delta)} \frac{a\,|x'|^4}{(\varepsilon+|x'|^2)^n}\,dx' \\ &\geqslant C_0\varepsilon^{1/2}\,|\log\varepsilon|, \end{aligned}$$

we deduce that

$$K_1(\varepsilon) \leqslant \tfrac{1}{2}K_1 - C_0\varepsilon^{1/2}\,|\log\varepsilon| + o(\varepsilon^{1/2}). \tag{3.19}$$

Similarly,

$$\begin{aligned} K_2(\varepsilon) &= \frac{1}{2}K_2 - \int_{D(0,\delta)} dx' \int_0^{h(x')} u_\varepsilon^{p+1}\,dx_n + O(\varepsilon^{n/2}) \\ &\geqslant \frac{1}{2}K_2 - \int_{D(0,\delta)} \frac{A\varepsilon^{n/2}\,|x'|^2}{(\varepsilon+|x'|^2)^n}\,dx' + O(\varepsilon^{n/2}) \\ &= \frac{1}{2}K_2 - O(\varepsilon^{1/2}). \end{aligned} \tag{3.20}$$

Let $J(t_\varepsilon u_\varepsilon)=Y_\varepsilon=\sup_{t>0} J(tu_\varepsilon)$. From (3.14), (3.19), and (3.20), we see that t_ε are uniformly bounded for $\varepsilon\in(0,\varepsilon_0)$ for some $\varepsilon_0>0$. Thus

$$\begin{aligned} Y_\varepsilon &\leqslant \sup_{t>0}\left[\frac{1}{2}K_1(\varepsilon)t^2-\frac{1}{p+1}K_2(\varepsilon)t^{p+1}\right]+O(\varepsilon^{1/2})\\ &=\frac{1}{n}\left[K_1(\varepsilon)/(K_2(\varepsilon))^{(n-2)/n}\right]^{n/2}+O(\varepsilon^{1/2}). \end{aligned}$$

Consequently if

$$K_1(\varepsilon)/(K_2(\varepsilon))^{(n-2)/n}<2^{-2/n}S-O(\varepsilon^{1/2}) \qquad \text{for } \varepsilon>0 \text{ small,} \tag{3.21}$$

then (3.4) (and thereby (3.3)) follows.

By (3.19) and (3.20), (3.21) reduces to

$$\begin{aligned} \tfrac{1}{2}K_1-C_0\varepsilon^{1/2}\,|\log\varepsilon| &< 2^{-2/n}S[\tfrac{1}{2}K_2-O(\varepsilon^{1/2})]^{(n-2)/n}+O(\varepsilon^{1/2})\\ &=\tfrac{1}{2}SK_2^{(n-2)/n}+O(\varepsilon^{1/2}). \end{aligned}$$

Since $K_1/K_2^{(n-2)/n}=S$, we obtain (3.21) immediately. ∎

We now turn to the problem

$$\begin{aligned} -\Delta u &= u^p+f(x,u) && \text{in } \Omega,\\ D_\gamma u &= 0 && \text{on } \partial\Omega,\\ u &> 0 && \text{in } \Omega, \end{aligned} \tag{3.22}$$

where Ω is a bounded domain in R^n with C^2 boundary, $n\geqslant 3$, and $f(x,u)$ satisfies (2.2) and (2.3) with

$$a(x)\leqslant 0, \qquad a(x)\not\equiv 0. \tag{3.23}$$

THEOREM 3.2. *Suppose* (2.2), (2.3), *and* (3.23) *hold. Moreover, suppose*

$$f(x,u)\geqslant -Au-Bu^q \qquad \forall x\in\Omega,\ u\geqslant 0 \tag{3.24}$$

for some $A, B\geqslant 0$, *and* $q\in(1, n/(n-2))$. *Then there exists a solution of* (3.22).

Proof. Let $x_0\in\partial\Omega$ such that the principal curvatures $\alpha_1,\dots,\alpha_{n-1}$ of $\partial\Omega$ at x_0 (relative to the inner normal) are positive. We may suppose that x_0 is the origin and $\Omega\subset\{x_n>0\}$. Define $K_1(\varepsilon)$, $K_2(\varepsilon)$ as in the proof of Theorem 3.1, and

$$K_3(\varepsilon)=K_3(\varepsilon,t)=\int_\Omega F(x,tu_\varepsilon)\,dx,$$

where

$$u_\varepsilon = \varepsilon^{(n-2)/4}(\varepsilon + |x|^2)^{-(n-2)/2}.$$

From (3.24) we have

$$K_3(\varepsilon) \geqslant \begin{cases} O(\varepsilon^{1/2}) & n=3, \\ o(\varepsilon^{1/2}) & n \geqslant 4 \end{cases} \tag{3.25}$$

for any fixed t. Moreover, from (2.2) and (2.3) we have $|f(x,t)| \leqslant \frac{1}{2}t^p + Ct$ for some $C > 0$. Hence

$$|K_3(\varepsilon, t)| < \tfrac{1}{2}K_2(\varepsilon)t^{p+1} + CK_1(\varepsilon)t^2. \tag{3.26}$$

Let

$$J(t_\varepsilon u_\varepsilon) = Y_\varepsilon = \sup_{t>0} J(tu_\varepsilon),$$

where

$$J(u) = \int_\Omega \left[\frac{1}{2}|Du|^2 - \frac{1}{p+1}u_+^{p+1} - F(x,u)\right] dx.$$

From (3.26) we see that t_ε is uniformly bounded for $\varepsilon > 0$ sufficiently small. Therefore,

$$\begin{aligned} Y_\varepsilon &\leqslant \sup_{t>0}\left[\frac{1}{2}K_1(\varepsilon)t^2 - \frac{1}{p+1}K_2(\varepsilon)t^{p+1}\right] - K_3(\varepsilon, t_\varepsilon) \\ &= \frac{1}{n}\left[K_1(\varepsilon)/|K_2(\varepsilon)|^{(n-2)/n}\right]^{n/2} - K_3(\varepsilon, t_\varepsilon). \end{aligned}$$

By virtue of (3.25), similarly to the proof of Theorem 3.1, we have $Y_\varepsilon < (1/2n)S^{n/2}$ for $\varepsilon > 0$ sufficiently small. Thus Theorem 3.2 follows. ∎

From the proof of Theorem 3.1 (and Theorem 3.2) we see that the C^2 regularity of $\partial\Omega$ can be weakened to:

there is a point $x_0 \in \partial\Omega$ where the principal curvatures $\alpha_1, ..., \alpha_{n-1}$ of $\partial\Omega$ are finite and satisfy $\sum_{i=1}^{n-1} \alpha_i > 0$. (3.27)

In this situation the condition (3.24) can be replaced by

$$f(x,u) \geqslant -Au - Bu^q \qquad \text{for a.e.} \quad x \in \omega,\ u \geqslant 0, \tag{3.24$'$}$$

where ω is a neighborhood of x_0, $A, B \geqslant 0$, and $q \in (1, n/(n-2))$. Indeed, we may suppose x_0 is the origin, and the x_n-axis is the inner normal to $\partial\Omega$ there. Then the boundary of Ω near x_0 is given by $x_n = h(x') = \frac{1}{2}\sum_{i=1}^{n-1} \alpha_i x_i^2 + o(|x'|^2)$, and the proof of Theorem 3.2 is still applicable.

A typical example of (3.22) is

$$f(x, u) = a(x)u + b(x)u^q,$$

where $a(x), b(x) \in L^\infty(\Omega)$, $a(x) \leqslant 0$, and $a(x) \not\equiv 0$. From the above we see that, if $b(x) \geqslant 0$ a.e. in ω, then there is a solution of (3.22) for each $q \in (1, (n+2)/(n-2))$. Otherwise there is a solution of (3.22) for $q \in (1, n/(n-2))$.

4. Existence of Solutions to (1.4)

In this section we are concerned with the problem

$$\begin{aligned} -\Delta u &= u^p && \text{in } \Omega, \\ D_\gamma u + \alpha(x)u &= 0 && \text{on } \partial\Omega, \\ u &> 0 && \text{in } \Omega, \end{aligned} \tag{4.1}$$

where $\Omega \subset R^n$ is a bounded domain with C^1 boundary, $n \geqslant 3$, $p = (n+2)/(n-2)$, $\alpha(x) \in L^\infty(\Omega)$, and $\alpha(x) \geqslant 0$.

It is well known from Pohozeav's identity that there is no solution of the problem

$$\begin{aligned} -\Delta u &= u^p && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \\ u &> 0 && \text{in } \Omega, \end{aligned} \tag{4.2}$$

where Ω is a bounded star-shaped domain. But for any $\alpha(x) \geqslant 0$, $\alpha(x) \not\equiv 0$, we have

THEOREM 4.1. *Suppose the origin $O \in \partial\Omega$, and the x_n-axis is the inner normal to $\partial\Omega$ there. Suppose also that the boundary $\partial\Omega$ near O is expressed by $x_n = h(x')$ for $x' \in D(0, \delta) = B(0, \delta) \cap \{x_n = 0\}$, and*

$$\lim_{x' \to 0} |x'|^{1-\alpha} h(x') = d > 0 \qquad \textit{for some} \quad \alpha \in (0, 1). \tag{4.3}$$

Then there exists a solution of (4.1).

Proof. The solutions of (4.1) correspond to the nonzero critical points of the functional

$$J(u)=\int_{\Omega}\left(\frac{1}{2}|Du|^2-\frac{1}{p+1}u_+^{p+1}\right)dx+\frac{1}{2}\int_{\partial\Omega}\alpha(x)u^2\,ds.$$

Let

$$u_\varepsilon(x)=\varepsilon^{(n-2)/4}(\varepsilon+|x|^2)^{-(n-2)/2}.$$

We claim that

$$Y_\varepsilon=\sup_{t>0}J(tu_\varepsilon)<\frac{1}{2n}S^{n/2} \tag{4.4}$$

for $\varepsilon>0$ sufficiently small, which implies (2.20), and consequently by Theorem 2.1 we obtain a solution of (4.1). Indeed, denote

$$K_1(\varepsilon)=\int_{\Omega}|Du_\varepsilon|^2\,dx,\qquad K_2(\varepsilon)=\int_{\Omega}u_\varepsilon^{p+1}\,dx.$$

We have

$$\begin{aligned}K_1(\varepsilon)&=\tfrac{1}{2}K_1-\int_{D(0,\delta)}dx'\int_0^{h(x')}|Du_\varepsilon|^2\,dx_n+O(\varepsilon^{(n-2)/2})\\&=\tfrac{1}{2}K_1-\int_{R^{n-1}}dx'\int_0^{g(x')}|Du_\varepsilon|^2\,dx_n\\&\quad-\int_{D(0,\delta)}dx'\int_{g(x')}^{h(x')}|Du_\varepsilon|^2\,dx_n+O(\varepsilon^{(n-2)/2})\end{aligned}$$

where $g(x')=d\,|x'|^{1+\alpha}$. Similarly to (3.8) we have

$$\left|\int_{D(0,\delta)}dx'\int_{g(x')}^{h(x')}|Du_\varepsilon|^2\,dx_n\right|=o(\varepsilon^{\alpha/2}).$$

Thus

$$K_1(\varepsilon)=\tfrac{1}{2}K_1-I(\varepsilon)+o(\varepsilon^{\alpha/2}), \tag{4.5}$$

where

$$\begin{aligned}I(\varepsilon)&=\int_{R^{n-1}}dx'\int_0^{g(x')}|Du_\varepsilon|^2\,dx_n\\&=(n-2)^2\,\varepsilon^{(n-2)/2}\int_{R^{n-1}}dx'\int_0^{g(x')}\frac{|x|^2}{(\varepsilon+|x|^2)^n}\,dx_n\\&=(n-2)^2\int_{R^{n-1}}dx'\int_0^{\varepsilon^{\alpha/2}g(x')}\frac{|x|^2}{(\varepsilon+|x|^2)^n}\,dx_n.\end{aligned} \tag{4.6}$$

We also have

$$K_2(\varepsilon) = \tfrac{1}{2}K_2 - \int_{D(0,\delta)} dx' \int_0^{h(x')} u_\varepsilon^{p+1}\, dx_n + O(\varepsilon^{n/2})$$

$$= \tfrac{1}{2}K_2 - \int_{R^{n-1}} dx' \int_0^{g(x')} u_\varepsilon^{p+1}\, dx_n$$

$$- \int_{D(0,\delta)} dx' \int_{g(x')}^{h(x')} u_\varepsilon^{p+1}\, dx_n + O(\varepsilon^{n/2})$$

$$= \tfrac{1}{2}K_2 - II(\varepsilon) + o(\varepsilon^{\alpha/2}), \tag{4.7}$$

where K_1 and K_2 were defined in (3.5) and (3.10), respectively,

$$II(\varepsilon) = \int_{R^{n-1}} dx' \int_0^{g(x')} \frac{\varepsilon^{n/2}}{(\varepsilon + |x|^2)^n}\, dx_n$$

$$= \int_{R^{n-1}} dx' \int_0^{\varepsilon^{\alpha/2} g(x')} \frac{1}{(1+|x|^2)^n}\, dx_n. \tag{4.8}$$

Observing that

$$\lim_{\varepsilon \to 0} \varepsilon^{-\alpha/2} II(\varepsilon) = d \int_{R^{n-1}} \frac{|x'|^{1+\alpha}}{(1+|x'|^2)^n}\, dx', \tag{4.9}$$

we have $II(\varepsilon) = O(\varepsilon^{\alpha/2})$. Moreover,

$$K_3(\varepsilon) = \int_{\partial\Omega} \alpha(x) u_\varepsilon^2\, ds \leqslant M \int_{\partial\Omega} \varepsilon^{(n-2)/2} \frac{1}{(\varepsilon + |x|^2)^{n-2}}\, ds$$

$$= M \int_{D(0,\delta)} \varepsilon^{(n-2)/2} \frac{1}{(\varepsilon + |x'|^2 + |h(x')|^2)^{n-2}}\, dx' + O(\varepsilon^{(n-2)/2})$$

$$\leqslant M \int_{D(0,\delta)} \varepsilon^{(n-2)/2} \frac{1}{(\varepsilon + |x'|^2)^{n-2}}\, dx' + O(\varepsilon^{(n-2)/2})$$

$$= o(\varepsilon^{\alpha/2}). \tag{4.10}$$

let $J(t_\varepsilon u_\varepsilon = Y_\varepsilon = \sup_{t>0} J(tu_\varepsilon)$. From (4.5), (4.7), and (4.10) it follows that t_ε are uniformly bounded for $\varepsilon > 0$ sufficiently small. Hence

$$Y_\varepsilon \leqslant \sup_{t>0} \left[\frac{1}{2} K_1(\varepsilon) t^2 - \frac{1}{p+1} K_2(\varepsilon) t^{p+1} \right] + o(\varepsilon^{\alpha/2})$$

$$= \frac{1}{n} \left[K_1(\varepsilon) / |K_2(\varepsilon)|^{(n-2)/n} \right]^{n/2} + o(\varepsilon^{\alpha/2}) \qquad \text{as} \quad \varepsilon \to 0.$$

In order to prove (4.4), it suffices to verify

$$K_1(\varepsilon)/|K_2(\varepsilon)|^{(n-2)/n} < 2^{-2/n}S + o(\varepsilon^{\alpha/2}) \tag{4.11}$$

for $\varepsilon > 0$ sufficiently small. By (4.5), (4.7), and $II(\varepsilon) = O(\varepsilon^{\alpha/2})$, (4.11) reduces to

$$\lim_{\varepsilon \to 0} \frac{I(\varepsilon)}{II(\varepsilon)} > \frac{n-2}{n} K_1/K_2. \tag{4.12}$$

From (4.6) and (4.8), we have

$$\begin{aligned}\lim_{\varepsilon \to 0} \frac{I(\varepsilon)}{II(\varepsilon)} &= \lim_{\varepsilon \to 0} \frac{I'(\varepsilon)}{II'(\varepsilon)} \\ &= (n-2)^2 \int_{R^{n-1}} \frac{|x'|^{3+\alpha}}{(1+|x'|^2)^n} dx' \Big/ \int_{R^{n-1}} \frac{|x'|^{1+\alpha}}{(1+|x'|^2)^n} dx' \\ &= (n-2)^2 \int_0^\infty \frac{r^{n+1+\alpha}}{(1+r^2)^n} dr \Big/ \int_0^\infty \frac{r^{n-1+\alpha}}{(1+r^2)^n} dr.\end{aligned}$$

Using (3.18), we obtain

$$\begin{aligned}\lim_{\varepsilon \to 0} \frac{I(\varepsilon)}{II(\varepsilon)} &= (n-2)^2 \frac{n+\alpha}{n-2-\alpha} \\ &> (n-2)^2 = \frac{n-2}{n} K_1/K_2.\end{aligned} \tag{4.13}$$

This completes the proof of Theorem 4.1. ∎

Condition (4.3) seems somewhat strange, but it plays a crucial rule in the proof. On the other hand, we have the following example.

THEOREM 4.2. *There exists a radial solution of*

$$\begin{aligned} -\Delta u &= u^p && \text{in } B(0,1), \\ D_\nu u + \lambda u &= 0 && \text{on } \partial B(0,1), \\ u &> 0 && \text{in } B(0,1)\end{aligned} \tag{4.14}$$

if and only if $\lambda \in (0, n-2)$.

Proof. Suppose $u = u(r)$ satisfies (4.14), $r = |x|$. Integrating the equation in (4.14) we obtain

$$\int_B u^p\, dx = -\int_{\partial B} D_\nu u\, ds = \lambda \int_{\partial B} u\, ds,$$

hence $\lambda > 0$. Next we prove $\lambda < n-2$. Multiplying the equation in (4.14) by $\sum_{j=1}^n x_j u_j$, we have

$$-\int_B u_{ii} x_j u_j \, dx = -\frac{1}{2}\lambda^2 \int_{\partial B} u^2 \, dx - \frac{n-2}{2} \int_B u_i^2 \, dx$$

$$= \int_B u^p x_j u_j \, dx = \frac{n-2}{2n} \int_{\partial B} u^{p+1} \, ds$$

$$-\frac{n-2}{2} \int_B u^{p+1} \, dx.$$

Note that

$$\int_B u_i^2 \, dx + \lambda \int_{\partial B} u^2 \, ds = \int_B u^{p+1} \, dx.$$

We obtain

$$-\frac{1}{2}\lambda^2 \int_{\partial B} u^2 \, ds = \frac{n+2}{2n} \int_{\partial B} u^{p+1} \, ds - \frac{n-2}{2} \lambda \int_{\partial B} u^2 \, ds,$$

that is,

$$\lambda^2 - (n-2)\lambda + \frac{n-2}{n} u^{p-1}(1) = 0.$$

Since $u(1) > 0$, it follows $\lambda < n-2$.

On the other hand, $\forall \lambda \in (0, n-2)$, the function

$$u(x) = C(1 + \mu |x|^2)^{-(n-2)/2}$$

satisfies (4.14), where $\mu = \lambda/(n-2-\lambda)$, $C = (\mu(n-2)n)^{(n-2)/4}$. ∎

Finally we return to the problem (2.1). We have

THEOREM 4.3. *Suppose the hypotheses of Theorem* 4.1, *and* (2.2)-(2.4) *hold. Suppose also that*

$$f(x, u) \geqslant -Au - Bu^q \qquad \forall (x, u) \in \Omega \times [0, \infty)$$

for some A, $B \geqslant 0$ *and* $q \geqslant (0, n/(n-2))$. *Then there exists a solution of* (2.1).

The proof is similar to that of Theorem 3.2, and is omitted here.

We now give a lemma to verify condition (2.11). Its proof is similar to that of Lemma 2.1 in [3] (but some computations in the proofs of Theorem 3.1 and Theorem 4.1 are needed), and is also omitted here.

LEMMA 4.1. *Suppose* $\partial\Omega \in C^2$, (2.2)–(2.4) *hold. Suppose also that there is a function* $f(u)$ *such that*

$$f(x, u) \geqslant f(u) \geqslant 0 \qquad \text{for a.e.} \quad x \in \omega, \text{ and for all } u \geqslant M_0,$$

where ω *is some nonempty open set in* $\bar{\Omega}$ *with* $\omega \cap \partial\Omega \neq \emptyset$, $M_0 > 0$ *is a constant, and the primitive* $F(u) = \int_0^u f(t)\, dt$ *satisfies*

$$\lim_{\varepsilon \to 0} \varepsilon^{(n-1)/2} \int_0^{\varepsilon^{-1/2}} F\left[\left(\frac{\varepsilon^{-1/2}}{1+s^2}\right)^{(n-2)/2}\right] s^{n-1}\, ds = \infty. \tag{4.15}$$

Then (2.11) *holds.*

Consequently we have

COROLLARY 4.1. *Assume that* $f(x, u) = a(x)u + b(x)u^q$, *where* $a(x), b(x)$ *are bounded measurable functions,* $b(x) \geqslant \delta > 0$ *in a neighborhood of* x_0 *for some* $x_0 \in \partial\Omega$, *and* (2.4) *holds; then there is a solution of* (2.1) *for all* $q \in (n/(n-2), (n+2)/(n-2))$.

Proof. We use Theorem 2.1 and Lemma 4.1 with $f(u) = (\delta/2)u^q$. It is easy to check (4.15) when $q > n/(n-2)$. Thus Corollary 4.1 follows. ∎

5. SOME OTHER RESULTS

(1) *Regularity of Solutions*

The solution u of (1.1) lies in $H(\Omega)$. In fact, u belongs to $L^\infty(\Omega)$. We first prove

LEMMA 5.1. *Suppose* $\partial\Omega \in C^1$, $u \in H(\Omega)$ *is a weak solution of*

$$\begin{aligned} -\Delta u &= a(x)u \qquad \text{in } \Omega, \\ D_\gamma u &= \alpha(x)u \qquad \text{on } \partial\Omega, \end{aligned} \tag{5.1}$$

where $a(x) \in L^{n/2}(\Omega)$, $\alpha(x) \in L^\infty(\Omega)$; *then* $u \in L^t(\Omega)$ *for all* $t \geqslant 1$.

Proof. For any fixed $x_0 \in \bar{\Omega}$, let $\eta(x) \geqslant 0$ be a smooth function with $\operatorname{supp} \eta \subset B(x_0, \delta)$ and $\eta(x) = 1$ for $x \in B(x_0, \frac{1}{2}\delta)$, where δ is so small that (with the help of Lemma 2.1)

$$\int_\Omega |Dv|^2\, dx > \frac{1}{4} S\left[\int_\Omega |v|^{p+1}\, dx\right]^{2/(p+1)}, \qquad p = \frac{n+2}{n-2} \tag{5.2}$$

for any $v \in H(\Omega)$ with $\operatorname{supp} v \subset B(x_0, \delta) \cap \bar{\Omega}$.

Let $\beta > 1$ and $N > 0$ be given. Define $G \in C^1([0, \infty))$ by $G(t) = t^\beta$ if $0 \leq t \leq N$ and $G(t)$ is linear if $t > N$. If u is a solution of (5.1), then $G(u)$, $G'(u)$, and $F(u) = \int_0^u |G'|^2 \, dt$ all belong to $H(\Omega)$. Since

$$\int_\Omega [Du \cdot Dv - a(x)uv] \, dx - \int_{\partial\Omega} \alpha(x)uv \, ds = 0 \qquad \forall v \in H(\Omega),$$

let $v = F(u)\eta^2$; we obtain

$$\int_\Omega |D(G\eta)|^2 \, dx \leq C \left[\int_\Omega G^2 \, |D\eta|^2 \, dx + \int_{\partial\Omega} |\alpha(x)| \, G^2\eta^2 \, ds + \int_\Omega |a(x)| \, G^2\eta^2 \, dx \right], \tag{5.3}$$

where C is a constant independent of δ. Let δ be so small that

$$\|a(x)\|_{L^{n/2}(B(x_0,\delta))} \leq S/8C.$$

From (5.2) and by $\operatorname{supp} \eta \subset B(x_0, \delta)$, it follows that

$$\int_\Omega |a(x)| \, G^2\eta^2 \, dx \leq \|a(x)\|_{L^{n/2}(B(x_0,\delta))} \cdot \|G\eta\|^2_{L^{p+1}(\Omega)} \leq \frac{1}{2C} \|D(G\eta)\|^2_{L^2(\Omega)}.$$

Hence (5.3) reads

$$\int_\Omega |D(G\eta)|^2 \, dx \leq C \left[\int_\Omega G^2 \, |D\eta|^2 \, dx + \int_{\partial\Omega} |\alpha(x)| \, G^2\eta^2 \, ds \right]. \tag{5.4}$$

By the Sobolev imbedding $H(\Omega) \subset L^{(2n-2)/(n-2)}(\partial\Omega)$, we may choose $\beta = (n-1)/(n-2) > 1$ in (5.4). Let $N \to \infty$; we obtain $u^\beta \eta \in H(\Omega)$. Since x_0 is arbitrary, it follows that $u^\beta \in H(\Omega)$.

Choose again that $\beta = \beta_k = ((n-1)/(n-2))^k$ in (5.4). Let $N \to \infty$, we get $u^{\beta_k} \in H(\Omega)$, $k = 2, 3, \ldots$ Thus $u \in L^t(\Omega)$ for all $t > 1$. ∎

For our purpose we use Lemma 5.1 with $a(x) = u^{p-1} + u^{-1}f(x, u) \in L^{n/2}(\Omega)$, then the regularity of solutions to (2.1) can be obtained by virtue of the following L^p estimates [1].

Lemma 5.2. *Suppose $\partial\Omega \in C^2$, $f(x) \in L^p(\Omega)$, $\varphi(x) \in W^{1,p}(\Omega)$, $p \in (1, \infty)$. If u is a solution of*

$$-\Delta u = f(x) \qquad \text{in } \Omega,$$
$$D_\gamma u = \varphi(x) \qquad \text{on } \partial\Omega,$$

then

$$\|u\|_{W^{2,p}(\Omega)} \leq C(\|f\|_{L^p} + \|\varphi\|_{W^{1,p}(\Omega)}).$$

From this lemma we see that the solutions of (2.1) belong to $C^{1+\alpha}(\bar{\Omega})$ for any $\alpha \in (0, 1)$ if $\alpha(x) \in W^{1,\infty}(\partial\Omega)$ and $\partial\Omega \in C^2$. We can further improve the smoothness of solutions by means of the Schauder estimates [7]. Consequently by the strong maximum principle it follows that any (weak) solution of (2.1) is positive everywhere in $\bar{\Omega}$.

(2) *Equations with Variable Coefficients*

Let $\Omega \subset R^n$ be a bounded domain with C^1 boundary, $n \geq 3$, and let $Lu = -\sum_{i,j=1}^n D_i(a_{ij}(x) D_j u)$ be a uniformly elliptic operator. We consider the conormal derivative problem

$$\begin{aligned} &Lu = b(x)u^p + f(x, u) && \text{in } \Omega, \\ &Bu = \sum_{i,j=1}^n a_{ij}(x)\gamma_i D_j u + \alpha(x)u = 0 && \text{on } \partial\Omega, \\ &u > 0 && \text{in } \Omega, \end{aligned} \tag{5.5}$$

where $p = (n+2)/(n-2)$, $a_{ij}(x)$, $b(x)$, and $\alpha(x)$ are bounded measurable functions, $\alpha(x) \geq 0$, $b(x) > 0$, and $\gamma = (\gamma_1, ..., \gamma_n)$ is the unit outward normal to $\partial\Omega$. We suppose $f(x, u)$ satisfies (2.2), (2.3), and the first eigenvalue of the following problem is positive:

$$Lu - a(x)u = \lambda u \text{ in } \Omega, \qquad Bu = 0 \text{ on } \partial\Omega.$$

That is

$$\begin{aligned} \lambda_1 = \inf \Big\{ &\int_\Omega [a_{ij}(x) D_i u D_j u - a(x)u^2] \, dx \\ &+ \int_{\partial\Omega} \alpha(x)u^2 \, ds; \int_\Omega u^2 \, dx = 1 \Big\} > 0. \end{aligned} \tag{5.6}$$

The solutions of (5.5) correspond to the nonzero critical points of the functional

$$\begin{aligned} J(u) = &\int_\Omega \left[\frac{1}{2} a_{ij}(x) D_i u D_j u - \frac{1}{p+1} b(x) u_+^{p+1} - F(x, u)\right] dx \\ &+ \frac{1}{2} \int_{\partial\Omega} \alpha(x)u^2 \, ds, \end{aligned} \tag{5.7}$$

where the summation convention is used. Set

$$c = \inf_{\psi \in \Psi} \sup_{t \in (0,1)} J(\psi(t)), \tag{5.8}$$

where $\Psi = \{\psi \in C([0, 1], H(\Omega));\ \psi(0) = 0,\ \psi(1) = \psi_0 \equiv t_0\}$, and the constant t_0 is so large that $J(t\psi_0) \leqslant 0$ for all $t \geqslant 1$. We have

THEOREM 5.1. *Suppose $a_{ij}(x) \in C(\bar{\Omega})$, (2.2), (2.3), and (5.6) hold. If*

$$c < \frac{1}{2n} S^{n/2} \operatorname{ess} \inf_{x \in \Omega} [\det(a_{ij}(x))/|b(x)|^{n-2}]^{1/2}, \tag{5.9}$$

then there exists a solution of (5.5).

The proof is a slight modification of that of Theorem 2.1 and is omitted here. By virtue of this theorem we can easily extend the results of Theorem 3.1 and Theorem 4.1 to the problem (5.5). For convenience we consider the simple case

$$\begin{aligned} -\sum_{i=1}^{n} D_i(a(x)\, D_i u) &= b(x)u^p + f(x)u && \text{in } \Omega, \\ D_\gamma u &= 0 && \text{on } \partial\Omega, \\ u &> 0, && \text{in } \Omega. \end{aligned} \tag{5.10}$$

THEOREM 5.2. *Suppose $\partial\Omega \subset C^2$, $a(x), b(x) \in C^1(\bar{\Omega})$, $a(x) \geqslant a' > 0$, and $f(x) \leqslant 0$, $f(x) \not\equiv 0$. Suppose also that there exists a point $x_0 \in \partial\Omega$ such that the principal curvatures $\alpha_1, ..., \alpha_{n-1}$ of $\partial\Omega$ at x_0 satisfy $\sum_{i=1}^{n-1} \alpha_i > 0$, and*

$$\begin{aligned} &a(x) \geqslant a(x_0), \qquad b(x) \leqslant b(x_0) \qquad \text{for all} \quad x \in \Omega, \\ &a(x) = a(x_0) + o(|x - x_0|), \\ &b(x) = b(x_0) + o(|x - x_0|) \qquad \text{as} \quad x \to x_0. \end{aligned} \tag{5.11}$$

Then there is a solution of (5.10).

Proof. Without loss of generality we may suppose x_0 is the origin and the x_n-axis is the inner normal of $\partial\Omega$ there. After stretching $u(x) = kv(x)$ for suitable constant k, we may also suppose $a(x_0) = b(x_0) = 1$. Let $u_\varepsilon = \varepsilon^{(n-2)/4}(\varepsilon + |x|^2)^{-(n-2)/2}$; by virtue of Theorem 5.1, it suffices to verify

$$Y_\varepsilon = \sup_{t > 0} J(tu_\varepsilon) < \frac{1}{2n} S^{n/2} \qquad \text{for } \varepsilon > 0 \text{ small.} \tag{5.12}$$

Set

$$K_1(\varepsilon)=\int_\Omega a(x)\,|Du_\varepsilon|^2\,dx, \qquad K_2(\varepsilon)=\int_\Omega b(x)u_\varepsilon^{p+1}\,dx.$$

By (5.11) we have

$$K_1(\varepsilon)=\int_\Omega |Du_\varepsilon|^2\,dx+o(\varepsilon^{1/2}), \qquad K_2(\varepsilon)=\int_\Omega u_\varepsilon^{p+1}\,dx+o(\varepsilon^{1/2}).$$

From the proof of Theorem 3.1 we thus have

$$\begin{aligned} K_1(\varepsilon)&=\tfrac{1}{2}K_1-I(\varepsilon)+o(\varepsilon^{1/2}),\\ K_2(\varepsilon)&=\tfrac{1}{2}K_2-II(\varepsilon)+o(\varepsilon^{1/2}) \qquad \text{if} \quad n\geqslant 4, \end{aligned}$$

or

$$\begin{aligned} K_1(\varepsilon)&\leqslant\tfrac{1}{2}K_1-C_0\varepsilon^{1/2}\,|\log\varepsilon|+O(\varepsilon^{1/2}),\\ K_2(\varepsilon)&\geqslant\tfrac{1}{2}K_2-O(\varepsilon^{1/2}) \qquad \text{if} \quad n=3, \end{aligned}$$

where K_1, K_2, $I(\varepsilon)$, and $II(\varepsilon)$ were defined in (3.5), (3.10), (3.6), and (3.11), respectively. Moreover, from (3.14) we have

$$K_3(\varepsilon)=\int_\Omega f(x)u_\varepsilon^2\,dx=\begin{cases} o(\varepsilon^{1/2}) & n\geqslant 4,\\ O(\varepsilon^{1/2}) & n=3.\end{cases}$$

Therefore similarly to (3.4) we obtain (5.12). ∎

We conclude this paper with the following example.

EXAMPLE 5.1. We give positive functions $a(x), b(x)\in C^1(\bar{\Omega})$ such that the problem

$$\begin{aligned} -\sum_{i=1}^{n} D_i(a(x)\,D_iu)&=b(x)u^p && \text{in } \Omega,\\ D_\gamma u+\alpha(x)u&=0 && \text{on } \partial\Omega, \qquad (5.13)\\ u&>0, && \text{in } \Omega. \end{aligned}$$

possesses a solution for any $\alpha(x)\in L^\infty(\Omega)$, $\alpha(x)\geqslant 0$, and $\alpha(x)\not\equiv 0$.

Indeed, we may suppose $B(0,2)\subset\Omega$. Choose $a(x)$ smooth and radially decreasing with $a(x)=1$ if $|x|<\frac{1}{2}$, $a(x)<1/N$ if $|x|\geqslant 1$. And choose $b(x)=|a(x)|^{n/(n-2)}$. Let $u_0(x)=1$ if $|x|\leqslant 1$, and $u_0(x)=\max(0,2-|x|)$ if $|x|>1$. Then $J(tu_0)$ is independent of $\alpha(x)$. Set $Y=\sup_{t>0}J(tu_0)$; simple computations show that $Y \to 0$ as $N \to \infty$. Hence we can fix N such that $Y < (1/2n) S^{n/2}$. Using Theorem 5.1 we therefore obtain a solution of (5.13).

References

1. S. Agmon, A. Douglis, and L. Nirenberg, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions, I, *Comm. Pure Appl. Math.* **12** (1959), 623–727.
2. H. Brezis and E. Lieb, A relation between pointwise convergence of functions and convergence of functionals, *Proc. Amer. Math. Soc.* **88** (1983), 486–490.
3. H. Brezis and L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents, *Comm. Pure Appl. Math.* **36** (1983), 436–477.
4. B. Gidas, W. M. Ni, and L. Nirenberg, Symmetry and related properties via the maximum principle, *Comm. Math. Phys.* **68** (1979), 209–243.
5. H. Egnell, Semilinear elliptic equations involving critical Sobolev exponents, *Arch. Rational Mech. Anal.* **104** (1988), 27-56.
6. H. B. Keller and D. S. Cohen, Some positone problems suggested by nonlinear heat generations, *J. Math. Mech.* **16** (1967), 1361–1376.
7. G. M. Lieberman, Intermediate Schauder estimates for oblique derivative problems, *Arch. Rational Mech. Anal.* **93** (1986), 129–136.
8. C. S. Lin, W. M. Ni, and I. Takagi, Large amplitude stationary solutions to a chemotaxis system, *J. Differential Equations* **72** (1988), 1–27.
9. W. M. Ni, "Recent Progress in Semilinear Elliptic Equations," Math. Report, Univ. of Minnessota, pp. 88–117.
10. W. M. Ni and I. Takagi, On the Neumann problem for some semilinear elliptic equations and systems of activator- inhibitor type, *Trans. Amer. Math. Soc.* **297** (1986), 351–368.
11. G. Talenti, Best constants in Sobolev inequality, *Ann. Mat. Pura Appl.* **110** (1976), 353 372.