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On the least squares estimator in a nearly unstable sequence of stationary spatial AR models

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ABSTRACT

A nearly unstable sequence of stationary spatial autoregressive processes is investigated, when the sum of the absolute values of the autoregressive coefficients tends to one. It is shown that after an appropriate normalization the least squares estimator for these coefficients has a normal limit distribution. If none of the parameters equals zero then the typical rate of convergence is n.

1. Introduction

Spatial autoregressive models have a great importance in many different fields of science such as geography, geology, biology and agriculture, see e.g. [1] for a detailed discussion, where the authors considered a general unilateral model having the form

$$X_{k,\ell} = \sum_{i=0}^{p_1} \sum_{j=0}^{p_2} \alpha_{i,j} X_{k-i,\ell-j} + \varepsilon_{k,\ell}, \qquad \alpha_{0,0} = 0. \tag{1.1}$$

A particular case of the model (1.1) is the so-called doubly geometric spatial autoregressive model

$$X_{k,\ell} = \alpha X_{k-1,\ell} + \beta X_{k,\ell-1} - \alpha\beta X_{k-1,\ell-1} + \varepsilon_{k,\ell},$$

introduced by Martin [12]. In fact, this is the simplest spatial model, since its nice product structure ensures that it can be considered as some kind of combination of two autoregressive processes on the line, and several properties can be derived by the analogy of one-dimensional autoregressive processes. The doubly geometric model was the first one for which the nearly unstability has been studied. Bhattacharyya et al. [8] showed that in the case when a sequence of stable models with $\alpha_n \to 1$, $\beta_n \to 1$ was considered, in contrast to the AR(1) model, the sequence of Gauss–Newton estimators $(\widehat{\alpha}_n, \widehat{\beta}_n)$ of (α_n, β_n) were asymptotically normal, namely,

$$n^{3/2} \begin{pmatrix} \widehat{\alpha}_n - \alpha_n \\ \widehat{\beta}_n - \beta_n \end{pmatrix} \xrightarrow{\mathcal{D}} \mathcal{N}(0, \Sigma)$$

with some covariance matrix Σ.

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The doubly geometric model has several applications. Jain [11] used it in the study of image processing, Martin [13], Cullis and Gleeson [10], Basu and Reinsel [2] in agricultural trials, while Tjøstheim [16] in digital filtering.

In the present paper we study another special case of the model (1.1). We consider the spatial autoregressive process $\{X_{k,\ell} : k, \ell \in \mathbb{Z}\}$ which is a solution of the spatial stochastic difference equation

$$X_{k,\ell} = \alpha X_{k-1,\ell} + \beta X_{k,\ell-1} + \varepsilon_{k,\ell} \tag{1.2}$$

with parameters $(\alpha, \beta) \in \mathbb{R}^2$. This model is stable (i.e. has a stationary solution) in case $|\alpha| + |\beta| < 1$ (see [1]), and unstable if $|\alpha|+|\beta| = 1$. In a recent paper Paulauskas [14] determined the exact asymptotic behaviour of the variances of a nonstationary solution of (1.2) with $X_{k,\ell} = 0$ for $k + \ell \le 0$, while Baran et al. [5] in the same model clarified the asymptotic properties of the least squares estimator (LSE) of (α, β) both in stable and unstable cases.

We remark, that in case $|\alpha| + |\beta| < 1$, if $\{\varepsilon_{k,\ell} : k, \ell \in \mathbb{Z}\}$ are independent and identically distributed random variables, a stationary solution can be given by

$$X_{k,\ell} = \sum_{(i,j)\in U_{k,\ell}} \binom{k+\ell-i-j}{k-i} \alpha^{k-i}\beta^{\ell-j}\varepsilon_{i,j}, \tag{1.3}$$

where $U_{k,\ell} := \{(i, j) \in \mathbb{Z}^2 : i \le k \text{ and } j \le \ell\}$ and the convergence of the series is understood in L_2-sense.

We are interested in the asymptotic behaviour of the stationary solution of (1.2) in the case when the parameters approach the boundary $|\alpha| + |\beta| = 1$. In order to determine the appropriate speed of parameters one may use the idea of Chan and Wei [9] and consider the order of

$$\mathbb{I}_n := \mathsf{E}\left(\sum_{(k,\ell)\in H_n} \begin{pmatrix} \left(X_{k-1,\ell}\right)^2 & X_{k-1,\ell}X_{k,\ell-1} \\ X_{k-1,\ell}X_{k,\ell-1} & \left(X_{k,\ell-1}\right)^2 \end{pmatrix}\right)$$

that is exactly the observed Fisher information matrix about (α, β) when the innovations $\varepsilon_{k,\ell}$ are normally distributed and the process is observed on a set $H_n \subset \mathbb{Z}^2$, $n \in \mathbb{N}$. From Theorem 1.1 of [5] we obtain that

$$\mathbb{I}_n \sim \begin{cases} n^2\sigma^2_{\alpha,\beta}\varGamma_{\alpha,\beta}, & \text{if } |\alpha| + |\beta| < 1, \\ n^{5/2}\sigma^2_{\alpha}\varPsi_{\alpha,\beta}, & \text{if } |\alpha| + |\beta| = 1,\ 0 < |\alpha| < 1, \\ n^3(4/3)\mathcal{I}, & \text{if } |\alpha| + |\beta| = 1,\ |\alpha| \in \{0, 1\}, \end{cases}$$

where

$$\varGamma_{\alpha,\beta} := 2\begin{pmatrix} 1 & -\varrho_{\alpha,\beta} \\ -\varrho_{\alpha,\beta} & 1 \end{pmatrix}, \qquad \varPsi_{\alpha,\beta} := \begin{pmatrix} 1 & \operatorname{sign}(\alpha\beta) \\ \operatorname{sign}(\alpha\beta) & 1 \end{pmatrix},$$

$\mathcal{I}$ denotes the two-by-two unit matrix and

$$\sigma^2_{\alpha,\beta} := ((1+\alpha+\beta)(1+\alpha-\beta)(1-\alpha+\beta)(1-\alpha-\beta))^{-1/2},$$

$$\varrho_{\alpha,\beta} := \begin{cases} \dfrac{(1-\alpha^2-\beta^2)\sigma^2_{\alpha,\beta} - 1}{2\alpha\beta\sigma^2_{\alpha,\beta}}, & \text{if } \alpha\beta \ne 0, \\ 0 & \text{otherwise}, \end{cases}$$

$$\sigma^2_{\alpha} := \frac{2^{9/2}}{15\sqrt{\pi\,|\alpha|(1-|\alpha|)}}.$$

Now, let $\alpha_n := \alpha - \gamma/a_n$, $\beta_n := \beta - \delta/a_n$, $|\alpha| + |\beta| = 1$, $|\alpha_n| + |\beta_n| < 1$. As nonstationary behaviour of $X_{k,\ell}$ becomes dominant when (α_n, β_n) is near the border, a reasonable choice for the sequence a_n should retain the order of $\mathbb{I}_n$ to be $n^{5/2}$ if $0 < |\alpha| < 1$ and n^3 if $|\alpha| \in \{0, 1\}$. Since we have $\sigma^2_{\alpha_n,\beta_n} \sim a_n^{1/2}$ for $0 < |\alpha| < 1$ and $\sigma^2_{\alpha_n,\beta_n} \sim a_n$ for $|\alpha| \in \{0, 1\}$ while $\varrho_{\alpha_n,\beta_n} \sim \mathit{const}$ in both cases, the above consideration yields $a_n = n$.

In what follows we consider a nearly unstable sequence of stationary processes, i.e. for each $n \in \mathbb{N}$, we take a stationary solution $\{X^{(n)}_{k,\ell} : k, \ell \in \mathbb{Z}\}$ of Eq. (1.2) with parameters (α_n, β_n) defined as

$$\alpha_n := \alpha - \frac{\gamma_n}{n}, \qquad \beta_n := \beta - \frac{\delta_n}{n}, \qquad |\alpha_n| + |\beta_n| < 1, \tag{1.4}$$

where $0 \le |\alpha| \le 1$, $|\beta| = 1 - |\alpha|$ and $\gamma_n \to \gamma$, $\delta_n \to \delta$ as $n \to \infty$, $(\gamma, \delta) \in \mathbb{R}^2$. We remark that in an earlier paper [3] the authors considered a similar sequence of stationary processes where the autoregressive parameters were equal and their sum converged to 1.

For a set $H \subset \mathbb{Z}^2$, the LSE $(\widehat{\alpha}^{(n)}_H, \widehat{\beta}^{(n)}_H)$ of (α_n, β_n) based on the observations $\{X^{(n)}_{k,\ell} : (k, \ell) \in H\}$ has the form

$$\begin{pmatrix} \widehat{\alpha}^{(n)}_H \\ \widehat{\beta}^{(n)}_H \end{pmatrix} = \left(\sum_{(k,\ell)\in H} \begin{pmatrix} \left(X^{(n)}_{k-1,\ell}\right)^2 & X^{(n)}_{k-1,\ell}X^{(n)}_{k,\ell-1} \\ X^{(n)}_{k-1,\ell}X^{(n)}_{k,\ell-1} & \left(X^{(n)}_{k,\ell-1}\right)^2 \end{pmatrix}\right)^{-1} \sum_{(k,\ell)\in H} \begin{pmatrix} X^{(n)}_{k-1,\ell}X^{(n)}_{k,\ell} \\ X^{(n)}_{k,\ell-1}X^{(n)}_{k,\ell} \end{pmatrix}.$$

Consider the triangles $T_{k,\ell} := \{(i, j) \in \mathbb{Z}^2 : i + j \ge 1,\ i \le k \text{ and } j \le \ell\}$ for $k, \ell \in \mathbb{Z}$. Note that $T_{k,\ell} = \emptyset$ if $k + \ell \le 0$.

Theorem 1.1. *For each $n \in \mathbb{N}$, let $\{X^{(n)}_{k,\ell} : k, \ell \in \mathbb{N}\}$ be a stationary solution of Eq.* (1.2) *with parameters (α_n, β_n) given by* (1.4), *and with independent and identically distributed random variables $\{\varepsilon^{(n)}_{k,\ell} : k, \ell \in \mathbb{Z}\}$ such that $\mathbb{E}\varepsilon^{(n)}_{0,0} = 0$, $\operatorname{Var}\varepsilon^{(n)}_{0,0} = 1$ and $M := \sup_{n\in\mathbb{N}} \mathbb{E}\left|\varepsilon^{(n)}_{0,0}\right|^8 < \infty$. Let (k_n) and (ℓ_n) be sequences of integers such that $k_n + \ell_n \to \infty$ as $n \to \infty$.*

If $0 < |\alpha| < 1$, $|\beta| = 1 - |\alpha|$ and

$$\lim_{n\to\infty} (k_n + \ell_n) n^{-1/2} \left(|\gamma_n| + |\delta_n|\right)^{1/2} = \infty \tag{1.5}$$

holds then

$$(k_n + \ell_n) \begin{pmatrix} \widehat{\alpha}_{T_{k_n,\ell_n}} - \alpha_n \\ \widehat{\beta}_{T_{k_n,\ell_n}} - \beta_n \end{pmatrix} \xrightarrow{\mathcal{D}} \mathcal{N}_2\left(0, |\alpha||\beta|\bar{\Psi}_{\alpha,\beta}\right)$$

as $n \to \infty$, where $\bar{\Psi}_{\alpha,\beta}$ denotes the adjoint matrix of $\Psi_{\alpha,\beta}$.

If $|\alpha| \in \{0, 1\}$, $|\beta| = 1 - |\alpha|$ and

$$\lim_{n\to\infty} (k_n + \ell_n) n^{-1} \left|\gamma_n^2 - \delta_n^2\right|^{1/2} = \infty \tag{1.6}$$

holds then let

$$[-\infty, \infty] \ni \omega := \lim_{n\to\infty} \omega_n, \qquad \omega_n := \alpha\frac{\gamma_n}{\delta_n} + \beta\frac{\delta_n}{\gamma_n}.$$

If $|\omega| > 1$ then

$$(k_n + \ell_n) n^{1/2} \left|\gamma_n^2 - \delta_n^2\right|^{-1/4} \begin{pmatrix} \widehat{\alpha}_{T_{k_n,\ell_n}} - \alpha_n \\ \widehat{\beta}_{T_{k_n,\ell_n}} - \beta_n \end{pmatrix} \xrightarrow{\mathcal{D}} \mathcal{N}_2\left(0, \Theta^{-1}_{\alpha,\beta,\omega}\right)$$

as $n \to \infty$, where

$$\Theta_{\alpha,\beta,\omega} := \frac{1}{4}\begin{pmatrix} 1 & \theta(\alpha,\beta,\omega) \\ \theta(\alpha,\beta,\omega) & 1 \end{pmatrix}, \qquad \theta(\alpha,\beta,\omega) := \begin{cases} \dfrac{-(\alpha+\beta)\operatorname{sign}(\omega)}{|\omega| + \sqrt{\omega^2 - 1}} & \text{if } |\omega| < \infty, \\ 0 & \text{if } |\omega| = \infty. \end{cases}$$

Remark 1.2. Obviously, $|\omega_n| > 1$, so $|\omega| \geq 1$. Condition $|\omega| > 1$ in Theorem 1.1 is needed to ensure the regularity of $\Theta_{\alpha,\beta,\omega}$. However, this condition can be omitted and using similar arguments as in the proof of the second statement of Theorem 1.1, one can easily show that if $|\alpha| \in \{0, 1\}$, $|\beta| = 1 - |\alpha|$ and (1.6) holds then

$$(k_n + \ell_n) n^{1/2} \left|\gamma_n^2 - \delta_n^2\right|^{-1/4} \Theta^{1/2}_{\alpha,\beta,\omega_n} \begin{pmatrix} \widehat{\alpha}_{T_{k_n,\ell_n}} - \alpha_n \\ \widehat{\beta}_{T_{k_n,\ell_n}} - \beta_n \end{pmatrix} \xrightarrow{\mathcal{D}} \mathcal{N}_2\left(0, \mathcal{I}\right),$$

where $\Theta^{1/2}_{\alpha,\beta,\omega_n}$ denotes the symmetric positive semidefinite square root of $\Theta_{\alpha,\beta,\omega_n}$.

Remark 1.3. Theorem 1.1 shows that in the typical case $k_n = \ell_n = n$ and $\gamma_n = \gamma \neq 0$, $\delta_n = \delta \neq 0$ if $0 < |\alpha| < \infty$, $|\beta| = 1 - |\alpha|$ then the rate of convergence is n.

We may suppose that $(k_n + \ell_n)$ is monotone increasing. Observe, that $\left(\widehat{\alpha}^{(n)}_{T_{k_n,\ell_n}}, \widehat{\beta}^{(n)}_{T_{k_n,\ell_n}}\right)$ and $\left(\widehat{\alpha}^{(n)}_{T_{\widetilde{k}_n,\widetilde{\ell}_n}}, \widehat{\beta}^{(n)}_{T_{\widetilde{k}_n,\widetilde{\ell}_n}}\right)$ have the same distribution, where $\widetilde{k}_n := [(k_n + \ell_n)/2]$ and $\widetilde{\ell}_n := [(k_n + \ell_n + 1)/2]$. As $\widetilde{k}_n + \widetilde{\ell}_n = k_n + \ell_n$, in Theorem 1.1 we may substitute $(\widetilde{k}_n, \widetilde{\ell}_n)$ for (k_n, ℓ_n). The sequence $(\widetilde{k}_n, \widetilde{\ell}_n)$ can be embedded into the sequence (k'_n, ℓ'_n), where $k'_n := [n/2]$ and $\ell'_n := [(n+1)/2]$, namely, $k'_{q_n} = \widetilde{k}_n$ and $\ell'_{q_n} = \widetilde{\ell}_n$ with $q_n := \widetilde{k}_n + \widetilde{\ell}_n$. Clearly $k'_n + \ell'_n = n$. Consider the sequence (r_n) defined by $r_n := k$ for $q_k \leq n < q_{k+1}$. Then $r_{q_n} = n$, and conditions (1.5) and (1.6) can be replaced by

$$\lim_{n\to\infty} n r_n^{-1/2} \left(|\gamma_{r_n}| + |\delta_{r_n}|\right)^{1/2} = \infty \tag{1.7}$$

and

$$\lim_{n\to\infty} n r_n^{-1} \left|\gamma_{r_n}^2 - \delta_{r_n}^2\right|^{1/2} = \infty, \tag{1.8}$$

respectively.

Thus, to prove Theorem 1.1 it suffices to show that if $0 < |\alpha| < 1$, $|\beta| = 1 - |\alpha|$ and (1.7) holds then

$$n \begin{pmatrix} \widehat{\alpha}_{T_{[n/2],[(n+1)/2]}} - \alpha_{r_n} \\ \widehat{\beta}_{T_{[n/2],[(n+1)/2]}} - \beta_{r_n} \end{pmatrix} \xrightarrow{\mathcal{D}} \mathcal{N}_2\left(0, |\alpha||\beta|\bar{\Psi}_{\alpha,\beta}\right),$$

while in the case $|\alpha| \in \{0, 1\}$, $|\beta| = 1 - |\alpha|$, $|\omega| > 1$ and (1.8) holds we have

$$n r_n^{1/2} \left|\gamma_{r_n}^2 - \delta_{r_n}^2\right|^{-1/4} \begin{pmatrix} \widehat{\alpha}_{T_{[n/2],[(n+1)/2]}} - \alpha_{r_n} \\ \widehat{\beta}_{T_{[n/2],[(n+1)/2]}} - \beta_{r_n} \end{pmatrix} \xrightarrow{\mathcal{D}} \mathcal{N}_2\left(0, \Theta_{\alpha,\beta,\omega}^{-1}\right).$$

We remark that conditions (1.5) and (1.7) are exactly the same as conditions (4) and (5) of [3], respectively.

To simplify notation we assume that $k_n = [n/2]$, $\ell_n = [(n+1)/2]$ and (r_n) is a monotone increasing sequence of positive integers. One can write

$$\begin{pmatrix} \widehat{\alpha}_{T_{k_n,\ell_n}} - \alpha_{r_n} \\ \widehat{\beta}_{T_{k_n,\ell_n}} - \beta_{r_n} \end{pmatrix} = B_n^{-1} A_n,$$

with

$$A_n := \sum_{(k,\ell)\in T_{k_n,\ell_n}} \begin{pmatrix} X_{k-1,\ell}^{(r_n)} \varepsilon_{k,\ell}^{(r_n)} \\ X_{k,\ell-1}^{(r_n)} \varepsilon_{k,\ell}^{(r_n)} \end{pmatrix}, \qquad B_n := \sum_{(k,\ell)\in T_{k_n,\ell_n}} \begin{pmatrix} \left(X_{k-1,\ell}^{(r_n)}\right)^2 & X_{k-1,\ell}^{(r_n)} X_{k,\ell-1}^{(r_n)} \\ X_{k-1,\ell}^{(r_n)} X_{k,\ell-1}^{(r_n)} & \left(X_{k,\ell-1}^{(r_n)}\right)^2 \end{pmatrix}.$$

Concerning the asymptotic behaviour of the random vector A_n and random matrix B_n we can formulate the following two propositions.

Proposition 1.4. *If* $0 < |\alpha| < 1$, $|\beta| = 1 - |\alpha|$ *and* (1.7) *holds then*

$$n^{-2} r_n^{-1/2} \left(|\gamma_{r_n}| + |\delta_{r_n}|\right)^{1/2} B_n \xrightarrow{L_2} (32|\alpha||\beta|)^{-1/2} \Psi_{\alpha,\beta} \quad \text{as } n \to \infty.$$

If $|\alpha| \in \{0, 1\}$, $|\beta| = 1 - |\alpha|$ *and* (1.8) *holds then*

$$n^{-2} r_n^{-1} \left|\gamma_{r_n}^2 - \delta_{r_n}^2\right|^{1/2} B_n \xrightarrow{L_2} \Theta_{\alpha,\beta,\omega}$$

as $n \to \infty$*, where*

$$\omega := \lim_{n\to\infty} \omega_{r_n}, \qquad \omega_{r_n} := \alpha \frac{\gamma_{r_n}}{\delta_{r_n}} + \beta \frac{\delta_{r_n}}{\gamma_{r_n}}. \tag{1.9}$$

Proposition 1.5. *If* $0 < |\alpha| < 1$, $|\beta| = 1 - |\alpha|$ *and* (1.7) *holds then*

$$n^{-1} r_n^{-1/4} \left(|\gamma_{r_n}| + |\delta_{r_n}|\right)^{1/4} A_n \xrightarrow{\mathcal{D}} \mathcal{N}_2\left(0, (32|\alpha||\beta|)^{-1/2} \Psi_{\alpha,\beta}\right) \quad \text{as } n \to \infty.$$

If $|\alpha| \in \{0, 1\}$, $|\beta| = 1 - |\alpha|$ *and* (1.8) *holds then*

$$n^{-1} r_n^{-1/2} \left|\gamma_{r_n}^2 - \delta_{r_n}^2\right|^{1/4} A_n \xrightarrow{\mathcal{D}} \mathcal{N}_2\left(0, \Theta_{\alpha,\beta,\omega}\right) \quad \text{as } n \to \infty.$$

In case $|\alpha| \in \{0, 1\}$, $|\beta| = 1 - |\alpha|$, and $|\omega| \neq 1$, $\Theta_{\alpha,\beta,\omega}$ is a regular matrix, so Propositions 1.4 and 1.5 imply the corresponding statement of Theorem 1.1. In the case $0 < |\alpha| < 1$, $|\beta| = 1 - |\alpha|$ we have $B_n^{-1} = \bar{B}_n / \det B_n$, and in this situation the statement of Theorem 1.1 is a consequence of the following propositions.

Proposition 1.6. *If* $0 < |\alpha| < 1$, $|\beta| = 1 - |\alpha|$ *and* (1.7) *holds then*

$$n^{-4} r_n^{-1/2} \left(|\gamma_{r_n}| + |\delta_{r_n}|\right)^{1/2} \det B_n \xrightarrow{L_2} 2\,(8|\alpha||\beta|)^{-3/2} \quad \text{as } n \to \infty.$$

Proposition 1.7. *If* $0 < |\alpha| < 1$, $|\beta| = 1 - |\alpha|$ *and* (1.7) *holds then*

$$n^{-3} r_n^{-1/2} \left(|\gamma_{r_n}| + |\delta_{r_n}|\right)^{1/2} \bar{B}_n A_n \xrightarrow{\mathcal{D}} \mathcal{N}_2\left(0, \left(2(8\alpha\beta)^2\right)^{-1} \bar{\Psi}_{\alpha,\beta}\right) \quad \text{as } n \to \infty.$$

Obviously, in the case $0 \le |\alpha| \le 1$, $|\beta| = 1 - |\alpha|$ if n is large enough, the corresponding sequences α_{r_n} and β_{r_n} have the same signs as α and β, respectively. Hence, similarly to [5], it suffices to prove Propositions 1.6 and 1.7 for $0 < \alpha, \beta < 1$, $\alpha + \beta = 1$.

2. Covariance structure

Let $\{X_{k,\ell} : k, \ell \in \mathbb{Z}\}$ be a stationary solution of Eq. (1.2) with parameters (α, β), $|\alpha| + |\beta| < 1$. Clearly $\mathsf{Cov}(X_{i_1,j_1}, X_{i_2,j_2}) = \mathsf{Cov}(X_{i_1-i_2,j_1-j_2}, X_{0,0})$ for all $i_1, j_1, i_2, j_2 \in \mathbb{Z}$. Let $R_{k,\ell} := \mathsf{Cov}(X_{k,\ell}, X_{0,0})$ for $k, \ell \in \mathbb{Z}$. The following lemma is a natural generalization of Lemma 4 of [3] (see also [1]).

Lemma 2.1. *Let $\alpha \neq 0$ and $\beta \neq 0$. If $k, \ell \in \mathbb{Z}$ with $k \cdot \ell \leq 0$ then*

$$R_{k,\ell} = \sigma_{\alpha,\beta}^2 \left(\frac{1+\alpha^2-\beta^2-\sigma_{\alpha,\beta}^{-2}}{2\alpha} \right)^{|k|} \left(\frac{2\beta}{1+\beta^2-\alpha^2+\sigma_{\alpha,\beta}^{-2}} \right)^{|\ell|}. \tag{2.1}$$

If $k, \ell \in \mathbb{Z}$ with $k \cdot \ell \geq 0$ then

$$R_{k,\ell} = R_{0,|k-\ell|} - \sum_{i=0}^{|k|\wedge|\ell|-1} \binom{|k-\ell|+2i}{i} \alpha^i \beta^{|k-\ell|+i}. \tag{2.2}$$

Remark 2.2. If $\alpha > 0$ and $\beta > 0$ then $R_{k,\ell} \geq 0$. If $\alpha < 0$ or $\beta < 0$ we have

$$0 \leq |R_{k,\ell}| \leq \widetilde{R}_{k,\ell} := \mathsf{Cov}(\widetilde{X}_{k,\ell}, \widetilde{X}_{0,0}), \quad k, \ell \in \mathbb{Z},$$

where $\{\widetilde{X}_{k,\ell} : k, \ell \in \mathbb{Z}\}$ is a stationary solution of Eq. (1.2) with parameters $(|\alpha|, |\beta|)$.

Besides representations (2.1) and (2.2) one can express the covariances as special cases of Appell's hypergeometric series $F_4(a, b, c, d; x, y)$ defined by

$$F_4(a, b, c, d; x, y) := \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}}{(c)_m (d)_n m! n!} x^m y^n, \quad \sqrt{|x|} + \sqrt{|y|} < 1,$$

where $a, b, c, d \in \mathbb{N}$ and $(a)_n := a(a+1)\ldots(a+n-1)$ [7].

Lemma 2.3. *Let $\alpha \neq 0$ and $\beta \neq 0$. If $k, \ell \in \mathbb{Z}$ with $k \cdot \ell \leq 0$ then*

$$R_{k,\ell} = \alpha^{|k|}\beta^{|\ell|} F_4\left(|k|+1, |\ell|+1, |k|+1, |\ell|+1; \alpha^2, \beta^2\right). \tag{2.3}$$

If $k, \ell \in \mathbb{Z}$ with $k \cdot \ell \geq 0$ then

$$R_{k,\ell} = \alpha^{|k|}\beta^{|\ell|} \binom{|k|+|\ell|}{|k|} F_4\left(|k|+|\ell|+1, 1, |k|+1, |\ell|+1; \alpha^2, \beta^2\right).$$

Moreover, in this case we have

$$R_{k,\ell} = (\mathrm{sign}(\alpha))^{|k|} (\mathrm{sign}(\beta))^{|\ell|} \sum_{i=0}^{\infty} (|\alpha|+|\beta|)^{|k|+|\ell|+2i} \, \mathsf{P}\left(S_{i,|k|+|\ell|+i}^{(\nu)} = |\ell|+i\right), \tag{2.4}$$

where $S_{n,m}^{(\nu)} := S_n^{(\nu)} + S_m^{(1-\nu)}$, $\nu := |\alpha|/(|\alpha|+|\beta|)$ and $S_n^{(\nu)}$ and $S_m^{(1-\nu)}$ are independent binomial random variables with parameters (n, ν) and $(m, 1-\nu)$, respectively.

Proof. The statements directly follow from representation (1.3) and from the independence of the error terms $\varepsilon_{i,j}$. $\square$

We remark, that as

$$F_4\left(a, b, a, b; \frac{-x}{(1-x)(1-y)}, \frac{-y}{(1-x)(1-y)}\right) = \frac{(1-x)^b(1-y)^a}{1-xy},$$

representation (2.1) directly follows from (2.3).

Proposition 2.4. *If $\alpha\beta > 0$, $|\alpha| + |\beta| < 1$ then there exists a universal positive constant K such that*

$$\left| R_{k-1,\ell+1} - R_{k,\ell} \right| \leq \frac{K}{(\alpha\beta)^{3/2}}, \quad k, \ell \in \mathbb{Z}.$$

Proof. Without loss of generality we may assume $\alpha > 0$ and $\beta > 0$.

Suppose $k > 0$, $\ell \geq 0$, so $(k-1)(\ell+1) \geq 0$ and $k \cdot \ell \geq 0$. Using notations introduced in Lemma 2.3 with the help of (2.4) we obtain

$$R_{k-1,\ell+1} - R_{k,\ell} = \sum_{i=0}^{\infty} (\alpha+\beta)^{k+\ell+2i} \Delta_{k,\ell,i}(\nu), \tag{2.5}$$

where

$$\Delta_{i,k,\ell}(\nu) := \mathsf{P}\left(S_{i,k+\ell+i}^{(\nu)} = \ell+i+1\right) - \mathsf{P}\left(S_{i,k+\ell+i}^{(\nu)} = \ell+i\right).$$

According to Theorem 2.6 of [5] $\Delta_{i,k,\ell}(\nu)$ can be approximated by

$$\widetilde{\Delta}_{i,k,\ell}(\nu) := \frac{1}{(2\pi\nu(1-\nu)(k+\ell+2i))^{1/2}}\left(\exp\left\{-\frac{(\nu\ell-(1-\nu)k+1)^2}{2\nu(1-\nu)(k+\ell+2i)}\right\} - \exp\left\{-\frac{(\nu\ell-(1-\nu)k)^2}{2\nu(1-\nu)(k+\ell+2i)}\right\}\right)$$

where

$$\left|\widetilde{\Delta}_{i,k,\ell}(\nu) - \Delta_{i,k,\ell}(\nu)\right| \le \frac{\widetilde{C}}{(\nu(1-\nu)(k+\ell+2i))^{3/2}}$$

with some positive constant $\widetilde{C}$. Thus, if on the right-hand side of (2.5) we replace $\Delta_{i,k,\ell}(\nu)$ with $\widetilde{\Delta}_{i,k,\ell}(\nu)$, the error of the approximation is

$$\sum_{i=0}^{\infty}(\alpha+\beta)^{k+\ell+2i}\left|\widetilde{\Delta}_{i,k,\ell}(\nu) - \Delta_{i,k,\ell}(\nu)\right| \le \frac{\widetilde{C}}{(\nu(1-\nu))^{3/2}}\zeta(3/2) \le \frac{C}{(\alpha\beta)^{3/2}},$$

where $\zeta(x)$ denotes Riemann's zeta function.

To find an upper bound for the approximating sum consider first the case $\nu\ell-(1-\nu)k \ge 0$. In this case

$$\sum_{i=0}^{\infty}(\alpha+\beta)^{k+\ell+2i}\left|\widetilde{\Delta}_{i,k,\ell}(\nu)\right| \le \sum_{i=0}^{\infty}\frac{2(\nu\ell-(1-\nu)k)+1}{\pi^{1/2}(2\nu(1-\nu)(k+\ell+2i))^{3/2}}\exp\left\{-\frac{(\nu\ell-(1-\nu)k)^2}{2\nu(1-\nu)(k+\ell+2i)}\right\}$$
$$\le \frac{\zeta(3/2)+1}{(\nu(1-\nu))^{3/2}} + \frac{1}{2\nu(1-\nu)}\widetilde{\Phi}\left(\frac{\nu\ell-(1-\nu)k}{(2\nu(1-\nu)(k+\ell))^{1/2}}\right) \le \frac{\zeta(3/2)+2}{(\nu(1-\nu))^{3/2}} \le \frac{\zeta(3/2)+2}{(\alpha\beta)^{3/2}},$$

where $\widetilde{\Phi}(x)$ is the error function defined by

$$\widetilde{\Phi}(x) := \frac{2}{\pi^{1/2}}\int_0^x e^{-t^2/2}dt, \quad x > 0.$$

Case $\nu\ell-(1-\nu)k < 0$ follows by symmetry.

In case $k \le 0$, $\ell < 0$ implying $(k-1)(\ell+1) \ge 0$ and $k\cdot\ell > 0$, we have

$$R_{k-1,\ell+1} - R_{k,\ell} = \sum_{i=0}^{\infty}(\alpha+\beta)^{-k-\ell+2i}\left(\mathsf{P}\left(S^{(\nu)}_{i,-k-\ell+i} = -\ell+i-1\right) - \mathsf{P}\left(S^{(\nu)}_{i,-k-\ell+i} = -\ell+i\right)\right)$$

and the statement can be proved similarly to the previous case.

Now, suppose $k > 0$, $\ell < 0$, so $(k-1)(\ell+1) \le 0$ and $k\cdot\ell \le 0$. Using the form (2.1) of the covariances direct calculations show

$$R_{k-1,\ell+1} - R_{k,\ell} = R_{k,\ell}\frac{1-(\alpha+\beta)^2+\sigma^{-2}_{\alpha,\beta}}{2\alpha\beta}.$$

It is not difficult to see that $1-(\alpha+\beta)^2 \le \sigma^{-2}_{\alpha,\beta}$, so we have

$$\left|R_{k-1,\ell+1} - R_{k,\ell}\right| \le \left|R_{k,\ell}\right|\frac{\sigma^{-2}_{\alpha,\beta}}{\alpha\beta} \le \frac{1}{\alpha\beta}.$$

In a similar way one can obtain the result for $k \le 0$, $\ell \ge 0$ that completes the proof. □

Using the notations of Lemma 2.3 with the help of the exponential approximation one can easily have the analogue of Corollary 2.7 of [5].

Corollary 2.5. *If $\alpha\beta > 0$, $|\alpha|+|\beta| < 1$ then there exists a constant $C > 0$ such that for all $k, \ell > 1$ and $0 \le i \le k+\ell-1$ we have*

$$\left|\mathsf{P}\left(S^{(\nu)}_{k,\ell} = i+1\right) - \mathsf{P}\left(S^{(\nu)}_{k,\ell} = i\right)\right| \le \frac{C}{\alpha\beta(k+\ell)}.$$

Remark 2.6. Using Theorem 2.4 of [5] it is not difficult to show that under conditions of Corollary 2.5 there exists a constant $D > 0$ such that for all $k, \ell > 1$ and $0 \le i \le k+\ell$ we have

$$\left|\mathsf{P}\left(S^{(\nu)}_{k,\ell} = i\right)\right| \le \frac{D}{\alpha\beta(k+\ell)^{1/2}}.$$

Now, let $\{X^{(n)}_{k,\ell} : k, \ell \in \mathbb{Z}\}$, $n \in \mathbb{N}$, be a nearly unstable sequence of stationary processes described in Theorem 1.1. For each $n \in \mathbb{N}$ let us introduce the piecewise constant random fields

$$Z_{1,0}^{(n)}(s,t) := r_n^{-1/4} X_{[ns]+1,[nt]}^{(r_n)}, \qquad Z_{0,1}^{(n)}(s,t) := r_n^{-1/4} X_{[ns],[nt]+1}^{(r_n)},$$
$$Y_{1,0}^{(n)}(s,t) := r_n^{-1/2} X_{[ns]+1,[nt]}^{(r_n)}, \qquad Y_{0,1}^{(n)}(s,t) := r_n^{-1/2} X_{[ns],[nt]+1}^{(r_n)}, \quad s,t \in \mathbb{R}.$$

Proposition 2.7. *Let $s_1, t_1, s_2, t_2 \in \mathbb{R}$.*

If $0 < |\alpha| < 1$, $|\beta| = 1 - |\alpha|$ and (1.7) holds then for all $(i_1, j_1), (i_2, j_2) \in \{(1,0), (0,1)\}$ we have

$$\lim_{n\to\infty} \left(|\gamma_{r_n}| + |\delta_{r_n}|\right)^{1/2} \operatorname{Cov}\left(Z_{i_1,j_1}^{(n)}(s_1,t_1), Z_{i_2,j_2}^{(n)}(s_2,t_2)\right) = 0 \quad \text{if } s_1 - s_2 \neq t_1 - t_2,$$
$$\limsup_{n\to\infty} \left(|\gamma_{r_n}| + |\delta_{r_n}|\right)^{1/2} \left|\operatorname{Cov}\left(Z_{i_1,j_1}^{(n)}(s_1,t_1), Z_{i_2,j_2}^{(n)}(s_2,t_2)\right)\right| \le \frac{1}{\sqrt{8|\alpha||\beta|}} \quad \text{if } s_1 - s_2 = t_1 - t_2.$$

If $|\alpha| \in \{0, 1\}$, $|\beta| = 1 - |\alpha|$ and (1.8) holds then for all $(i_1, j_1), (i_2, j_2) \in \{(1,0), (0,1)\}$ we have

$$\lim_{n\to\infty} \left|\gamma_{r_n}^2 - \delta_{r_n}^2\right|^{1/2} \operatorname{Cov}\left(Y_{i_1,j_1}^{(n)}(s_1,t_1), Y_{i_2,j_2}^{(n)}(s_2,t_2)\right) = 0 \quad \text{if } s_1 - s_2 \neq t_1 - t_2,$$
$$\limsup_{n\to\infty} \left|\gamma_{r_n}^2 - \delta_{r_n}^2\right|^{1/2} \left|\operatorname{Cov}\left(Y_{i_1,j_1}^{(n)}(s_1,t_1), Y_{i_2,j_2}^{(n)}(s_2,t_2)\right)\right| \le \frac{1}{2} \quad \text{if } s_1 - s_2 = t_1 - t_2.$$

Moreover, if $s_1 - s_2 \neq t_1 - t_2$ then the convergence to 0 in both cases has an exponential rate.

Proof. For simplicity we consider only the case $0 \le \alpha, \beta \le 1$. The other cases can be handled in a similar way.

First, let $0 < \alpha < 1$, so $\beta = 1 - \alpha$. Without loss of generality we may assume $\alpha_{r_n} > 0$, $\beta_{r_n} > 0$ and $\delta_{r_n} > 0$, $\gamma_{r_n} > 0$. As

$$r_n^{-1/2} \sigma_{\alpha_{r_n},\beta_{r_n}}^2 = \left(\left(\gamma_{r_n} + \delta_{r_n}\right) \left(2 - \frac{\gamma_{r_n} + \delta_{r_n}}{r_n}\right) \left(2\alpha - \frac{\gamma_{r_n} - \delta_{r_n}}{r_n}\right) \left(2(1-\alpha) + \frac{\gamma_{r_n} - \delta_{r_n}}{r_n}\right)\right)^{-1/2}$$

we have

$$\lim_{n\to\infty} \left(\gamma_{r_n} + \delta_{r_n}\right)^{1/2} r_n^{-1/2} \sigma_{\alpha_{r_n},\beta_{r_n}}^2 = \frac{1}{\sqrt{8\alpha(1-\alpha)}} = \frac{1}{\sqrt{8\alpha\beta}}. \tag{2.6}$$

Suppose $s_1 - s_2 \ge 0 \ge t_1 - t_2$, so $[ns_1] - [ns_2] \ge 0 \ge [nt_1] - [nt_2]$. By (2.1)

$$0 \le \operatorname{Cov}\left(Z_{1,0}^{(n)}(s_1,t_1), Z_{1,0}^{(n)}(s_2,t_2)\right) \le r_n^{-1/2} \sigma_{\alpha_{r_n},\beta_{r_n}}^2 \left(1 - \frac{1}{\varrho_{r_n}}\right)^{\frac{n}{2}|s_1 - s_2|} \left(1 + \frac{1}{\tau_{r_n}}\right)^{-\frac{n}{2}|t_1 - t_2|}$$

if n is large enough, where

$$\varrho_{r_n} := \frac{2\alpha_{r_n}}{2\alpha_{r_n} - 1 - \alpha_{r_n}^2 + \beta_{r_n}^2 + \sigma_{\alpha_{r_n},\beta_{r_n}}^{-2}}, \qquad \tau_{r_n} := \frac{2\beta_{r_n}}{1 + \beta_{r_n}^2 - \alpha_{r_n}^2 + \sigma_{\alpha_{r_n},\beta_{r_n}}^{-2} - 2\beta_{r_n}}. \tag{2.7}$$

As

$$\sigma_{\alpha,\beta}^2 = \left((1 + \alpha^2 - \beta^2)^2 - 4\alpha^2\right)^{-1/2},$$

it is easy to see that $\varrho_{r_n} \to \infty$ and $\tau_{r_n} \to \infty$ as $n \to \infty$. Moreover, condition (1.7) ensures that $n\varrho_{r_n}^{-1} \to \infty$ and $n\tau_{r_n}^{-1} \to \infty$ as $n \to \infty$. Hence, if $s_1 = s_2$ and $t_1 = t_2$,

$$\lim_{n\to\infty} \left(\gamma_{r_n} + \delta_{r_n}\right)^{1/2} \operatorname{Cov}\left(Z_{1,0}^{(n)}(s_1,t_1), Z_{1,0}^{(n)}(s_2,t_2)\right) = \frac{1}{\sqrt{8\alpha\beta}},$$

otherwise it converges to 0 in exponential rate.

Further, let $s_1 - s_2 > 0$ and $t_1 - t_2 > 0$. In this case $[ns_1] - [ns_2] \ge 0$ and $[nt_1] - [nt_2] \ge 0$, so by (2.2) we have

$$0 \le \operatorname{Cov}\left(Z_{1,0}^{(n)}(s_1,t_1), Z_{1,0}^{(n)}(s_2,t_2)\right) \le r_n^{-1/2} \sigma_{\alpha_{r_n},\beta_{r_n}}^2 \left(1 + \frac{1}{\tau_{r_n}}\right)^{-|[ns_1] - [ns_2] - [nt_1] + [nt_2]|}. \tag{2.8}$$

If $s_1 - s_2 \neq t_1 - t_2$ then similarly to the previous case one can show that the right-hand side of (2.8) converges to 0 in exponential rate as $n \to \infty$.

In case $s_1 - s_2 = t_1 - t_2$ we have $|[ns_1] - [ns_2] - [nt_1] + [nt_2]| \le 2$, so by (2.8)

$$\limsup_{n\to\infty} \left(\gamma_{r_n} + \delta_{r_n}\right)^{1/2} \operatorname{Cov}\left(Z_{1,0}^{(n)}(s_1,t_1), Z_{1,0}^{(n)}(s_2,t_2)\right) \le \frac{1}{\sqrt{8\alpha\beta}}.$$

Obviously, the same results hold for the covariances $\operatorname{Cov}\left(Z_{1,0}^{(n)}(s_1,t_1), Z_{0,1}^{(n)}(s_2,t_2)\right)$, $\operatorname{Cov}\left(Z_{0,1}^{(n)}(s_1,t_1), Z_{1,0}^{(n)}(s_2,t_2)\right)$ and $\operatorname{Cov}\left(Z_{0,1}^{(n)}(s_1,t_1), Z_{0,1}^{(n)}(s_2,t_2)\right)$.

Now, consider for example the case $\alpha = 1$, $\beta = 0$. Without loss of generality we may assume $\alpha_{r_n} > 0$. Furthermore, $|\alpha_{r_n}| + |\beta_{r_n}| < 1$ implies $\gamma_{r_n} > 0$ and $|\delta_{r_n}| < \gamma_{r_n}$. As

$$r_n^{-1}\sigma^2_{\alpha_{r_n},\beta_{r_n}} = \left(\left(\gamma_{r_n}^2 - \delta_{r_n}^2\right)\left(2 - \frac{\gamma_{r_n} + \delta_{r_n}}{r_n}\right)\left(2 - \frac{\gamma_{r_n} - \delta_{r_n}}{r_n}\right)\right)^{-1/2}$$

we have

$$\lim_{n\to\infty} \left(\gamma_{r_n}^2 - \delta_{r_n}^2\right)^{1/2} r_n^{-1}\sigma^2_{\alpha_{r_n},\beta_{r_n}} = \frac{1}{2}. \tag{2.9}$$

Again, suppose $s_1 - s_2 \ge 0 \ge t_1 - t_2$. The form of covariances (2.1) implies that if n is large enough

$$0 \le \left|\mathrm{Cov}\left(Y_{1,0}^{(n)}(s_1, t_1), Y_{1,0}^{(n)}(s_2, t_2)\right)\right| \le r_n^{-1}\sigma^2_{\alpha_{r_n},\beta_{r_n}} \left(1 - \frac{1}{\varrho_{r_n}}\right)^{\frac{n}{2}|s_1-s_2|}\left(1 + \frac{1}{|\tau_{r_n}|}\right)^{-\frac{n}{2}|t_1-t_2|}, \tag{2.10}$$

where ϱ_{r_n} and τ_{r_n} are defined by (2.7). Obviously, if $s_1 = s_2$ and $t_1 = t_2$ then (2.9) implies

$$\limsup_{n\to\infty} \left(\gamma_{r_n}^2 - \delta_{r_n}^2\right)^{1/2} \left|\mathrm{Cov}\left(Y_{1,0}^{(n)}(s_1, t_1), Y_{1,0}^{(n)}(s_2, t_2)\right)\right| \le \frac{1}{2}. \tag{2.11}$$

Further, we have $\varrho_{r_n} \to \infty$ as $n \to \infty$ and now (1.8) ensures $n\varrho_{r_n}^{-1} \to \infty$. Thus, as $1 + 1/|\tau_{r_n}| \ge 1$, if $s_1 \ne s_2$ then

$$\left(\gamma_{r_n}^2 - \delta_{r_n}^2\right)^{1/2} \left|\mathrm{Cov}\left(Y_{1,0}^{(n)}(s_1, t_1), Y_{1,0}^{(n)}(s_2, t_2)\right)\right| \to 0 \tag{2.12}$$

as $n \to \infty$ in exponential rate. Now, let us assume $s_1 = s_2$ and $t_1 \ne t_2$. Short calculation shows

$$\left(1 + \frac{1}{|\tau_{r_n}|}\right)^{-1} = \frac{2|\delta_{r_n}|}{2\gamma_{r_n} - \frac{\gamma_{r_n}^2 - \delta_{r_n}^2}{r_n} + \left(\gamma_{r_n}^2 - \delta_{r_n}^2\right)^{1/2}\left(\frac{\gamma_{r_n}^2 - \delta_{r_n}^2}{r_n} - 4\frac{\gamma_{r_n}}{r_n} + 4\right)^{1/2}}. \tag{2.13}$$

If $|\delta| < \gamma$ then

$$\lim_{n\to\infty} \left(1 + \frac{1}{|\tau_{r_n}|}\right)^{-1} = \frac{|\delta|}{\gamma + (\gamma^2 - \delta^2)^{1/2}} < 1,$$

so using (2.9) and (2.10) we obtain again (2.12). Further, condition (1.8) implies

$$\lim_{n\to\infty} n\left(\gamma_{r_n}^2 - \delta_{r_n}^2\right)^{1/2} = \infty.$$

Hence, with the help of (2.13) one can easily see that if $|\delta| = \gamma \ne 0$, or $\delta = \gamma = 0$ and $\lim_{n\to\infty} \gamma_{r_n}|\delta_{r_n}|^{-1} = 1$, we obtain $|\tau_{r_n}| \to \infty$ and $n|\tau_{r_n}|^{-1} \to \infty$ as $n \to \infty$. Thus, (2.9) and (2.10) imply (2.12) and the rate of convergence is again exponential. In case $\delta = \gamma = 0$ and $\lim_{n\to\infty} \gamma_{r_n}|\delta_{r_n}|^{-1} = |\omega| > 1$ we have

$$\lim_{n\to\infty} \left(1 + \frac{1}{|\tau_{r_n}|}\right)^{-1} = \frac{1}{|\omega| + (\omega^2 - 1)^{1/2}} < 1,$$

that implies (2.12). Finally, if $\delta = \gamma = 0$ and $\lim_{n\to\infty} \gamma_{r_n}|\delta_{r_n}|^{-1} = \infty$ then (2.12) follows from

$$\lim_{n\to\infty} \left(1 + \frac{1}{|\tau_{r_n}|}\right)^{-1} = 0.$$

Now, let $s_1 - s_2 > 0$ and $t_1 - t_2 > 0$. Lemma 2.1 and Remark 2.2 imply

$$0 \le \left|\mathrm{Cov}\left(Y_{1,0}^{(n)}(s_1, t_1), Y_{1,0}^{(n)}(s_2, t_2)\right)\right| \le r_n^{-1}\sigma^2_{\alpha_{r_n},\beta_{r_n}} \left(1 + \frac{1}{|\tau_{r_n}|}\right)^{-|[ns_1]-[ns_2]-[nt_1]+[nt_2]|},$$

where τ_{r_n} is defined by (2.7). If $s_1 - s_2 = t_1 - t_2$ then as $|[ns_1] - [ns_2] - [nt_1] + [nt_2]| \le 2$ and $1 + 1/|\tau_{r_n}| \ge 1$, using (2.9) we obtain (2.11). Finally, if $s_1 - s_2 \ne t_1 - t_2$ then to prove (2.11) one has to do the same considerations as in the case $s_1 = s_2$ and $t_1 \ne t_2$. □

In order to estimate the covariances we make use of the following lemma which is a natural generalization of Lemma 2.8 of [5].

Lemma 2.8. *Let $\xi_1, \xi_2, \ldots$ be independent random variables with $\mathsf{E}\xi_i = 0$, $\mathsf{E}\xi_i^2 = 1$ for all $i \in \mathbb{N}$, and $M_4 := \sup_{i\in\mathbb{N}} \mathsf{E}\xi_i^4 < \infty$. Let $a_1, a_2, \ldots, b_1, b_2, \ldots, c_1, c_2 \ldots, d_1, d_2 \ldots \in \mathbb{R}$, such that $\sum_{i=1}^\infty a_i^2 < \infty$, $\sum_{i=1}^\infty b_i^2 < \infty$, $\sum_{i=1}^\infty c_i^2 < \infty$ and $\sum_{i=1}^\infty d_i^2 < \infty$. Let*

$$X := \sum_{i=1}^{\infty} a_i \xi_i, \qquad Y := \sum_{i=1}^{\infty} b_i \xi_i, \qquad Z := \sum_{i=1}^{\infty} c_i \xi_i, \qquad W := \sum_{i=1}^{\infty} d_i \xi_i,$$

where the convergence of the infinite sums is understood in L_2*-sense. Then*

$$\mathsf{Cov}(XY, ZW) = \sum_{i=1}^{\infty} (\mathsf{E}\xi_i^4 - 3)\, a_i b_i c_i d_i + \mathsf{Cov}(X, Z)\,\mathsf{Cov}(Y, W) + \mathsf{Cov}(X, W)\,\mathsf{Cov}(Y, Z). \tag{2.14}$$

Moreover, if $a_i, b_i, c_i, d_i \geq 0$ *then*

$$0 \leq \mathsf{Cov}(XY, ZW) \leq M_4\, \mathsf{Cov}(X, Z)\,\mathsf{Cov}(Y, W) + M_4\, \mathsf{Cov}(X, W)\,\mathsf{Cov}(Y, Z),$$

and

$$0 \leq \mathsf{E}XYZW \leq M_4 \left(\mathsf{E}XZ\, \mathsf{E}YW + \mathsf{E}XW\, \mathsf{E}YZ + \mathsf{E}XY\, \mathsf{E}ZW \right).$$

Remark 2.9. Using the definitions of Lemma 2.8 from (2.14) one can easily see, that

$$|\,\mathsf{Cov}(XY, ZW)| \leq \mathsf{Cov}(\widetilde{X}\widetilde{Y}, \widetilde{Z}\widetilde{W}),$$

where

$$\widetilde{X} := \sum_{i=1}^{\infty} |a_i| \xi_i, \qquad \widetilde{Y} := \sum_{i=1}^{\infty} |b_i| \xi_i, \qquad \widetilde{Z} := \sum_{i=1}^{\infty} |c_i| \xi_i, \qquad \widetilde{W} := \sum_{i=1}^{\infty} |d_i| \xi_i.$$

3. Proof of Proposition 1.4

Let us assume $\alpha_{r_n} \neq 0$ and $\beta_{r_n} \neq 0$. Using the stationarity of $\left\{X_{k,\ell}^{(r_n)} : k, \ell \in \mathbb{Z}\right\}$ and Lemma 2.1 we obtain

$$\begin{aligned}
\mathsf{E}B_n &= \sum_{(k,\ell)\in T_{k_n,\ell_n}} \begin{pmatrix} \mathsf{Var}\left(X_{0,0}^{(r_n)}\right) & \mathsf{Cov}\left(X_{0,0}^{(r_n)}, X_{1,-1}^{(r_n)}\right) \\ \mathsf{Cov}\left(X_{0,0}^{(r_n)}, X_{1,-1}^{(r_n)}\right) & \mathsf{Var}\left(X_{0,0}^{(r_n)}\right) \end{pmatrix} \\
&= \frac{(k_n+\ell_n)(k_n+\ell_n+1)}{2} \sigma^2_{\alpha_{r_n},\beta_{r_n}} \begin{pmatrix} 1 & D_{r_n} \\ D_{r_n} & 1 \end{pmatrix} = \frac{n(n+1)}{2} \sigma^2_{\alpha_{r_n},\beta_{r_n}} \begin{pmatrix} 1 & D_{r_n} \\ D_{r_n} & 1 \end{pmatrix},
\end{aligned}$$

where

$$D_{r_n} = \left(\frac{1 + \alpha_{r_n}^2 - \beta_{r_n}^2 - \sigma^{-2}_{\alpha_{r_n},\beta_{r_n}}}{2\alpha_{r_n}} \right) \left(\frac{2\beta_{r_n}}{1 + \beta_{r_n}^2 - \alpha_{r_n}^2 + \sigma^{-2}_{\alpha_{r_n},\beta_{r_n}}} \right).$$

If $0 < |\alpha| < 1$ and $|\beta| = 1 - |\alpha|$ then it is not difficult to see that $\sigma^{-2}_{\alpha_{r_n},\beta_{r_n}} \to 0$ and in this way $D_{r_n} \to \mathrm{sign}(\alpha\beta)$ as $n \to \infty$. Hence, using the same arguments as in the proof of (2.6) we obtain

$$\lim_{n\to\infty} n^{-2} r_n^{-1/2} \left(|\gamma_{r_n}| + |\delta_{r_n}| \right)^{1/2} \mathsf{E}B_n = (32|\alpha||\beta|)^{-1/2}\, \Psi_{\alpha,\beta}.$$

If $|\alpha| \in \{0, 1\}$ and $|\beta| = 1 - |\alpha|$, again, we have $\sigma^{-2}_{\alpha_{r_n},\beta_{r_n}} \to 0$ as $n \to \infty$, and similarly to the proof of (2.9) one can see

$$\lim_{n\to\infty} n^{-2} r_n^{-1} \left| \gamma_{r_n}^2 - \delta_{r_n}^2 \right|^{1/2} \frac{n(n+1)}{2} \sigma^2_{\alpha_{r_n},\beta_{r_n}} = \frac{1}{4}.$$

Concerning the limit of D_{r_n} from the four possible cases that can be handled in the same way we consider only the case $\alpha = 1, \beta = 0$. In this case $\alpha \frac{\gamma_{r_n}}{\delta_{r_n}} + \beta \frac{\delta_{r_n}}{\gamma_{r_n}} = \frac{\gamma_{r_n}}{\delta_{r_n}}$ and we may assume $\alpha_{r_n} > 0$ and thus $|\delta_{r_n}| \leq \gamma_{r_n}$ (hence $\gamma_{r_n} > 0$). Obviously,

$$\lim_{n\to\infty} \frac{1 + \alpha_{r_n}^2 - \beta_{r_n}^2 - \sigma^{-2}_{\alpha_{r_n},\beta_{r_n}}}{2\alpha_{r_n}} = 1,$$

and

$$\begin{aligned}
\frac{2\beta_{r_n}}{1 + \beta_{r_n}^2 - \alpha_{r_n}^2 + \sigma^{-2}_{\alpha_{r_n},\beta_{r_n}}} &= \Bigg(\frac{\gamma_{r_n} - \delta_{r_n}}{2r_n} - \mathrm{sign}(\omega_{r_n}) \left(1 - \frac{\gamma_{r_n} - \delta_{r_n}}{2r_n} \right)^{1/2} \\
&\quad \times \left(\frac{\gamma_{r_n}}{|\delta_{r_n}|} \left(1 - \frac{\gamma_{r_n} - \delta_{r_n}}{2r_n} \right)^{1/2} + \left(\frac{\gamma_{r_n}^2}{\delta_{r_n}^2} - 1 \right)^{1/2} \left(1 - \frac{\gamma_{r_n} + \delta_{r_n}}{2r_n} \right)^{1/2} \right) \Bigg)^{-1}.
\end{aligned}$$

Hence,

$$\lim_{n\to\infty} D_{r_n} = \begin{cases} -\operatorname{sign}(\omega)\left(|\omega| + (\omega^2-1)^{1/2}\right)^{-1} & \text{if } |\omega| < \infty, \\ 0 & \text{if } |\omega| = \infty, \end{cases}$$

where ω is the limit defined by (1.9) satisfying $|\omega| \geq 1$. Thus, we have

$$\lim_{n\to\infty} n^{-2} r_n^{-1} \left|\gamma_{r_n}^2 - \delta_{r_n}^2\right|^{1/2} \mathsf{E} B_n = \Theta_{\alpha,\beta,\omega}.$$

Observe, that $\lim_{n\to\infty} D_{r_n} = \lim_{n\to\infty} \theta(\alpha, \beta, \omega_{r_n})$.

By Remark 2.9 in the remaining part of the proof we may assume $\alpha_{r_n} \geq 0$, $\beta_{r_n} \geq 0$. Hence, using Lemma 2.8 we have

$$\operatorname{Var}\left(\sum_{(i,j)\in T_{k_n,\ell_n}} \left(X_{i-1,j}^{(r_n)}\right)^2\right) \leq 2M_4 \sum_{(i_1,j_1)\in T_{k_n,\ell_n}} \sum_{(i_2,j_2)\in T_{k_n,\ell_n}} \operatorname{Cov}\left(X_{i_1-1,j_1}^{(r_n)}, X_{i_2-1,j_2}^{(r_n)}\right)^2, \tag{3.1}$$

where $M_4 := \sup_{n\in\mathbb{N}} \mathsf{E}(\varepsilon_{0,0}^{(n)})^4$, and from the stationarity of $\left\{X_{k,\ell}^{(r_n)} : k, \ell \in \mathbb{Z}\right\}$ follows that the triangle T_{k_n,ℓ_n} can be replaced by $T_{n,0}$.

Now, (3.1) implies that if $0 < |\alpha| < 1$ and $|\beta| = 1 - |\alpha|$

$$n^{-4} r_n^{-1} \left(|\gamma_{r_n}| + |\delta_{r_n}|\right) \operatorname{Var}\left(\sum_{(i,j)\in T_{k_n,\ell_n}} \left(X_{i-1,j}^{(r_n)}\right)^2\right)$$
$$\leq 2M_4 \iint_T \iint_T \left(\left(|\gamma_{r_n}| + |\delta_{r_n}|\right)^{1/2} \operatorname{Cov}\left(Z_{0,1}(s_1,t_1), Z_{0,1}(s_2,t_2)\right)\right)^2 \mathrm{d}s_1 \mathrm{d}t_1 \mathrm{d}s_2 \mathrm{d}t_2, \tag{3.2}$$

while for $|\alpha| \in \{0, 1\}$, $|\beta| = 1 - |\alpha|$ we have

$$n^{-4} r_n^{-2} \left|\gamma_{r_n}^2 - \delta_{r_n}^2\right| \operatorname{Var}\left(\sum_{(i,j)\in T_{k_n,\ell_n}} \left(X_{i-1,j}^{(r_n)}\right)^2\right)$$
$$\leq 2M_4 \iint_T \iint_T \left(\left|\gamma_{r_n}^2 - \delta_{r_n}^2\right|^{1/2} \operatorname{Cov}\left(Y_{0,1}(s_1,t_1), Y_{0,1}(s_2,t_2)\right)\right)^2 \mathrm{d}s_1 \mathrm{d}t_1 \mathrm{d}s_2 \mathrm{d}t_2, \tag{3.3}$$

where $T := \left\{(s,t) \in \mathbb{R}^2 : 0 \leq s \leq 1, -s \leq t \leq 0\right\}$. As the area of the triangle T is finite and the integrands in both cases are uniformly bounded on $T \times T$, Fatou's lemma and Proposition 2.7 imply that the right-hand sides of (3.2) and (3.3) converge to 0 as $n \to \infty$. In a similar way one can show

$$n^{-4} \kappa_n \operatorname{Var}\left(\sum_{(i,j)\in T_{k_n,\ell_n}} X_{i-1,j}^{(r_n)} X_{i,j-1}^{(r_n)}\right) \to 0 \quad \text{and} \quad n^{-4} \kappa_n \operatorname{Var}\left(\sum_{(i,j)\in T_{k_n,\ell_n}} \left(X_{i,j-1}^{(r_n)}\right)^2\right) \to 0,$$

as $n \to \infty$, where

$$\kappa_n = \begin{cases} r_n^{-1}\left(|\gamma_{r_n}| + |\delta_{r_n}|\right) & \text{if } 0 < |\alpha| < 1, |\beta| = 1 - |\alpha|, \\ r_n^{-2}\left|\gamma_{r_n}^2 - \delta_{r_n}^2\right| & \text{if } |\alpha| \in \{0,1\}, |\beta| = 1 - |\alpha|. \end{cases}$$

that finishes the proof of Proposition 1.4. □

4. Proof of Proposition 1.5

To prove Proposition 1.5 we are going to use the same technique as in [3–5]. For a given $n \in \mathbb{N}$ and $1 \leq m \leq n$, let

$$A_{n,m} = \begin{pmatrix} A_{n,m}^{(1)} \\ A_{n,m}^{(2)} \end{pmatrix} := \sum_{(k,\ell)\in T_{k_m,\ell_m}} \begin{pmatrix} X_{k-1,\ell}^{(r_n)} \varepsilon_{k,\ell}^{(r_n)} \\ X_{k,\ell-1}^{(r_n)} \varepsilon_{k,\ell}^{(r_n)} \end{pmatrix},$$

where $A_{n,0} := (0,0)^\top$. Let $\mathcal{F}_m^n$ denote the σ-algebra generated by the random variables $\left\{\varepsilon_{k,\ell}^{(r_n)} : (k,\ell) \in U_{k_m,\ell_m}\right\}$. Obviously, $A_{n,n} = A_n = \sum_{m=1}^n (A_{n,m} - A_{n,m-1})$. First we show that $\left(A_{n,m} - A_{n,m-1}, \mathcal{F}_m^n\right)$ is a square integrable martingale difference. Let $R_m := T_{k_m,\ell_m} \setminus T_{k_{m-1},\ell_{m-1}}$, where $R_1 := T_{k_1,\ell_1}$. Short calculation shows

$$A_{n,m} - A_{n,m-1} = A_{n,m,1} + \sum_{(k,\ell)\in R_m} \varepsilon_{k,\ell}^{(r_n)} A_{n,m,2,k,\ell}, \tag{4.1}$$

where $A_{n,m,1} = \left(A^{(1)}_{n,m,1}, A^{(2)}_{n,m,1}\right)^\top$ and $A_{n,m,2,k,\ell} = \left(\widetilde{A}_{n,m,2,k-1,\ell}, \widetilde{A}_{n,m,2,k,\ell-1}\right)^\top$ with

$$A^{(1)}_{n,m,1} := \sum_{(k,\ell)\in R_m} \varepsilon^{(r_n)}_{k,\ell} \sum_{(i,j)\in U_{k-1,\ell}\setminus U_{k_{m-1},\ell_{m-1}}} \binom{k+\ell-1-i-j}{k-1-i} \alpha_{r_n}^{k-1-i}\beta_{r_n}^{\ell-j}\varepsilon^{(r_n)}_{i,j},$$

$$A^{(2)}_{n,m,1} := \sum_{(k,\ell)\in R_m} \varepsilon^{(r_n)}_{k,\ell} \sum_{(i,j)\in U_{k,\ell-1}\setminus U_{k_{m-1},\ell_{m-1}}} \binom{k+\ell-1-i-j}{k-i} \alpha_{r_n}^{k-i}\beta_{r_n}^{\ell-1-j}\varepsilon^{(r_n)}_{i,j},$$

$$\widetilde{A}_{n,m,2,k,\ell} := \sum_{(i,j)\in U_{k,\ell}\cap U_{k_{m-1},\ell_{m-1}}} \binom{k+\ell-i-j}{k-i} \alpha_{r_n}^{k-i}\beta_{r_n}^{\ell-j}\varepsilon^{(r_n)}_{i,j}.$$

We remark that for the odd values of m we have $R_m = \bigcup_{i=-\ell_m+1}^{k_m}\{(i,\ell_m)\}$, and

$$A^{(1)}_{n,m,1} = \sum_{k=-\ell_m+2}^{k_m}\sum_{i=-\infty}^{k-1} \alpha_{r_n}^{k-1-i}\varepsilon^{(r_n)}_{k,\ell_m}\varepsilon^{(r_n)}_{i,\ell_m}, \qquad A^{(2)}_{n,m,1} = 0,$$

while for the even values $R_m = \bigcup_{j=-k_m+1}^{\ell_m}\{(k_m,j)\}$, and

$$A^{(2)}_{n,m,1} = \sum_{\ell=-k_m+2}^{\ell_m}\sum_{j=-\infty}^{\ell-1} \beta_{r_n}^{\ell-1-j}\varepsilon^{(r_n)}_{k_m,\ell}\varepsilon^{(r_n)}_{k_m,j}, \qquad A^{(1)}_{n,m,1} = 0.$$

The components of $A_{n,m,1}$ are quadratic forms of the variables $\left\{\varepsilon^{(r_n)}_{i,j} : (i,j)\in R_m\right\}$, hence $A_{n,m,1}$ is independent of $\mathcal{F}^n_{m-1}$. Further, the terms $\widetilde{A}_{n,m,2,k,\ell}$ are linear combinations of the variables $\{\varepsilon^{(r_n)}_{i,j} : (i,j)\in U_{k_{m-1},\ell_{m-1}}\}$, thus they are measurable with respect to $\mathcal{F}^n_{m-1}$. Hence,

$$\mathsf{E}\left(A_{n,m} - A_{n,m-1} \mid \mathcal{F}^n_{m-1}\right) = \mathsf{E}A_{n,m,1} + \sum_{(k,\ell)\in R_m} A_{n,m,2,k,\ell}\mathsf{E}\left(\varepsilon^{(r_n)}_{p,q} \mid \mathcal{F}^n_{m-1}\right) = 0.$$

By the Martingale Central Limit Theorem (see, e.g. [15, Theorem 4, p. 511]), the statement in Proposition 1.5 is a consequence of the following two propositions, where $\mathbb{1}_H$ denotes the indicator function of the set H.

Proposition 4.1. *If* $0 < |\alpha| < 1$, $|\beta| = 1-|\alpha|$ *and* (1.7) *holds then*

$$n^{-2}r_n^{-1/2}\left(|\gamma_{r_n}| + |\delta_{r_n}|\right)^{1/2}\sum_{m=1}^n \mathsf{E}\left((A_{n,m}-A_{n,m-1})(A_{n,m}-A_{n,m-1})^\top|\mathcal{F}^n_{m-1}\right) \xrightarrow{\mathsf{L}_2} (32|\alpha||\beta|)^{-1/2}\,\Psi_{\alpha,\beta}$$

as $n\to\infty$.

If $0 < |\alpha| \in \{0,1\}$, $|\beta| = 1-|\alpha|$ *and* (1.8) *holds then*

$$n^{-2}r_n^{-1}\left|\gamma_{r_n}^2 - \delta_{r_n}^2\right|^{1/2}\sum_{m=1}^n \mathsf{E}\left((A_{n,m}-A_{n,m-1})(A_{n,m}-A_{n,m-1})^\top|\mathcal{F}^n_{m-1}\right) \xrightarrow{\mathsf{L}_2} \Theta_{\alpha,\beta,\omega}$$

as $n\to\infty$.

Proposition 4.2. *If* $0 < |\alpha| < 1$, $|\beta| = 1-|\alpha|$ *and* (1.7) *holds then for all* $\delta > 0$

$$n^{-2}r_n^{-1/2}\left(|\gamma_{r_n}| + |\delta_{r_n}|\right)^{1/2}\sum_{m=1}^n \mathsf{E}\left(\|A_{n,m}-A_{n,m-1}\|^2 \times \mathbb{1}_{\left\{\|A_{n,m}-A_{n,m-1}\|\ge\delta n r_n^{1/4}(|\gamma_{r_n}|+|\delta_{r_n}|)^{-1/4}\right\}}|\mathcal{F}^n_{m-1}\right)$$

converges to 0 *in probability as* $n\to\infty$.

If $0 < |\alpha| \in \{0,1\}$, $|\beta| = 1-|\alpha|$ *and* (1.8) *holds then for all* $\delta > 0$

$$n^{-2}r_n^{-1}\left|\gamma_{r_n}^2 - \delta_{r_n}^2\right|^{1/2}\sum_{m=1}^n \mathsf{E}\left(\|A_{n,m}-A_{n,m-1}\|^2 \times \mathbb{1}_{\left\{\|A_{n,m}-A_{n,m-1}\|\ge\delta n r_n^{1/2}|\gamma_{r_n}^2-\delta_{r_n}^2|^{-1/4}\right\}}|\mathcal{F}^n_{m-1}\right)$$

converges to 0 *in probability as* $n\to\infty$.

The proofs of Propositions 4.1 and 4.2 follow the same line as the proof of Propositions 13 and 14 of [4], respectively. For more details the authors refer to [6]. □

5. Proof of Propositions 1.6 and 1.7

The proofs of Propositions 1.6 and 1.7 are very similar to the proofs of Propositions 1.4 and 1.5 of [5], respectively, so here we merely recall the main ideas. The detailed proofs can be found in [6].

In what follows we will assume $0 < \alpha < 1$ and $\beta = 1-\alpha$, so without loss of generality we may suppose that α_{r_n}, β_{r_n}, γ_{r_n} and δ_{r_n} are all positive. Consider the following expression of $\det B_n$

$$\det B_n = \sum_{(i_1,j_1)\in T_{k_n,\ell_n}} \sum_{(i_2,j_2)\in T_{k_n,\ell_n}} W^{(n)}_{i_1,j_1,i_2,j_2},$$

where

$$W^{(n)}_{i_1,j_1,i_2,j_2} := \left(X^{(r_n)}_{i_1,j_1-1}\right)^2 \left(X^{(r_n)}_{i_2-1,j_2}\right)^2 - X^{(r_n)}_{i_1-1,j_1} X^{(r_n)}_{i_1,j_1-1} X^{(r_n)}_{i_2-1,j_2} X^{(r_n)}_{i_2,j_2-1}.$$

With the help of Lemma 2.8, Propositions 2.4 and 2.7 and Fatou's lemma one can show

$$\lim_{n\to\infty} n^{-4} r_n^{-1/2} \left(\gamma_{r_n} + \delta_{r_n}\right)^{1/2} \mathsf{E} \det B_n = \frac{2}{(8\alpha\beta)^{3/2}}.$$

Further, after tedious but straightforward calculations, using the independence of the error terms $\varepsilon^{(r_n)}_{i,j}$, Corollary 2.5, Lemma 2.1, Propositions 2.4 and 2.7 and Remark 2.6 one obtains

$$\lim_{n\to\infty} n^{-8} r_n \left(\gamma_{r_n} + \delta_{r_n}\right) \mathsf{Var}\, (\det B_n) = 0$$

that completes the proof of Proposition 1.6.

Concerning the statement of Proposition 1.7 we have

$$n^{-3} r_n^{-1/2} \left(\gamma_{r_n} + \delta_{r_n}\right)^{1/2} \bar{B}_n A_n = \left(n^{-2} r_n^{-1/2} \left(\gamma_{r_n} + \delta_{r_n}\right)^{1/2} \bar{B}_n - \frac{1}{\sqrt{32\alpha\beta}} \bar{1} \right) \frac{1}{n} A_n + \frac{1}{\sqrt{32\alpha\beta}} \frac{1}{n} \bar{1} A_n,$$

where 1 denotes the two-by-two matrix of ones. Short straightforward calculations show

$$\left(n^{-2} r_n^{-1/2} \left(\gamma_{r_n} + \delta_{r_n}\right)^{1/2} \bar{B}_n - \frac{1}{\sqrt{32\alpha\beta}} \bar{1} \right) \frac{1}{n} A_n = C_n + D_n,$$

where

$$C_n := n^{-1} r_n^{-1/4} \left(\gamma_{r_n} + \delta_{r_n}\right)^{1/4} \mathrm{diag}(A_n) n^{-2} r_n^{-1/4} \left(\gamma_{r_n} + \delta_{r_n}\right)^{1/4} \bar{B}_n (1,1)^\top,$$

$$D_n := \left(n^{-2} r_n^{-1/2} \left(\gamma_{r_n} + \delta_{r_n}\right)^{1/2} \sum_{(i,j)\in T_{k_n,\ell_n}} X^{(r_n)}_{i-1,j} X^{(r_n)}_{i,j-1} - \frac{1}{\sqrt{32\alpha\beta}} \right) \frac{1}{n} (1,-1) A_n (1,-1)^\top$$

and $\mathrm{diag}(A_n)$ denotes the two-by-two diagonal matrix having A_n in its main diagonal. From Proposition 1.4, representation (1.3), independence of the error terms $\varepsilon^{(r_n)}_{i,j}$ and (2.6) we obtain $D_n \stackrel{\mathsf{P}}{\longrightarrow} (0,0)^\top$ as $n \to \infty$.

Further, using direct calculations one can see

$$n^{-2} r_n^{-1/4} \left(\gamma_{r_n} + \delta_{r_n}\right)^{1/4} \bar{B}_n (1,1)^\top \stackrel{\mathsf{L}_2}{\longrightarrow} (0,0)^\top \quad \text{as } n \to \infty$$

that together with Proposition 1.5 implies $C_n \stackrel{\mathsf{P}}{\longrightarrow} (0,0)^\top$ as $n \to \infty$. Hence, to prove the asymptotic normality of $n^{-3} r_n^{-1/2} \left(\gamma_{r_n} + \delta_{r_n}\right)^{1/2} \bar{B}_n A_n$ it suffices to show the asymptotic normality of $n^{-1} \bar{1} A_n$.

For a given $n \in \mathbb{N}$ and $1 \le m \le n$ let $Q_{n,m} := (1,-1) A_{n,m}$. Obviously $Q_{n,n} = (1,-1) A_n$ and from Eq. (4.1) we have

$$Q_{n,m} - Q_{n,m-1} = A^{(1)}_{n,m,1} - A^{(2)}_{n,m,1} + \sum_{(k,\ell)\in R_m} \varepsilon^{(r_n)}_{k,\ell} \left(\widetilde{A}_{n,m,2,k-1,\ell} - \widetilde{A}_{n,m,2,k,\ell-1} \right).$$

As $\left(Q_{n,m} - Q_{n,m-1}, \mathcal{F}^n_m\right)$ is a square integrable martingale difference, similarly to the proof of Proposition 1.5 the statement of Proposition 1.7 follows from the Martingale Central Limit Theorem. □

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References

[1] S. Basu, G.C. Reinsel, Properties of the spatial unilateral first-order ARMA model, Adv. Appl. Probab. 25 (1993) 631–648.
[2] S. Basu, G.C. Reinsel, Regression models with spatially correlated errors, J. Amer. Statist. Assoc. 89 (1994) 88–99.
[3] S. Baran, G. Pap, M.v. Zuijlen, Asymptotic inference for a nearly unstable sequence of stationary spatial AR models, Statist. Probab. Lett. 69 (2004) 53–61.
[4] S. Baran, G. Pap, M.v. Zuijlen, Asymptotic inference for an unstable spatial AR model, Statistics 38 (2004) 465–482.
[5] S. Baran, G. Pap, M.v. Zuijlen, Asymptotic inference for unit roots in spatial triangular autoregression, Acta Appl. Math. 96 (2007) 17–42.
[6] S. Baran, G. Pap, On the least squares estimator in a nearly unstable sequence of stationary spatial AR models. arXiv:0803.2486v1.
[7] H. Bateman, A. Erdélyi, Higher Transcendental Functions, vol. 1, Mc Graw-Hill, New York, 1953.
[8] B.B. Bhattacharyya, G.D. Richardson, L.A. Franklin, Asymptotic inference for near unit roots in spatial autoregression, Ann. Statist. 25 (1997) 1709–1724.
[9] N.H. Chan, C.Z. Wei, Asymptotic inference for nearly nonstationary AR(1) processes, Ann. Statist. 15 (1987) 1050–1063.
[10] B.R. Cullis, A.C. Gleeson, Spatial analysis of field experiments — an extension to two dimensions, Biometrics 47 (1991) 1449–1460.
[11] A.K. Jain, Advances in mathematical models for image processing, Proc. IEEE 69 (1981) 502–528.
[12] R.J. Martin, A subclass of lattice processes applied to a problem in planar sampling, Biometrika 66 (1979) 209–217.
[13] R.J. Martin, The use of time-series models and methods in the analysis of agricultural field trials, Comm. Statist. Theory Methods 19 (1990) 55–81.
[14] V. Paulauskas, On unit roots for spatial autoregressive models, J. Multivariate Anal. 98 (2007) 209–226.
[15] A.N. Shiryayev, Probability, Springer-Verlag, New York, 1984.
[16] D. Tjøstheim, Autoregressive modelling and spectral analysis of array data in the plane, IEEE Trans. Geosci. Remote Sensing 19 (1981) 15–24.