On Perron complements of inverse $N_0$-matrices

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ARTICLE INFO

Article history:
Received 5 June 2005
Accepted 5 December 2010
Available online 11 January 2011
Submitted by M. Neumann

AMS Classification:
15B48

Keywords:
$N_0$-matrices
Inverse $Z$-matrices
Inverse $N_0$-matrices
Perron complements

ABSTRACT

In this paper, we show that the Perron complements of irreducible $N_0$-matrices are $N_0$-matrices. We also demonstrate the Perron complements of irreducible inverse $N_0$-matrices are inverse $N_0$-matrices with a little restriction. In addition, we give some related inequalities about inverse $N_0$-matrices.

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1. Introduction

If $X = (x_{ij}) \in \mathbb{R}^{n \times n}$ and $Y = (y_{ij}) \in \mathbb{R}^{n \times n}$, we write $X \geq 0$ (respectively, $X \leq 0$) if and only if $x_{ij} \geq 0$ (respectively, $x_{ij} \leq 0$). Similarly, we write $X \geq Y$ (respectively, $X \leq Y$) if and only if $X - Y \geq 0$ (respectively, $X - Y \leq 0$). A matrix $A \in \mathbb{R}^{n \times n}$ is called an $N_0$-matrix (respectively, an $N$-matrix) if $A \leq 0$ and all principal minors of $A$ are nonpositive (respectively, negative) (see [2,5]). A $Z$-matrix $B \in \mathbb{R}^{n \times n}$ is a matrix whose off-diagonal entries are nonpositive. The representation $B = tl - P$ for a $Z$-matrix is often convenient, where $P \geq 0$. A $Z$-matrix $B$ is called an $N_0$-matrix (respectively, an $N$-matrix) if it satisfies $\rho_{n-1}(P) \leq t < \rho(P)$ (respectively, $\rho_{n-1}(P) < t < \rho(P)$), where $\rho_{n-1}(P)$ denotes that maximal spectral radius of all principal submatrices of $P$ of order $n - 1$ and $\rho(\cdot)$ denotes the spectral radius of a matrix. Moreover, a $Z$-matrix $B$ is called an $M$-matrix if it satisfies $\rho(P) < t$. A nonsingular matrix $C \in \mathbb{R}^{n \times n}$ is said to be an inverse $Z$-matrix (respectively, inverse $M$-matrix, inverse $N_0$-matrix,

* This research was supported by NSFC (60973015), Sichuan Province Sci. and Tech. Research Project (2009GZ0004).

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doi:10.1016/j.laa.2010.12.004
inverse $N$-matrix) if $C^{-1}$ is a $Z$-matrix (respectively, an $M$-matrix, an $N_0$-matrix, an $N$-matrix) (see [6,7,8,9]). In fact, it is known that inverse $N_0$-matrices are the subclass of $N_0$-matrices.

We define $(\langle n \rangle) = \{1, \ldots, n\}$ and denote the empty set by $\phi$. Let $A$ be a real square matrix of order $n$ and $\alpha$, $\beta$ be nonempty subset of $(\langle n \rangle)$, both of strictly increasing integers. Denote the submatrix of $A$ by $A[\alpha, \beta]$ whose rows and columns are determined by $\alpha$ and $\beta$. If $\alpha = \beta$, we abbreviate $A[\alpha, \beta]$ to be $A[\alpha]$.

We partition

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where $A_{11}$ and $A_{22}$ are square matrices of orders $k$ and $n - k$, respectively. In general, the matrix

$$A/A_{11} = A_{22} - A_{21}A_{11}^{-1}A_{12}$$

is called the Schur complement of $A_{11}$ in $A$, where $A_{11}$ is invertible. It is well known that if $A$ and $A_{11}$ are nonsingular, then $A/A_{11}$ is also nonsingular and

$$A^{-1} = \begin{bmatrix} A_{11}^{-1} + A_{11}^{-1}A_{12}(A/A_{11})^{-1}A_{21}A_{11}^{-1} & -A_{11}^{-1}A_{12}(A/A_{11})^{-1} \\ -(A/A_{11})^{-1}A_{21}A_{11}^{-1} & (A/A_{11})^{-1} \end{bmatrix}.$$ (2)

For a given class of matrices, it is interesting to study the properties inherited by their submatrices and related matrices, such as the Schur complement and the Perron complement. Meyer introduced the concept of the Perron complement of a nonnegative and irreducible matrix in 1989 and used it to construct an algorithm for computing the stationary distribution vector for the Markov chains [4]. Neumann extended the Perron complement and showed that the extended Perron complements of irreducible inverse $M$-matrices are inverse $M$-matrices [1].

Next we extend the Perron complements of nonnegative and irreducible matrices to the non-positive and irreducible matrices. This is motivated by the extended Perron complements of inverse $M$-matrices [1] and the Perron complements of totally nonnegative matrices [2]. We prove that the extended Perron complements of $N_0$-matrices are $N_0$-matrices in Section 2. We obtain that the extended Perron complements of inverse $N_0$-matrices are inverse $N_0$-matrices in Section 3.

For an $n \times n$ nonnegative and irreducible matrix $A$, by Meyer’s definition, for any $\alpha \subseteq (\langle n \rangle)$ and $\beta = (\langle n \rangle) \setminus \alpha$, then the Perron complement of $A[\beta]$ in $A$ is given by

$$P(A/A[\beta]) = A[\alpha] + A[\alpha, \beta](\rho(A)I - A[\beta])^{-1}A[\beta, \alpha].$$ (3)

By Neumann’s definition, for any $\alpha \subseteq (\langle n \rangle)$, $\beta = (\langle n \rangle) \setminus \alpha$, and for any $t \geq \rho(A)$, let the extended Perron complement of $A[\beta]$ in $A$ at $t$ be the matrix

$$P_t(A/A[\beta]) = A[\alpha] + A[\alpha, \beta](tI - A[\beta])^{-1}A[\beta, \alpha].$$ (4)

For a non-positive and irreducible matrix $A$, we extend the concept of the Perron complement as follows:

Let $\alpha \subset (\langle n \rangle)$, $\beta = (\langle n \rangle) \setminus \alpha$. The Perron complement of $A[\beta]$ in $A$ is given by

$$P(A/A[\beta]) = A[\alpha] - A[\alpha, \beta](\rho(A)I + A[\beta])^{-1}A[\beta, \alpha].$$ (5)

For any $t \geq \rho(A)$, let the extended Perron complement of $A[\beta]$ in $A$ at $t$ be the matrix

$$P_t(A/A[\beta]) = A[\alpha] - A[\alpha, \beta](tI + A[\beta])^{-1}A[\beta, \alpha].$$ (6)

It is known that for a nonnegative and irreducible matrix $A$, $P(A/A[\beta])$ and $P_t(A/A[\beta])$ in (3) and (4) are nonnegative and irreducible matrices [4]. In fact, for a non-positive and irreducible matrix, we
have the same results as above, that is, $\mathcal{P}(A/A[\beta])$ and $\mathcal{P}_t(A/A[\beta])$ in (5) and (6) are non-positive and irreducible matrices.

2. Properties and lemmas

The problem of determining whether a matrix is an inverse $N_0$-matrix or not is called the inverse $N_0$-matrix problem. There are some references about the inverse $N_0$-matrix problem, see [8, 9]. By Theorem 2.7 of [7], we easily obtain the following conclusion.

Theorem 1. Let $A$ be an $n \times n$ matrix. Then the following are equivalent:

1. $A$ is an inverse $N_0$-matrix;
2. All principal submatrices of $A^{-1}$ are M-matrices and $\det(A) < 0$;
3. $A \leq 0$ and is irreducible;
4. All principal minors of $A$ are non-positive;
5. $\forall \varepsilon > 0$, $A^{-1} + \varepsilon I$ is an $N_0$-matrix.

Ky Fan has shown that if $A$ is an $N$-matrix, then $A^{-1} < 0$. Moreover, by Theorem 1, we have:

Corollary 2. Let $A$ be an $n \times n$ matrix. Then the following are equivalent:

1. $A$ is an inverse $N$-matrix;
2. $A < 0$ and $A^{-1}$ is a Z-matrix.

Next we will prove that the extended Perron complements of $N_0$-matrices are $N_0$-matrices. We first need a lemma as follows.

Lemma 3. Let $A$ be an $n \times n$ irreducible $N_0$-matrix, $\beta = \{i\} \subset \langle n \rangle$ and $\alpha = \langle n \rangle \setminus \beta$. Then for any $t \in [\rho(A), \infty)$, the matrix $\mathcal{P}_t(A/A[\beta])$ is an irreducible $N_0$-matrix. In particular, for $\beta = \{i\}$, the Perron complement $\mathcal{P}(A/A[\beta])$ is an irreducible $N_0$-matrix.

Proof. Without loss of generality, we assume $\beta = \{n\}$. Let $A$ be partitioned as (1), where $A_{11}$ is an $(n-1) \times (n-1)$ matrix and $A_{22} = a_{nn}$. It is evident that

$$t \geq \rho(A) > -a_{nn},$$

then

$$\mathcal{P}_t(A/a_{nn}) = A_{11} - \frac{A_{12}A_{21}}{t + a_{nn}}.$$

Consider the matrix

$$B = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & t + a_{nn} \end{bmatrix}.$$

Observe that

$$B/(t + a_{nn}) = \mathcal{P}_t(A/a_{nn}).$$

Then we can compute any principal minor of $\mathcal{P}_t(A/a_{nn})$ by computing principal minors of a related Schur complement $B/(t + a_{nn})$. We have

$$\det((B/(t + a_{nn}))[\alpha]) = \frac{\det B[\alpha \cup \{n\}]}{t + a_{nn}}, \quad \alpha \subset \langle n \rangle.$$
Observe that
\[
\det B[\alpha \cup \{n\}] = \det \begin{bmatrix}
A_{11}[\alpha] & A_{12}[\alpha] \\
A_{21}[\alpha] & t + a_{nn}
\end{bmatrix}
\]
\[
= t \det A[\alpha] + \det \begin{bmatrix}
A_{11}[\alpha] & A_{12}[\alpha] \\
A_{21} & a_{nn}
\end{bmatrix}
\]
\[
= t \det A[\alpha] + \det A[\alpha \cup \{n\}] \leq 0.
\]

We obtain that all principal minors of \( P_t(A/\text{ann}) \) are non-positive and it is evident that \( P_t(A/\text{ann}) \) is irreducible. This completes the proof. □

Let \( A \) be a real square matrix of order \( n \) and \( \phi \neq \alpha \subset \beta \subset \langle n \rangle \). We have the quotient formulas [2]:
\[
A/\text{ann} = (A/A[\alpha])/\text{ann}((A/\beta)/(A[\alpha])).
\]
(7)

We can also state the quotient formula as follows [2]:
If \( \phi \neq \gamma_1, \gamma_2 \subset \beta \subset \langle n \rangle \), with \( \gamma_1 \cup \gamma_2 = \beta \) and \( \gamma_1 \cap \gamma_2 = \phi \), then
\[
A/\text{ann} = (A/A[\gamma_1])/\text{ann}(A[\gamma_2]).
\]

By Theorem 2.4 in [2], we easily obtain the following conclusion.

**Lemma 4.** Let \( A \) be an \( n \times n \) irreducible non-positive matrix. If \( \phi \neq \gamma_1, \gamma_2 \subset \beta \subset \langle n \rangle \), with \( \gamma_1 \cup \gamma_2 = \beta \) and \( \gamma_1 \cap \gamma_2 = \phi \), then for any \( t \in [\rho(A), \infty) \), we have
\[
P_t(A/\text{ann}) = P_t(P_t(A/\beta)/\text{ann}(A[\gamma_2])).
\]

Using this quotient formula for the extended Perron complements and Lemma 3 and Lemma 4 we have the following result.

**Theorem 5.** Let \( A \) be an \( n \times n \) irreducible \( N_0 \)-matrix, \( \phi \neq \beta \subset \langle n \rangle \) and define \( \alpha = \langle n \rangle \setminus \beta \). Then for any \( t \in [\rho(A), \infty) \), the matrix \( P_t(A/\text{ann}) \) is an irreducible \( N_0 \)-matrix. In particular, the Perron complement \( P(A/\text{ann}) \) is an irreducible \( N_0 \)-matrix.

For an inverse \( N_0 \)-matrix, its submatrices and the Schur complements have the following properties.

**Lemma 6** [8]. Let \( A \) be an inverse \( N_0 \)-matrix, \( B \) be a \( k \times k \) principal submatrix of \( A \) with \( k \geq 2 \). Then \( B \) is an inverse \( N_0 \)-matrix.

**Lemma 7** [7]. Let \( A \) be an inverse \( N_0 \)-matrix and be partitioned as (1). If \( A_{11} \) is nonsingular, then the Schur complement of \( A_{11} \) in \( A \) is an inverse \( M \)-matrix.

If \( A \) is an inverse \( N_0 \)-matrix with \( \text{diag}A < 0 \), we have the following observation.

**Lemma 8.** Let \( A = (a_{ij}) \) be an \( n \times n \) inverse \( N_0 \)-matrix and \( \text{diag}A < 0 \). Then \( A < 0 \), that is, \( A \) is an inverse \( N \)-matrix.

**Proof.** We consider any \( 2 \times 2 \) submatrix of \( A \) and let \( \alpha = \{i, j\} \subset \langle n \rangle \). Then
\[
A[\alpha] = \begin{bmatrix}
a_{ii} & a_{ij} \\
 a_{ji} & a_{jj}
\end{bmatrix}.
\]
By Lemma 6,
\[
\det A[\alpha] = a_\alpha a_{jj} - a_j a_{ij} \leq 0.
\]
We suppose \(a_j = 0\) or \(a_{ji} = 0\), then \(a_\alpha a_{jj} \leq 0\). This is a contradiction to diagA < 0. Hence
\[
a_j < 0, \quad \text{and} \quad a_{ji} < 0,
\]
that is, \(A < 0\). By Corollary 2, \(A\) is an inverse \(N\)-matrix. \(\square\)

3. Perron complements of inverse \(N_0\)-matrices

We know that extended Perron complements of irreducible inverse \(M\)-matrices are inverse \(M\)-matrices. Next we show that, with a little restriction, extended Perron complements of irreducible inverse \(N_0\)-matrices have the similar properties with inverse \(M\)-matrices.

**Theorem 9.** Let \(A\) be an \(n \times n\) irreducible inverse \(N_0\)-matrix, \(\beta = \{i\} \subset \langle n \rangle\) and \(\alpha = \langle n \rangle \setminus \beta\). If \(A[\beta]\) is nonzero, then for any \(t \in [\rho(A), \infty)\), the matrix \(P_t(A/A[\beta])\) is invertible and is an irreducible inverse \(N_0\)-matrix. In particular, the Perron complement \(P(A/A[\beta])\) is an irreducible inverse \(N_0\)-matrix, for \(\beta = \{i\}\).

**Proof.** Without loss of generality, we assume \(\beta = \{n\}\). Let \(A\) be partitioned as (1), where \(A_{11}\) is an \((n - 1) \times (n - 1)\) matrix and \(A_{22} = a_{nn}\). We begin with showing that the matrix
\[
P_t(A/A_{22}) = A_{11} - A_{12}(t + A_{22})^{-1}A_{21}
\]
is nonsingular by computing its inverse. Here we use a consequence of the Woodbury formula:

If \(E\) and \(F\) are \(m \times n\) and \(k \times k\) matrices and \(U\) and \(V\) are \(m \times k\) and \(k \times m\) matrices, for which the matrix \(F^{-1} + VE^{-1}U\) is invertible, then the matrix \(E + UFV\) is invertible and the inverse matrix is given by
\[
(E + UFV)^{-1} = E^{-1} - E^{-1}U(F^{-1} + VE^{-1}U)^{-1}VE^{-1}.
\]

By Lemma 6, we know \(A_{11}\) is an inverse \(N_0\)-matrix. Since \(A\) and \(A_{22}\) are nonsingular, the inverse matrix of \(A\) is in the block form (2). Again, because \(A\) is an inverse \(N_0\)-matrix, we have
\[
-(A^{-1})_{12} = A_{11}^{-1}A_{12}(A/A_{11})^{-1} \geq 0.
\]

By Lemma 7, we know \((A/A_{11})^{-1}\) is an \(M\)-matrix, then \(A_{11}^{-1}A_{12} \geq 0\). So is \(A_{21}A_{11}^{-1}\). Because \(A/A_{11}\) is an inverse \(M\)-matrix, we obtain that \(t + A/A_{11}\) is an inverse \(M\)-matrix. Let
\[
E = A_{11}, \quad F = (t + A_{22})^{-1}, \quad U = A_{12}, \quad V = A_{21}.
\]
We know the matrix \(P_t(A/A_{22})\) is nonsingular, and
\[
(P_t(A/A_{22}))^{-1} = A_{11}^{-1} + A_{11}^{-1}A_{12}\left[t + (A_{22} - A_{21}A_{11}^{-1}A_{12})\right]^{-1}A_{21}A_{11}^{-1}
\]
\[
= A_{11}^{-1} + A_{11}^{-1}A_{12}(t + A/A_{11})^{-1}A_{21}A_{11}^{-1}
\]
\[
= A_{11}^{-1} + A_{11}^{-1}A_{12}(t + A/A_{11})^{-1}A_{21}A_{11}^{-1}
\]
\[
= A_{11}^{-1} + \frac{A_{11}^{-1}A_{21}A_{11}^{-1}}{t + a_{nn} - A_{21}A_{11}^{-1}A_{12}}
\]
\[
= A_{11}^{-1} + \frac{A_{11}^{-1}A_{21}A_{11}^{-1}}{a_{nn} - A_{21}A_{11}^{-1}A_{12}}
\]
\[
= (A/A_{22})^{-1},
\]
where
\[
A/A_{11} = a_{nn} - A_{21}A_{11}^{-1}A_{12} > 0.
\]
Since \((A/A_{22})^{-1}\) is an \(M\)-matrix, we have
\[
P_t(A/A[\beta])^{-1}_{ij} \leq (A/A_{22})^{-1}_{ij} \leq 0.
\]
Then \((P_t(A/A[\beta]))^{-1}\) is a \(Z\)-matrix. And it is evident that \(P_t(A/A_{22})\) is irreducible. By Theorem 1, we know that \((P_t(A/A[\beta]))^{-1}\) is an irreducible inverse \(N_0\)-matrix. This completes our proof. □

**Corollary 10.** Let \(A\) be an \(n \times n\) inverse \(N\)-matrix, \(\beta = \{i\} \subset \langle n \rangle\) and \(\alpha = \langle n \rangle \setminus \beta\). Then for any \(t \in [\rho(A), \infty)\), the matrix \(P_t(A/A[\beta])\) is invertible and is an inverse \(N\)-matrix. In particular, for \(\beta = \{i\}\), the Perron complement \(P(A/A[\beta])\) is an inverse \(N\)-matrix.

For \(\phi \neq \beta \subset \langle n \rangle\), the Perron complement \(P_t(A/A[\beta])\) can be obtained by Lemma 4 from a sequence of Perron complements, which are irreducible \(N_0\)-matrices by Theorem 9. So we obtain the following conclusion.

**Theorem 11.** Let \(A\) be an \(n \times n\) irreducible inverse \(N_0\)-matrix, \(\phi \neq \beta \subset \langle n \rangle\) and \(\alpha = \langle n \rangle \setminus \beta\). If all entries of \(\text{diag}(A[\beta])\) are nonzero, then for any \(t \in [\rho(A), \infty)\), the matrix \(P_t(A/A[\beta])\) is invertible and is an irreducible inverse \(N_0\)-matrix. In particular, the Perron complement \(P(A/A[\beta])\) is an irreducible inverse \(N_0\)-matrix.

**Theorem 12.** Let \(A\) be an \(n \times n\) inverse \(N\)-matrix, \(\phi \neq \beta \subset \langle n \rangle\) and define \(\alpha = \langle n \rangle \setminus \beta\). Then for any \(t \in [\rho(A), \infty)\), the matrix \(P_t(A/A[\beta])\) is invertible and is an inverse \(N\)-matrix. In particular, the Perron complement \(P(A/A[\beta])\) is an inverse \(N\)-matrix.

### 4. Inequalities of inverse \(N_0\)-matrices

According to the proof of Theorem 9, we now have the following conclusion.

**Theorem 13.** Let \(A\) be an \(n \times n\) irreducible inverse \(N_0\)-matrix, \(\beta = \{i\} \subset \langle n \rangle\) and \(\alpha = \langle n \rangle \setminus \beta\). If \(A[\beta]\) is nonzero, then for any \(t \in [\rho(A), \infty)\), the following ordering holds:
\[
(A[\alpha])^{-1} \leq (P_t(A/A[\beta]))^{-1} \leq (A/A[\beta])^{-1}.
\]

Moreover, as a function of \(t\), the matrix \((P_t(A/A[\beta]))^{-1}\) is entrywise decreasing in \([\rho(A), \infty)\) and
\[
\lim_{t \to \infty} (P_t(A/A[\beta]))^{-1} = (A[\alpha])^{-1}.
\]

By the definitions of (5) and (6), we easily obtain the following conclusion.

**Theorem 14.** Let \(A\) be an \(n \times n\) irreducible inverse \(N_0\)-matrix, \(\beta \subset \langle n \rangle\) and \(\alpha = \langle n \rangle \setminus \beta\). Then for any \(t \in [\rho(A), \infty)\),
\[
P(A/A[\beta]) \leq P_t(A/A[\beta]) \leq A[\alpha].
\]

Moreover, as a function of \(t\), the matrix \(P_t(A/A[\beta])\) is entrywise increasing in \([\rho(A), \infty)\) and
\[
\lim_{t \to \infty} P_t(A/A[\beta]) = A[\alpha].
\]

Relative to Theorem 13, we can make further result as follows.

**Theorem 15.** Let \(A\) be an \(n \times n\) inverse \(N_0\)-matrix, \(\phi \neq \beta \subset \langle n \rangle\) and define \(\alpha = \langle n \rangle \setminus \beta\). Then, for any \(t \in [\rho(A), \infty)\),
\[
(A[\alpha])^{-1} \leq (P_t(A/A[\beta]))^{-1}.
\]
Proof. Let $A_{11} = A[\alpha], A_{22} = A[\beta], A_{12} = A[\alpha, \beta]$ and $A_{21} = A[\beta, \alpha]$. By the proof of Theorem 9,

$$(P_t(A/A_{22}))^{-1} = A_{11}^{-1} + A_{11}^{-1}A_{12}(tI + A/A_{11})^{-1}A_{21}A_{11}^{-1}$$

$$= A_{11}^{-1} + A_{11}^{-1}A_{12}(I + t(A/A_{11})^{-1})^{-1}(A/A_{11})^{-1}A_{21}A_{11}^{-1}. $$

Because $(A/A_{11})^{-1}$ is an $M$-matrix, then $I + t(A/A_{11})^{-1}$ is an $M$-matrix. Moreover,

$$A_{11}^{-1}A_{12} \geq 0, \quad (A/A_{11})^{-1}A_{21}A_{11}^{-1} \geq 0.$$ 

Then

$$A_{11}^{-1}A_{12}(I + t(A/A_{11})^{-1})^{-1}(A/A_{11})^{-1}A_{21}A_{11}^{-1} \geq 0.$$ 

Hence, we have

$$(P_t(A/A[\beta]))^{-1} \geq A_{11}^{-1} = (A[\alpha])^{-1}. \quad \square $$

Theorem 16. Let $A = (a_{ij})$ be an $n \times n$ real matrix, $\phi \neq \alpha \subset (n)$, and define $\alpha' = (n) \setminus \alpha$. Then

1. If $A = (a_{ij})$ is an $N_0$-matrix, $A^{-1}[\alpha]$ and $A[\alpha]$ are all invertible, then

$$(A^{-1}[\alpha])^{-1} \leq A[\alpha] \quad \text{and} \quad A^{-1}[\alpha] \leq A[\alpha]^{-1};$$

2. If $A = (a_{ij})$ is an inverse $N_0$-matrix, $A^{-1}[\alpha]$ and $A[\alpha]$ are all invertible, then

$$(A^{-1}[\alpha])^{-1} \geq A[\alpha] \quad \text{and} \quad A^{-1}[\alpha] \geq A[\alpha]^{-1}. $$ 

Proof

1. It is known that

$$(A^{-1}[\alpha])^{-1} = A/A[\alpha'] = A[\alpha] - A[\alpha, \alpha']A[\alpha']^{-1}A[\alpha', \alpha].$$

Let $A = (a_{ij})$ be an $N_0$-matrix. Since

$$A[\alpha, \alpha'] \leq 0, \quad A[\alpha', \alpha] \leq 0, \quad A[\alpha']^{-1} \geq 0,$$

we have

$$(A^{-1}[\alpha])^{-1} \leq A[\alpha].$$

For an inverse $N_0$-matrix $A$, since $A^{-1}$ is an $N_0$-matrix, we have

$$A^{-1}[\alpha] \geq A[\alpha]^{-1}. $$

2. It is evident that the conclusion follows from part 1. \quad \square 

Acknowledgements

The authors would like to express their great thankfulness to the referees and Prof. Michael Neumann for their much constructive, detailed and helpful advice regarding revising this manuscript.

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