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Homotopy perturbation method to space–time fractional solidification in a finite slab

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A B S T R A C T

A mathematical model describing the space and time fractional solidification of fluid initially at its freezing temperature contained in a finite slab under the constant wall temperature is presented. The approximate analytical solution of this problem is obtained by the homotopy perturbation method. The results thus obtained are compared with exact solution of integer order ($\beta = 1, \alpha = 2$) and are good agreement. The problem has been studied in detail by considering different order time and space fractional derivatives. The temperature distribution and the moving interface position for different fractional order space and time derivatives are shown graphically. The model and the solution are the generalization of the previous works and include them as special cases.

1. Introduction

Melting and solidification process occur in numerous important areas of science, engineering and industry [1]. For Example freezing and thawing foods, production of steel, growing crystal for semiconductors, formation of monotectics, eutectics, chemical reactions and drug delivery are all involve either a moving freezing, moving melting or moving diffusion unknown front. The manufacturers are interested in controlling their processes so as to make moving boundary as smooth as possible. Many important physical processes that occur during melting or solidification have not been adequately studied and are not understood. Super computations with wrong models of some of the transport processes will yield meaningless results. For this reason interaction between those engaged in physical phenomena and analysis is much needed, because today many fundamental physical theories involve through numerical studies. The melting/solidification process is normally governed by heat conduction equation which is a combination of conservation equation of energy and the conduction law. In many one dimensional systems with total momentum conservation, the heat conduction equation does not obey the Fourier law and the heat conductivity depends on the system size [2]. Li and Wang [3,4] have found a simple formula which connects anomalous heat conductivity with anomalous diffusion. Povstenko [5] discusses a fractional heat conduction equation. Recently Jiang and Xu [6] obtained a time fractional heat conduction equation in the general orthogonal curvilinear co-ordinate and in cylindrical coordinate system.

The fractional (space–time) heat conduction equation is obtained from the standard heat conduction equation by replacing the second order space derivative with a Riesz-feller derivative of order $\alpha \in (0,2]$, and the first order time derivative with the Jumarie derivative of order $\beta \in (0,1]$. The moving boundary problems of fractional order are a special linear and non linear problem which is difficult to get the exact solution. A very few attention is occurred to apply fractional calculus

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Nomenclature

c	specific heat capacity
k	thermal conductivity
a	thermal diffusivity
R	position of fixed boundary
L	latent heat of fusion
r	position of solidified material
t	time
T	temperature distribution
T_0	reference temperature
T_f	freezing temperature
T_w	temperature at fixed boundary
S	Stefan number
x	dimensionless position

Greek symbols

α	space fractional
β	time fractional
$\lambda_0(t)$	position of freezing front
ρ	density
$\lambda(\tau)$	dimensionless solidification front defined in Eq. (7)
τ	dimensionless time defined in Eq. (7)
θ	dimensionless temperature defined in Eq. (7)

[7,8] with moving boundary condition by the authors due to the high nonlinearity of moving boundary problems. Approximate methods have been used to solve the moving boundary problems of the integer order e.g. the perturbation method [9–11] combination of variable method [12,13]. Many methods fails in fractional cases due to the fact that many of the useful properties of ordinary derivative are not known in case of fractional order derivative such as a clear geometric or physical meaning, product rules, chain rule and so on.

Hristov [14] solved half time fractional heat-diffusion sub-model using integral balance method. During study of a moving boundary problem, Hill [15] observed that the error in thickness of frozen layer obtained by integral balance method increases as Stefan number decreases. The Stefan number is strictly positive and signifies the importance of sensible heat relative to the latent heat. Integral balance method depends on the choice of a suitable approximate profile for the temperature and assumes a finite penetration depth, the temperature distribution away from the boundary remains unchanged until the diffusion front arrives. Therefore, The validity of the Integral balance method for small Stefan number and short time is restricted. The homotopy perturbation method was firstly presented by He [16–19] and applied to various nonlinear problem [20–25]. Many authors Odibat and Momani [26,27], Wang [28,29] and Ganji [30,31] also applied the homotopy perturbation method to nonlinear fractional equations which have nonlinear terms in the equations. Ganji et al. [32] solved a time-fraction generalized Hirota–Satsuma coupled K dV equation. Analytical study of Navier–Stokes Equation with Fractional Orders Using He's Homotopy Perturbation and Variational Iteration Methods done by Ali et al. [33]. Yaldirim [34] developed an Algorithm for solving the fractional nonlinear schrodinger equation. This Method is successfully applied to multi-order time fractional differential equations by Golbabai and Sayevand [35]. Li et al. [36] model a moving boundary problem and solve in terms of the Fox H functions. Li et al. [37] used homotopy perturbation method to solve time fractional moving boundary problems in case of drug delivery.

The mathematical model describing the space–time fractional solidification of fluid initially at its freezing temperature contained in a finite slab under the constant wall temperature have been considered and an approximate analytical solution is obtained by using Homotopy perturbation method. This Paper is a key step to use fractional calculus in freezing and melting process.

2. Mathematical model of the problem

A molten material initially at its freezing temperature is contained in a finite region slab length R. At the time greater than zero the boundary is cooled by imposing the boundary at temperature $T_w < T_f$. The liquid freezes and the solidification shell grow in a symmetric manner. Finite regions consists of two zones, the first zone $0 < x < \lambda_0(t)$ in which all liquid freeze and $\lambda_0(t) < x < R$ which contain molten. Two zones are separated by the solidification front $x = \lambda_0(t)$ which moves inward as time progresses. The Dynamics of freezing can be described by the space–time fractional heat conduction equation.

$$\frac{\partial^\beta T(r,t)}{\partial t^\beta} = a\frac{\partial^\alpha T(r,t)}{\partial r^\alpha} \quad 0 < \beta \leqslant 1 < \alpha \leqslant 2, \quad t > 0. \tag{1}$$

Initial condition

$$T(r,t) = T_f|_{t=0}. \tag{2}$$

The associated boundary conditions are

$$T(r,t) = T_w|_{r=0}. \tag{3}$$

The Energy balance at the solid–liquid interface yields

$$\frac{d^\beta \lambda_0(t)}{dt^\beta} = \frac{k}{\rho L}\frac{\partial T(r,t)}{\partial r}\bigg|_{r=\lambda_0(t)}, \tag{4}$$

$$T(r,t) = T_f|_{r=\lambda_0(t)}, \tag{5}$$

$$\lambda_0(t) = 0|_{t=0}. \tag{6}$$

Introducing the dimensionless variable and similarity criteria

$$\left.\begin{aligned} &x = \tfrac{r}{R}, \quad \lambda(\tau) = \tfrac{\lambda_0(t)}{R}, \quad S = \tfrac{c\Delta T}{L}, \quad c = \tfrac{k}{\rho a}, \\ &\tau = \left(\tfrac{k\Delta T}{\rho L R^\alpha}\right)^{\frac{1}{\beta}} t, \quad \theta = \tfrac{T-T_0}{\Delta T}, \end{aligned}\right\} \tag{7}$$

where $\Delta T = T_f - T_w$ and $T_0 = T_w$.

The system of Eqs. (1)–(6) reduce to the following form

$$S\frac{\partial^\beta \theta}{\partial \tau^\beta} = \frac{\partial^\alpha \theta}{\partial x^\alpha} \quad (0 < \beta \leqslant 1 < \alpha \leqslant 2), \tag{8}$$

$$\theta(x,\tau) = 1|_{\tau=0}, \tag{9}$$

$$\theta(x,\tau) = 0|_{x=0}. \tag{10}$$

The Energy balance at the solid–liquid interface yields

$${}_0D_\tau^\beta \lambda(\tau) = \frac{\partial \theta}{\partial x}\bigg|_{x=\lambda(\tau)}, \tag{11}$$

$$\theta(x,\tau) = 1|_{x=\lambda(\tau)}, \tag{12}$$

$$\lambda(\tau) = 0|_{\tau=0}. \tag{13}$$

3. Basic definition of fractional calculus

Recently, a New modified Riemann–Liouville fractional derivative is proposed by G. Jumarie (1993). Comparing with the classical caputo derivative, the definition of the fractional derivative is not required to satisfy higher integer-order derivative than α. Secondly, αth derivative of a constant is zero. For this merits, Jumarie's modified derivative was successfully applied in the probability calculus (Jumarie, 2006), Fractional Laplace problems (Jumarie, 2009a). Wu and He [38] use Jumarie fractional derivatives in fractal spacetime.

We give some basic definitions and properties of the fractional calculus theory which are used further in this paper:

Definition 3.1. A real function $f: R \to R,\ t \to f(t)$ denote a continuous (but not necessarily differentiable) function and let the partition $h > 0$ in the interval $[0,1]$. Jumarie derivative is defined through the fractional difference (Jumarie, 2009):

$$\Delta^\alpha f(t) = (FW - 1)^\alpha f(t) = \sum_0^\infty (-1)^k \binom{\alpha}{k} f[t + (\alpha - k)h], \tag{14}$$

where FWf (t) = f(t + h). Then the fractional derivative (Jumarie, 2009) is defined as the following limit.

$$f^{(\alpha)}(t) = \lim_{h\to 0}\frac{\Delta^\alpha[f(t) - f(0)]}{h^\alpha}. \tag{15}$$

This definition is close to the standard definition of derivatives, and as a direct result, the αth derivative of a constant, $0 < \alpha < 1$; is zero.

Definition 3.2. The Riemann–Liouville fractional integral operator of order $\alpha \geqslant 0$ is defined (Miller and Ross, 2003, Oldham and Spanier, 1999, Podlubry,1999) as

$$ {}_0I_t^{\alpha}h(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}h(\tau)d\tau, \quad (\alpha > 0), \tag{16}$$

$$ {}_0I_t^{0}h(t) = h(t), \tag{17}$$

where $\Gamma(z)$ is well-known Gamma function.

Definition 3.3. The modified Riemann–Liouville derivative (Jumarie, 2009) is defined as

$$ {}_0D_t^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t (t-\xi)^{n-\alpha}(f(\xi)-f(0))d\xi, \tag{18}$$

where $t \in [0,1]$, $n-1 \leqslant \alpha < n$ *and* $n \geqslant 1$.

The proposed modified Riemann–Liouville derivative as shown in Eq. (18) is strictly equivalent to Eq. (15). Mean-while, we would introduce some properties of the fractional modified Riemann–Liouville derivative in Eqs. (19) and (20).

(a) Fractional Leibniz product law

$$ {}_0D_t^{(\alpha)}(uv) = u^{(\alpha)}v + uv^{(\alpha)}. \tag{19}$$

(b) Fractional Leibniz Formulation

$$ {}_0I_t^{\alpha}D_t^{\alpha}f(t) = f(t) - f(0), \quad 0 < \alpha \leqslant 1. \tag{20}$$

Therefore, the integration by part can be used during the fractional calculus

$$ {}_0I_a^{\alpha}u^{(\alpha)}v = (uv)|_0^a - {}_0I_a^{\alpha}uv^{(\alpha)}. \tag{21}$$

Definition 3.4. Fractional derivative of compounded functions is defined as

$$ d^{(\alpha)}f \cong \Gamma(1+\alpha)df, \quad 0 < \alpha < 1. \tag{22}$$

Definition 3.5. The integral with respect to $(dx)^{\alpha}$ is defined as the solution of fractional differential equation

$$ dy \cong f(t)(dt)^{\alpha}, \quad t \geqslant 0, \quad y(0) = 0, \quad 0 < \alpha < 1. \tag{23}$$

Lemma 3.4. *Let $f(t)$ denote a continuous functions then the solution of Eq. (23) is defined as*

$$ y = \int_0^t f(\tau)(d\tau)^{\alpha} = \alpha\int_0^t (t-\tau)^{\alpha-1}f(\tau)d\tau, \quad 0 < \alpha \leqslant 1. \tag{24}$$

For example $f(t) = t^{\gamma}$ in Eq. (21) one obtains

$$ \int_0^t \tau^{\gamma}(d\tau)^{\alpha} = \frac{\Gamma(\alpha+1)\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}t^{\alpha+\gamma}, \quad 0 < \alpha \leqslant 1. \tag{25}$$

Definition 3.6. Assume that the continuous function $f : R \to R,\ t \to f(t)$ has a fractional derivative of order $k\alpha$, for any positive integer k and any α, $0 < \alpha \leqslant 1$; then the following equality holds, which is

$$ f(t+h) = \sum_{k=0}^{\infty}\frac{h^{\alpha k}}{\alpha k!}f^{(\alpha k)}(t), \quad 0 \leqslant t \leqslant 1, \quad 0 < \alpha \leqslant 1. \tag{26}$$

On making the substitution $h \to t$ and $t \to 0$ We obtain the fractional Mc-Laurin series

$$ f(t) = \sum_{j=0}^{\infty}\frac{t^{\alpha k}}{\alpha k!}f^{\alpha k}(0), \quad 0 < \alpha \leqslant 1. \tag{27}$$

4. Solution of the problem by HPM

The essential idea of this method is to introduce a homotopy parameter, say p, which takes the values from 0 to 1. When $p = 0$, the system of equation usually reduces to a sufficiently simplified form, which normally has simple solution. As p gradually increases to 1 the system goes through a sequence of deformation's the solution of each of which is close to that at the previous stage of deformation. Eventually at $p = 1$, the system takes the original form of the equation and the final stage of deformation gives the desired solution.

According to the homotopy perturbation method, we construct the following simple homotopy

$$(1-p)\frac{\partial^\alpha \theta}{\partial x^\alpha} + p\left[\frac{\partial^\alpha \theta}{\partial x^\alpha} - S_0 D_\tau^\beta \theta\right] = 0. \tag{28}$$

Or

$$\frac{\partial^\alpha \theta}{\partial x^\alpha} - pS_0 D_\tau^\beta \theta = 0, \tag{29}$$

where $p \in [0,1]$ is an embedding parameter. In case $p = 0$ Eq. (29) is an ordinary differential equation, ${}_0D_x^\alpha \theta = 0$, which is easy to solve and when $p = 1$ Eq. (29) turns out to be the original one. The basic concept behind the homotopy perturbation method is that the solution can be written as a power series in p for Eqs. (29) and (11).

$$\theta = \sum_{n=0}^{\infty} p^n \theta_n, \quad \lambda = \sum_{n=0}^{\infty} p^n \lambda_n. \tag{30}$$

The approximate solution of the original equations can be obtained by setting $p = 1$, i.e.

$$\theta = \sum_{n=0}^{\infty} \theta_n, \quad \lambda = \sum_{n=0}^{\infty} \lambda_n. \tag{31}$$

Substituting θ and λ from Eq. (30) into Eqs. (29) and (12), we obtain

$$\sum_{n=0}^{\infty} p^n \frac{\partial^\alpha \theta_n}{\partial x^\alpha} = S \sum_{n=0}^{\infty} p^{n+1} {}_0D_\tau^\alpha \theta_n, \tag{32}$$

$$\sum_{m=0}^{\infty} p^m \theta_m \left(\sum_{n=0}^{\infty} p^n \lambda_n, \tau\right) = 1. \tag{33}$$

The perturbation parameter p is both explicit and implicit parameter. The implicit part connect to the variable λ. In order to compare the coefficients of different powers of p, We need the explicit form of p, To do this, Taylor's series of $\theta_k(x,\tau)$ is used in a suitable neighborhood of a point (λ_0, τ). Thus $\theta_k(x,\tau)$ has a Taylor's series representation with respect to x as follow:

$$\theta_k(x,\tau) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^n \theta_k}{\partial x^n}\bigg|_{(\lambda_0,\tau)} (x-\lambda_0)^n, \quad k = 0,1,2,3\ldots \tag{34}$$

Applying this result in Eq. (33), we obtain

$$\sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \frac{p^j}{m!} \left(\sum_{n=1}^{\infty} p^n \lambda_n\right)^m \frac{\partial^m \theta_j}{\partial x^m} = 1 \quad (x = \lambda_0). \tag{35}$$

Similarly, moving boundary Eq. (11) becomes

$$\sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \frac{p^j}{m!} \left(\sum_{n=1}^{\infty} p^n \lambda_n\right)^m \frac{\partial^{m+1} \theta_j}{\partial x^{m+1}} = \sum_{n=0}^{\infty} p_0^n D_\tau^\beta \lambda_n \quad (x = \lambda_0). \tag{36}$$

Equating the terms with identical powers of p in (32), (35) and (36). We can obtain a series of equations of the form:

$$p^0: \quad \left.\begin{array}{l} {}_0D_x^\alpha \theta_0 = 0, \\ \theta_0(0,\tau) = 0, \\ \theta_0(\lambda_0,\tau) = 1, \\ \frac{\partial \theta_0}{\partial x} = {}_0^c D_\tau^\beta \lambda_0 \quad (x = \lambda_0), \\ \lambda_0(0) = 0, \end{array}\right\} \tag{37}$$

$$p^1: \quad \left.\begin{array}{l} {}_0D_x^\alpha = S_0 D_\tau^\beta \theta_0, \\ \theta_1(0,\tau) = 0, \\ \theta_1(\lambda_0,\tau) + \lambda_1(\tau)\frac{\partial \theta_0}{\partial x} = 0 \quad (x = \lambda_0), \\ \frac{\partial \theta_1}{\partial x} + \lambda_1(\tau)\frac{\partial^2 \theta_0}{\partial x^2} = {}_0D_\tau^\beta \lambda_1(\tau) \quad (x = \lambda_0), \\ \lambda_1(0) = 0. \end{array}\right\} \tag{38}$$

According to the first three equations of (37), we have

$$\theta_0 = \frac{x}{\lambda_0}. \tag{39}$$

Substituting it into the fourth equation of Eq. (37) we have

$$\frac{1}{\lambda_0} = {}_0D_\tau^\beta \lambda_0. \tag{40}$$

Considering the properties of fractional derivative and the initial condition of λ_0, we can assume

$$\lambda_0 = a_0\tau^\gamma, \tag{41}$$

where a_0 *and* γ are constants to be determined Substituting Eq. (41) into Eq. (40), and a_0 *and* γ can be obtained.

$$\theta_0 = \frac{x}{a_0\tau^\gamma}, \tag{42}$$

where

$$\gamma = \frac{\beta}{2}, \quad a_0 = \left[\frac{\Gamma(1-\frac{\beta}{2})}{\Gamma(1+\frac{\beta}{2})}\right]^{\frac{1}{2}}. \tag{43}$$

Substituting θ_0 and λ_0 into Eq. (38), the equations for θ_1 and λ_1 are obtained, Applying the method similar to the above process, we have

$$\theta_1 = \frac{Sa_1x^{1+\alpha}}{\tau^{\frac{3\beta}{2}}} - Sa_1a_0^\alpha\tau^{\frac{(\alpha-3)\beta}{2}}x - a_2a_0^{-2}\tau^{(\delta-\beta)}x, \tag{44}$$

$$\lambda_1(\tau) = a_2\tau^\delta, \tag{45}$$

where

$$a_1 = \frac{a_0^{-1}\Gamma(1-\frac{\beta}{2})}{\Gamma(1-\frac{3\beta}{2})\Gamma(\alpha+2)}, \tag{46}$$

$$a_2 = \frac{Sa_1a_0^\alpha\alpha}{\left[\frac{\Gamma(1+\delta)}{\Gamma(1+\delta-\beta)} + a_0^{-2}\right]}, \tag{47}$$

$$\delta = \frac{(\alpha-1)\beta}{2}. \tag{48}$$

Sequentially θ_i, λ_i, $i = 2,3,\ldots$ can be obtained.

Substituting θ_0, θ_1, λ_0 *and* λ_1 into Eq. (31) the first order approximate solution can be written as

$$\theta(x,\tau) = a_0^{-1}\tau^{-\frac{\beta}{2}}x + Sa_1\tau^{-\frac{3\beta}{2}}x^{1+\alpha} - Sa_1a_0^\alpha\tau^{\frac{(\delta-3)\beta}{2}}x - a_0^{-2}a_2\tau^{(\delta-\beta)}x, \tag{49}$$

$$\lambda(\tau) = a_0\tau^{\frac{\beta}{2}} + a_2\tau^{\frac{(\alpha-1)\beta}{2}}. \tag{50}$$

5. Numerical computation and discussion

The solution of this space–time fractional solidification problem will be discussed in detail by considering three particular cases:

Case 1: when $\alpha = 2$, $\beta = 1$ the governing Eqs. (8)–(13) degenerates into the standard heat conduction equation. In this case, the exact solution occurs in the form

$$\theta(x,\tau) = \frac{erf\left(\frac{x}{2}\sqrt{\frac{S}{\tau}}\right)}{erf\left(\frac{\lambda}{2}\sqrt{\frac{S}{\tau}}\right)}. \tag{51}$$

The whole liquid solidified at $\tau = 0.7$ when we consider $S = 1$. At this stage, the temperature distribution obtained by exact method and by HPM is approximately the same as shown in Fig. 1.

Case 2: when $\alpha = 2$, $0 < \beta \leqslant 1$ the governing equation reduces to time fractional heat conduction equation. Fig. 2 shows the variation on $\lambda(\tau)$ with τ for different values of β. As β increases, the time required for complete freezing increases. It is clear from Fig. 3, that when complete melts solidified, the temperature of solidified slab is higher for higher value of β.

Case 3: when $1 < \alpha \leqslant 2$, $\beta = 1$ the governing equation reduces to space fractional heat conduction equation. Fig. 4 shows the variation on $\lambda(\tau)$ with τ for different values of α. It is clear from Fig. 4 that as α increases, the time required for complete freezing decreases and that the temperature of the slab increases with α Fig. 5.

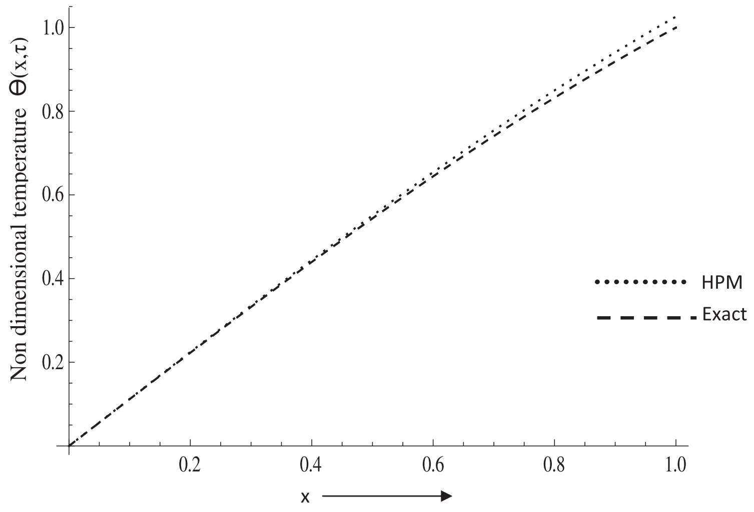

Fig. 1. $\theta(x,\tau)$ ver. x for ($\tau = 0.7, S = 1$ *and* $\alpha = 2, \beta = 1$).

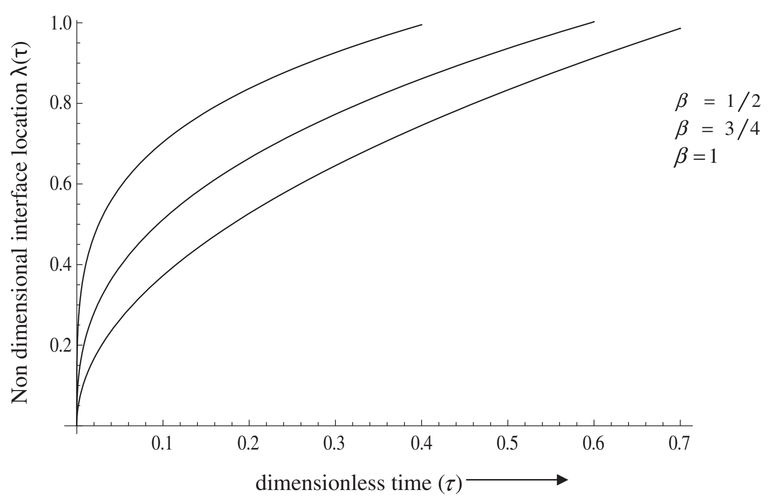

Fig. 2. $\lambda(\tau)$ ver. τ for different time fractional value (β), $S = 1$.

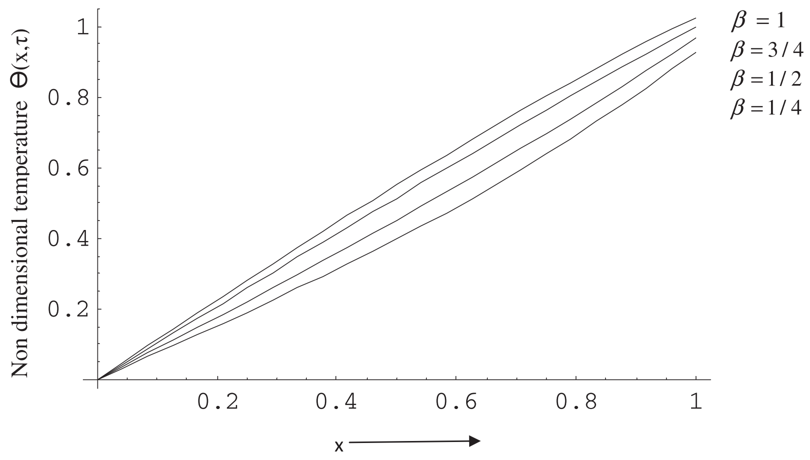

Fig. 3. $\theta(x,\tau)$ ver. x for different time fractional value and $S = 1$.

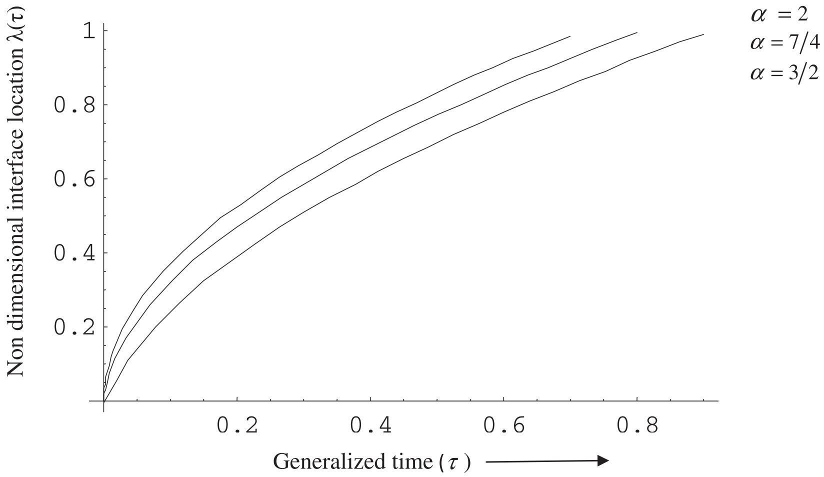

Fig. 4. $\lambda(\tau)$ ver. τ for different space fractional value (α) and $S = 1$.

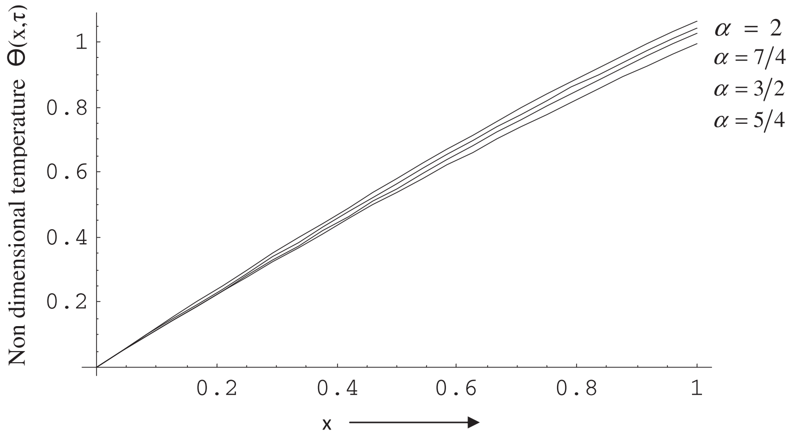

Fig. 5. $\theta(x,\tau)$ ver. x for different space fractional value and $S = 1$.

6. Conclusion

In this article, we have studied the effect of different order of fractional time and space derivatives on the freezing process. Our results includes all possible cases of diffusion occurring in freezing process like sub diffusion, normal diffusion and super diffusion. A normal diffusion leads to a normal heat conduction obeying the Fourier law. A super diffusion and sub diffusion leads to anomalous heat conduction equations. It has also been observed that time fractional β is more pronounced than space fractional α during freezing process. Advantage and Applications of new fractional heat conduction equation is described the sub diffusion process in many real physical systems such as highly ramified media in porous systems percolation clusters exact fractals the motion of the bead in a polymer semiconductors, and freezing of the material. Therefore, the study of fractional calculus in moving boundary problems would be of great interest to both scientists and engineers.

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