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Expansion Formulas. II. Variations on a Theme

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In "Expansion Formulas, I" [S. A. Joni, *J. Math. Anal. Appl.* **81** (1981)], it was shown that the Steffensen formula or polynomial sequences of binomial type gives rise to a method for generating a certain class of expansion identities. Special cases of this class of identities were studied by Carlitz [*SIAM J. Appl. Math.* **26** (1974), 431–436; **8** (1977), 320–336]. Since the Umbral calculus for polynomial sequences of binomial type has been generalized to encompass the theories of composition sequences [A. M. Garsia and S. A. Joni, *Comm. Algebra*, in press] and factor sequences [S. Roman and G.-C. Rota, *Adv. in Math.* **27** (1978), 95–188], we herein extend the results of part I to these two more general settings.

I. INTRODUCTION

In "Expansion Formulas. I. A General Method" [8], it has been shown that the diagonal multivariate Steffensen formula [5] for diagonal multivariate polynomial sequences of binomial type gives rise to a method for generating a certain class of expansion identities. Two of these identities, namely,

$$\sum_{n=0}^{\infty} (na+b)^n \frac{x^n}{n!} e^{-nax} = \frac{e^{bx}}{1-ax} \tag{1.1}$$

and

$$\sum_{n=0}^{\infty} [na+b]^n \frac{x^n}{n!} (1+x)^{-n(a+1)} = \frac{(1+x)^b}{1-ax} \tag{1.2}$$

(where $[\alpha]^n = \alpha(\alpha+1)\cdots(\alpha+n-1)$) and their multivariate analogs were studied by Carlitz [1, 2].

To date, two different generalizations of the theory of polynomial sequences of binomial type have been developed; both of these generalizations have "Steffensen-type" formula. The first generalization is known as the theory of *composition sequences* [6]. In the study of

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polynomials sequences of binomial type, the sequence $\{x^n/n!\}_{n \geqslant 0}$ plays a distinguished role. Essentially the theory of composition sequences arises when we assign this role to an arbitrary basis $\{b_n(x)\}_{n \geqslant 0}$, where for all n, the degree of $b_n(x)$ is equal to n. The second generalization is the theory of *special sequences* [10]. This theory has been recently developed by S. Roman; it began as an outgrowth of the study of factor sequences [11]. Roughly speaking, special sequences are sequences of formal series in the field of fractions of the polynomial algebra $K[1/x]$ which satisfy a certain "binomial-like" convolution. In this paper we shall extend the results of "Expansion Formula, I" to these two settings.

The Steffensen formula for polynomial sequences of binomial type has been shown to be equivalent to the Lagrange Inversion Theorem [4, 5], and indeed, this equivalence holds for both composition sequences and special sequences as well [6]. Therefore, in this paper we shall obtain our expansion formulas directly from the Lagrange Inversion Theorem.

We shall present only as much of the theories of composition and special sequences as is needed herein. The reader is referred to [6] and [10], respectively, for detailed expositions of these topics. For the sake of simplicity, we shall restrict our presentation to the univariate case. Multivariate techniques similar to those presented in "Expansion Formulas, I" [8] yield the analogous multivariate results. In Section II we give preliminary concepts and notational conventions. Section III is concerned with the expansion formula for composition sequences, and Section IV contains the development of this formula for special sequences.

II. Preliminaries

Let K denote a field of characteristic zero. We denote the algebra of polynomials in the variable α with coefficients in K by $K[\alpha]$, and the algebra of formal power series in the variable x with coefficients in K by $K[[x]]$. A formal power series $S(x)$ is said to be *multiplicatively invertible* in $K[[x]]$ if there exists a formal power series $R(x)$ such that

$$S(x)\,R(x) = 1. \tag{2.1}$$

The series $R(x)$ exists if and only if $S(0) \neq 0$; in this case we denote $R(x)$ by $S(x)^{-1}$. Functional composition of formal power series is a non-commutative associative operation that is well defined whenever all the series involved have zero constant terms. If $f(x)$ is a formal power series with $f(0) = 0$, we say that $f(x)$ is *invertible under functional composition* (or *compositionally invertible*) if there exists a formal power series $F(x)$ such that

$$f(F(x)) = F(f(x)) = x. \tag{2.2}$$

Note that (2.2) implies that $F(x)$ must have zero constant term. Moreover, it is well known and easy to prove that $F(x)$ exists if and only if the formal derivative of $f(x)$ evaluated at zero is non-zero, i.e., $f'(0) \neq 0$.

The field of fractions of $K[[x]]$, denoted by $K((x))$, is the field of all *formal x-laurent series*, where the function $h(x)$ is said to be a formal x-laurent series if it is of the form

$$h(x) = \sum_{n=k}^{\infty} h_n x^n, \tag{2.3}$$

where k is any integer. We shall say that the *index* of $h(x)$ equals k if k is the smallest integer such that the coefficient h_k of x^k in (2.3) is non-zero. Note that every formal power series is a formal x-laurent series, and if $f(x)$ is a formal power series of index k, then $f(x)^{-1}$ is a formal x-laurent series of index $-k$. The function $g(x)$ is said to be a *formal x^{-1}-laurent series* if $g(x)$ is of the form

$$g(x) = \sum_{n=-\infty}^{m} g_n x^n, \tag{2.4}$$

where m is any integer. We shall say that $g(x)$ is of *degree m* (or that the degree of $g(x)$ equals m) if m is the largest integer such that the coefficient g_m of x^m in (2.4) is non-zero. Finally, we shall say that $e(x)$ is a *formal series* if $e(x)$ is of the form

$$e(x) = \sum_{n=-\infty}^{\infty} e_n x^n. \tag{2.5}$$

Given a formal series $e(x)$ as in (2.5), we shall set for all n,

$$e(x)\,|_{x^n} = e_n. \tag{2.6}$$

That is, we shall use the symbol $e(x)\,|_{x^n}$ to denote the coefficient of the monomial x^n in $e(x)$. The *residue* of $e(x)$ is the coefficient of x^{-1} in $e(x)$, i.e.,

$$\operatorname{res}(e(x)) = e(x)\,|_{x^{-1}}. \tag{2.7}$$

There are many equivalent formulations of the Lagrange Inversion Theorem (see, for example, [3, 4, 6, 13]). We shall use the following formulation of this theorem in the subsequent sections of this paper.

THEOREM 2.1 (Lagrange Inversion Theorem). *Let $F(x)$ be a formal power series which is invertible under functional composition, and let $\Phi(x)$ denote any formal series. Then*

$$\Phi(x) = \sum_{n=-\infty}^{\infty} c_n(\Phi)\, F(x)^n, \tag{2.8}$$

where for all n,

$$c_n(\Phi) = \Phi(x)\, F'(x)(F(x)/x)^{-n-1}\,|_{x^n}. \tag{2.9}$$

Proof. Since by assumption, the index of $F(x)$ is equal to one, the sequence $\{F(x)^n\}_{n=-\infty}^{\infty}$ forms a basis for the vector space of formal series. By linearity, it is therefore sufficient to show that (2.9) holds for all $\Phi(x) = F(x)^m$, where m is an arbitrary integer. Thus, we need to show that for all m,

$$F'(x)\, F(x)^{m-n-1}\,|_{x^{-1}} = \delta_{n,m}, \tag{2.10}$$

where $\delta_{n,m}$ denotes the Kronecker δ function. If $m = n$, then a simple calculation gives that the residue of $F'(x)/F(x)$ is equal to one. Moreover, if $m \neq n$, then

$$\operatorname{res}(F'(x)\, F(x)^{m-n-1}) = \operatorname{res}\left(\frac{(F(x)^{m-n})'}{m-n}\right) = 0, \tag{2.11}$$

since the formal derivative of any formal x-laurent series must have zero residue. Thus (2.10) is established, and our proof is complete.

As an immediate consequence of Theorem 2.1, we have

COROLLARY 2.1. *Let $f(x)$ and $F(x)$ be a pair of formal power series with zero constant terms such that*

$$f(F(x)) = F(f(x)) = x. \tag{2.12}$$

Then, for all k,

$$f(x)^k\,|_{x^n} = F'(x)\, F(x)^{-n-1}\,|_{x^{-k-1}}. \tag{2.13}$$

Proof. Let $\Phi(x) = x^k$. Substituting $f(x)$ for x in both sides of (2.8) gives

$$(f(x))^k = \sum_{n=-\infty}^{\infty} c_n(x^k)\, x^n. \tag{2.14}$$

Formula (2.13) then follows immediately from formulas (2.9) and (2.14).

III. COMPOSITION SEQUENCES

A *polynomial sequence* $\{p_n(\alpha)\}_{n \geqslant 0}$ is a sequence of polynomials such that for all $n \geqslant 0$, $\deg p_n(\alpha) = n$. Let $\{b_n(\alpha)\}_{n \geqslant 0}$ denote a given, fixed polynomial sequence. A polynomial sequence $\{p_n(\alpha)\}_{n \geqslant 0}$ is said to be a $\{b_n\}$*-composition* *sequence* if there exists a formal power series $f(x)$ which is invertible under functional composition and

$$\sum_{n=0}^{\infty} p_n(\alpha)\, b_n(x) = \sum_{k=0}^{\infty} b_k(\alpha)\, f(x)^k. \tag{3.1}$$

If $F(x)$ is the compositional inverse of $f(x)$, and $\{p_n(\alpha)\}_{n\geqslant 0}$ satisfies (3.1), we shall say that $\{p_n(\alpha)\}_{n\geqslant 0}$ is the $\{b_n\}$-*basic sequence* for $F(x)$. Note that if for all $n \geqslant 0$, $b_n(\alpha) = \alpha^n/n!$, and $\{p_n(\alpha)\}_{n\geqslant 0}$ satisfies (3.1), then $\{p_n(\alpha)\}_{n\geqslant 0}$ is a polynomial sequence of binomial type [12]. Formula (3.1) is the generating function characterization of composition sequences. As in the theory of binomial type polynomials, an operator theoretic and a "binimial-like" convolution characterization of composition sequences is known [6].

Let $\lambda = \{\lambda_n\}_{n\geqslant 0}$ denote a sequence of non-zero constants in K. We shall work with $\{b_n\}$-composition sequences where for all $n \geqslant 0$,

$$b_n(\alpha) = \frac{\alpha^n}{\lambda_n}. \tag{3.2}$$

Such polynomial sequences are caled λ-*composition sequences*. Thus, $\{p_n(\alpha)\}_{n\geqslant 0}$ is a λ-composition sequence if there exists a compositionally invertible formal power series $g(x)$ such that

$$\sum_{n=0}^{\infty} p_n(\alpha) \frac{x^n}{\lambda_n} = \exp_\lambda(\alpha g(x)), \tag{3.3}$$

where for notational convenience, we have set

$$\exp_\lambda(x) = \sum_{n=0}^{\infty} \frac{x^n}{\lambda_n}. \tag{3.4}$$

In "Expansion Formulas, I," we were concerned with finding a generating function identity for the sequence $\{s_n(na)\}_{n\geqslant 0}$, where $\{s_n(\alpha)\}$ is a binomial-type sequence. The generalization of the operation of "substituting" na into $s_n(\alpha)$ needed for λ-composition sequences is as follows. For all non-negative n and k, let us set

$$\lambda_{n,k} = \sum_{\substack{v_1 + \cdots + v_n = k \\ v_i \geqslant 0}} \frac{\lambda_k}{\lambda_{v_1} \cdots \lambda_{v_n}} \tag{3.5}$$

and

$$\langle na\rangle^k = a^k \lambda_{n,k}. \tag{3.6}$$

For any polynomial $q(\alpha)$ where

$$q(\alpha)=\sum_{k=0}^{m} q_k \alpha^k, \tag{3.7}$$

we set

$$\begin{aligned} q\langle na\rangle &= \sum_{k=0}^{m} q_k \langle na\rangle^k \\ &= \sum_{k=0}^{m} \lambda_{n,k} q_k a^k. \end{aligned} \tag{3.8}$$

Our expansion formula for λ-composition sequences is as follows.

THEOREM 3.1. *Let* $\{p_n(\alpha)\}_{n\geqslant 0}$ *be the* λ*-composition sequence satisfying* (3.3). *Then*

$$\begin{aligned} &\sum_{n=0}^{\infty} p_n\langle na\rangle \frac{x^n}{\lambda_n} (\exp_\lambda(ag(x)))^{-n} \\ &= \frac{\exp_\lambda(ag(x))}{\exp_\lambda(ag(x)) - x(\exp_\lambda(ag(x)))'}. \end{aligned} \tag{3.9}$$

Proof. Let us set

$$F(x)=x(\exp_\lambda(ag(x)))^{-1}, \tag{3.10}$$

and

$$R(x)=\exp_\lambda(ag(x)).$$

Since $F(x)\,R(x)=x$, the product rule for differentiation gives

$$F'(x)\,R(x)=1-\frac{x(\exp_\lambda(ag(x)))'}{\exp_\lambda(ag(x))}. \tag{3.12}$$

If we set

$$\Phi(x)=\frac{\exp_\lambda(ag(x))}{\exp_\lambda(ag(x)) - x(\exp_\lambda(ag(x)))'} \tag{3.13}$$

and apply the Lagrange Inversion Theorem (see formulas (2.8) and (2.9)), we see that (3.9) is established once we have shown that for all $n\geqslant 0$,

$$p_n\langle na\rangle = (\exp_\lambda(ag(x)))^n\,|_{x^n/\lambda_n}. \tag{3.14}$$

For convenience, set for all $0 \leqslant k \leqslant n$,

$$p_{n,k} = p_n(\alpha) \,|_{\alpha^k/\lambda_k}. \tag{3.15}$$

By the definition (3.3) of $\{p_n(\alpha)\}_{n \geqslant 0}$, we have

$$p_{n,k} = g(x)^k \,|_{x^n/\lambda_n}. \tag{3.16}$$

Moreover, using (3.5),

$$(\exp_\lambda(ag(x)))^n = \sum_{k=0}^{\infty} \frac{g(x)^k}{\lambda_k} \lambda_{n,k} a^k. \tag{3.17}$$

Therefore, taking the coefficient of x^n/λ_n on both sides of (3.17) and using (3.16) and then (3.8), we see that

$$\begin{aligned}(\exp_\lambda(ag(x)))^n \,|_{x^n/\lambda_n} &= \sum_{k=0}^{n} \frac{p_{n,k}}{\lambda_k} a^k \lambda_{n,k} \\ &= p_n\langle na\rangle,\end{aligned}$$

which is the desired formula (3.14), and our proof is complete.

Remark 3.1. A polynomial sequence $\{s_n(x)\}$ is said to be a *Sheffer λ-composition sequence* if there exists a multiplicatively invertible formal power series $S(x)$ and a compositionally invertible formal power series $g(x)$ such that

$$\sum_{n=0}^{\infty} s_n(a) \frac{x^n}{\lambda_n} = S(x) \exp_\lambda(ag(x)). \tag{3.18}$$

Thus, λ-composition sequences are Sheffer λ-composition sequences with $S(x) = 1$. Theorem 3.1 for Sheffer λ-composition sequences becomes

$$\begin{aligned}&\sum_{n=0}^{\infty} s_n\langle na\rangle \frac{x^n}{\lambda_n} (\exp_\lambda(ag(x)))^{-n} \\ &\qquad = \frac{S(x)\exp_\lambda(ag(x))}{\exp_\lambda(ag(x)) - x(\exp_\lambda(ag(x)))'},\end{aligned} \tag{3.19}$$

and the proof follows the same arguments as those given above.

EXAMPLE 1. As previously remarked, if for all $n \geqslant 0$, $\lambda_n = n!$, then Theorem 3.1 is the univariate statement of Corollary 3.1 in "Expansion Formulas, I" [8].

EXAMPLE 2. Let $\lambda_n = 1$ for all $n \geqslant 0$. In this case, a λ-composition sequence $\{p_n(\alpha)\}$ is said to be a polynomial sequence of *newjonian type* [6, 9]. These sequences satisfy the convolution identity

$$\frac{xp_n(x) - yp_n(y)}{x - y} = \sum_{k=0}^{n} p_k(x)\, p_{n-k}(y). \tag{3.20}$$

Examples of such sequences include $\{x^n\}$, $\{x(x+\beta)^{n-1}\}$, $\{r_n(x)\}$, and $\{s_n(x)\}$ where for all $n \geqslant 0$,

$$r_n(x) = \sum_{v=1}^{n} \binom{2n-v}{n} \frac{n}{2n-v} \beta^{n-v} x^v,$$

$$s_n(x) = \sum_{v=1}^{n} \frac{v}{(n-v)!} x^v (-\beta n)^{n-v-1},$$

and β is an arbitrary element of K.

Note that

$$\exp_\lambda(\alpha) = \frac{1}{1-\alpha}, \tag{3.21}$$

and it is well known [3] that

$$\begin{aligned} \lambda_{n,k} &= \sum_{\substack{v_1 + \cdots + v_n = k \\ v_i \geqslant 0}} 1 \\ &= \binom{n+k-1}{k}. \end{aligned} \tag{3.22}$$

Thus, our "generalized powers," $\langle na \rangle^k$, are

$$\langle na \rangle^k = \prod_{j=1}^{k} a \left(\frac{n+j-1}{j} \right). \tag{3.23}$$

It is interesting to note that the formal techniques developed above seem to give "automatic" proofs of known identities. We close this section with one very simple example. Let $\lambda_n = 1$ for all $n \geqslant 0$, and let $g(x) = x$. Theorem 3.1, formula (3.9) gives

$$\sum_{n=0}^{\infty} \binom{2n-1}{n} (ax)^n (1-ax)^n = \frac{1-ax}{1-2ax}. \tag{3.24}$$

Therefore, equating coefficients of x^n on both sides of (3.24) we see that for all $m \geqslant 1$,

$$2^{m-1} = \sum_{n=\lceil m/2 \rceil}^{m} \binom{2n-1}{n} \binom{n}{m-n} (-1)^{m-n} \tag{3.25}$$

where $\lceil m/2 \rceil$ denotes the least integer k such that $k \geqslant m/2$.

IV. Special Sequences

The elegant theory of special sequences has been recently developed by Roman [10]. Here we shall show how the expansion formulas generalize to this setting.

Let us set, for all integers n,

$$\begin{aligned} \omega_n &= n! && \text{if} \quad n \geqslant 0, \\ &= \frac{(-1)^n}{(|n|-1)!} && \text{if} \quad n < 0, \end{aligned} \tag{4.1}$$

and

$$\exp_\omega(x) = \sum_{n=-\infty}^{\infty} \frac{x^n}{\omega_n}. \tag{4.2}$$

A sequence $\{q_n(x)\}_{n=-\infty}^{\infty}$ is said to be a *special sequence* [10][1] if

for all n, $q_n(x)$ is a formal x^{-1}-laurent series of degree n,

and (4.3)

there exists a formal power series $g(x)$ which is invertible under functional composition such that

$$\sum_{n=-\infty}^{\infty} q_n(\alpha) \frac{x^n}{\omega_n} = \exp_\omega(\alpha g(x)). \tag{4.4}$$

If $\{q_n(\alpha)\}_{n=-\infty}^{\infty}$ is a special sequence, we shall set, for all n,

$$q_n(\alpha) = \sum_{k=-\infty}^{n} q_{n,k} \frac{\alpha^k}{\omega_k}, \tag{4.5}$$

[1] Note that Roman's definition of special sequences is more general than the one we give here. We restrict our definition for the sake of the simplicity of the results. In addition, the extension of these results to Sheffer special sequences is as in Remark 3.1, and will not be repeated here.

and define the polynomial sequence $\{\tilde{q}_n(\alpha)\}_{n\geqslant 0}$ by setting

$$\tilde{q}_n(\alpha)=\sum_{k=0}^{n} q_{n,k}\frac{\alpha^k}{\omega_k}. \tag{4.6}$$

Thus, for $n\geqslant 0$, $\tilde{q}_n(\alpha)$ is the "polynomial part" of $q_n(\alpha)$.

THEOREM 4.1. *Let* $\{q_n(\alpha)\}_{n=-\infty}^{\infty}$ *be the special sequence satisfying* (4.4), *and let* $\{\tilde{q}_n(\alpha)\}_{n\geqslant 0}^{\infty}$ *be as in* (4.6). *Then*

$$\sum_{n=0}^{\infty}\tilde{q}_n(\alpha)\frac{x^n}{n!}=e^{\alpha g(x)}. \tag{4.7}$$

Proof. By formula (4.4), for all $k\leqslant n$, we have

$$q_{n,k}=g(x)^k\,|_{x^n/\omega_n}. \tag{4.8}$$

Therefore, using (4.6) we have that for all $n\geqslant 0$,

$$\begin{aligned}\tilde{q}_n(\alpha)&=\sum_{k=0}^{n}\frac{\alpha^k}{\omega_k}\left(g(x)^k\,|_{x^n/\omega_n}\right)\\&=\left(\sum_{k=0}^{\infty}\frac{\alpha^k}{\omega_k}g(x)^k\right)\Bigg|_{x^n/\omega_n}\end{aligned} \tag{4.9}$$

Whence, multiplying both sides of (4.9) by x^n/ω_n and summing over all $n\geqslant 0$ gives

$$\sum_{n=0}^{\infty}\tilde{q}_n(\alpha)\frac{x^n}{n!}=\sum_{k=0}^{\infty}g(x)^k\frac{\alpha^k}{k!}=e^{\alpha g(x)}.$$

Note that formula (4.7) is equivalent to stating that the polynomial sequence $\{q_n(\alpha)\}_{n\geqslant 0}$ is of binomial type [12]. We shall set, for all n,

$$\underline{(a+b)}^n=\sum_{k=0}^{\infty}\frac{\omega_n}{\omega_k\omega_{n-k}}a^kb^{n-k}. \tag{4.10}$$

Note that (4.10) is in general *not* symmetric in a and b, and that for $n<0$, it agrees with the usual definition of $(a+b)^n$. If $f(x)$ is a formal x^{-1}-laurent series, say,

$$f(x)=\sum_{k=-\infty}^{m} f_k x^k,$$

then we set

$$f(\underline{a+b}) = \sum_{k=-\infty}^{m} f_k(\underline{a+b})^k. \tag{4.11}$$

THEOREM 4.2. *The sequence* $\{q_n(\alpha)\}_{n=-\infty}^{\infty}$ *is a special sequence if and only if for all* a, b, *and* n,

$$q_n(\underline{a+b}) = \sum_{k=0}^{\infty} \tilde{q}_k(a)\, q_{n-k}(b) \frac{\omega_n}{\omega_k \omega_{n-k}}. \tag{4.12}$$

Proof. If $\{q_n(\alpha)\}$ satisfies (4.4), then using (4.5) and (4.11) we have

$$\begin{aligned} q_n(\underline{a+b}) &= \sum_{k=-\infty}^{\infty} \sum_{l=0}^{\infty} \frac{a^l}{\omega_l} \frac{b^{k-l}}{\omega_{k-l}} g(x)^k \,|_{x^n/\omega_n} \\ &= e^{ag(x)} \exp_\omega(bg(x)) \,|_{x^n/\omega_n}. \end{aligned} \tag{4.13}$$

Using (4.4) and (4.7), we have

$$e^{ag(x)} \exp_\omega(bg(x)) \,|_{x^n/\omega_n} = \sum_{k=0}^{\infty} \tilde{q}_k(a)\, q_{n-k}(b) \frac{\omega_n}{\omega_k \omega_{n-k}}, \tag{4.14}$$

which combined with (4.13) gives (4.12). The converse follows by reversing the above argument and setting $a=0$ in (4.13) whence, our proof is complete.

Our expansion formula for special sequences is as follows.

THEOREM 4.3. *Let* $\{q_n(x)\}_{n=-\infty}^{\infty}$ *be the special sequence satisfying* (4.4). *Then*

$$\sum_{n=-\infty}^{\infty} q_n(\underline{na+b}) \frac{(xe^{-ag(x)})^n}{\omega_n} = \frac{\exp_\omega(bg(x))}{1-xag'(x)}. \tag{4.15}$$

Proof. Let us set $F(x) = xe^{-ag(x)}$ and $R(x) = e^{ag(x)}$. Since $F(x)\,R(x) = x$, the product rule for differentiation gives

$$F'(x)\,R(x) = 1 - xag'(x). \tag{4.16}$$

Applying the Lagrange Inversion Theorem (see formulas (2.8) and (2.9)) with

$$\Phi(x) = \frac{\exp_\omega(bg(x))}{1-xag'(x)},$$

we see that our theorem is established once we have shown that for all n,

$$q_n(\underline{na+b}) = \exp_\omega(bg(x))\, e^{nag(x)}\,|_{x^n/\omega_n}. \tag{4.17}$$

But this is precisely the content of Theorem 4.2, and our result is established.

The special sequence analog. of Carlitz's formula (1.1) is obtained by setting $g(x) = x$. In this case we have

$$\sum_{n=-\infty}^{\infty} (\underline{na+b})^n \frac{x^n}{\omega_n} e^{-nax} = \frac{\exp_\omega(bg(x))}{1-ax}. \tag{4.18}$$

To obtain the special sequence analog of Carlitz's formula (1.2) we shall use

COROLLARY 4.1. *Let* $\{q_n(x)\}_{n=-\infty}^{\infty}$ *be the special sequence satisfying* (4.4), *and let* $h(x)$ *be the compositional inverse of* $xg'(x)$. *Then*

$$\sum_{n=-\infty}^{\infty} q_n(\underline{na+b}) \frac{(h(x)\, e^{-ag(h(x))})^n}{\omega_n} = \frac{\exp_\omega(bg(h(x)))}{1-ax}. \tag{4.19}$$

The *upper factorial* special sequence is the special sequence satisfying (4.4) with $g(x) = -\log(1-x)$ (see [11]). We denote this sequence by $\{[a]^n\}_{n=-\infty}^{\infty}$, and have $[a]^n = a(a+1)\cdots(a+n-1)$. The special sequence analog of Carlitz's formula (1.2) follows from (4.19) with $g(x) = -\log(1-x)$, $h(x) = x(1+x)$ and $g(h(x)) = \log(1+x)$. Thus we have

$$\sum_{n=-\infty}^{\infty} [\underline{na+b}]^n \frac{x^n}{\omega_n} (1+x)^{-n(a+1)} = \frac{\exp_\omega(b\log(1+x))}{1-ax}. \tag{4.20}$$

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