\begin{abstract}
Motivated from Wong [M.W. Wong, \(L^p\)-Boundedness of localization operators associated to left regular representations, Proc. Amer. Math. Soc. 130 (2002) 2911–2929], an \(L^p\)-boundedness of localization operators associated to Bessel’s left regular representations of locally compact Hausdorff group is obtained and an application of Bessel wavelet multipliers is investigated by using Zemanian theory of Hankel transformation.

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\textit{Keywords:} \(L^p\)-Space; Bessel operator; Hankel transformation; Localization operator; Hausdorff group
\end{abstract}

\section{Introduction}

The theory of Hankel transformation

\[
(h_\mu \phi)(x) = \int_0^\infty (xy)^{1/2} J_\mu(xy) \phi(y) dy, \quad \mu \geq -1/2,
\]  

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where \(0 < y < \infty\) and \(J_\mu\) is the Bessel function of the first kind and of order \(\mu\), has been extended by Zemanian [5] to the distributions belonging to \(H'_\mu(I)\) the dual of test function space \(H_\mu(I)\) which is defined as follows:

The space \(H_\mu(I)\) consists of all complex valued infinitely differentiable functions \(\phi\) on \(I = (0, \infty)\) such that

\[
y_{\mu,k}^{n}(\phi) = \sup_{x\in I} x^{n} \left( x^{-1} \frac{d}{dx} \right)^{k} x^{-\mu-1/2} \phi(x) < \infty
\]

for all \(n, k \in \mathbb{N}_0\).

Let \(f \in H'_\mu(I)\) and \(\phi \in H_\mu(I)\). Then the generalized Hankel transformation \(h'_\mu f\) is defined by

\[
\langle h'_\mu f, \phi \rangle = \langle f, h_\mu \phi \rangle, \quad \phi \in H_\mu(I).
\]

It was shown by Zemanian that the Hankel transformation \(h_\mu\) is an automorphism on the space \(H_\mu(I)\) and the generalized Hankel transformation \(h'_\mu\) is an automorphism on \(H'_\mu(I)\).

Now we recall the definition of pseudo-differential operators which was studied in [2]. The pseudo-differential operator associated with a symbol \(a(x, \xi)\) is defined by

\[
(h_\mu, a\phi)(x) = \int_{0}^{\infty} (x\xi)^{1/2} J_\mu(x\xi) a(x, \xi)(h_\mu \phi)(\xi) d\xi
\]

for \(\phi \in H_\mu(I)\), where the symbol \(a(x, \xi)\) is defined as follows:

The function \(a(x, \xi) : C^\infty(I \times I) \to \mathbb{C}\) belongs to the class \(H^m\) if and only if for each \(q \in \mathbb{N}_0\) there exists \(D = D_{\alpha, m, q, \beta}\) such that

\[
(1 + x)^q \left| \left( x^{-1} \frac{d}{dx} \right)^{\beta} \left( \xi^{-1} \frac{d}{d\xi} \right)^{\alpha} a(x, \xi) \right| \leq D(1 + \xi)^{m-2\alpha},
\]

where \(m\) is a fixed real number.

For \(1 \leq p < \infty\), we define \(L^p_\mu(I)\) as the Banach-space of measurable functions \(f\) on \(I\) such that

\[
\|f\|_p = \left( \int_{0}^{\infty} |f(x)|^p d\sigma(x) \right)^{1/p},
\]

where \(d\sigma(x) = [2^\mu \Gamma(\mu + 1)]^{-1} x^{2\mu+1} dx\).

Let \(\Delta(x, y, z)\) denote the area of a triangle with sides \(x, y, z\) if such a triangle exists. For fixed \(\mu \geq -\frac{1}{2}\), set

\[
D(x, y, z) = 2^{3\mu-\frac{1}{2}}(\pi)^{-\frac{1}{2}} \left[ \Gamma(\mu + 1) \right]^{2} \left[ \Gamma\left( \mu + \frac{1}{2} \right) \right]^{-1} (xyz)^{-2\mu} \times \left[ \Delta(x, y, z) \right]^{2\mu-1},
\]

if \(\Delta\) exists and zero otherwise. Then \(D(x, y, z) \geq 0\) and that \(D(x, y, z)\) is symmetric in \(x, y, z\).
Let \( f \in L^1_{\mu}(I) \), \( g \in L^p_{\mu}(I) \) and define the Hankel translation of \( f \in L^1_{\mu}(I) \) by

\[
(\tau_x f)(y) = \int_0^\infty f(z) D(x, y, z) d\sigma(z) \quad (x, y, z \in I),
\]

(1.8)

where

\[
D(x, y, z) = \int_0^\infty J(xt) J(yt) J(zt) d\sigma(t),
\]

\[
J(x) = 2^{\mu + 1} x^{-\mu} J_\mu(x).
\]

(1.9)

The Hankel convolution is defined by

\[
(f \ast g)(x) = \int_0^\infty f(x, y) g(y) d\sigma(y), \quad x \in I.
\]

(1.10)

The Hankel convolution satisfies the following norm inequalities:

\[
\| f \ast g \|_{L^p_{\mu}(I)} \leq \| f \|_{L^1_{\mu}(I)} \| g \|_{L^p_{\mu}(I)} \quad \text{and}
\]

\[
\| f \ast g \|_{L^r_{\mu}(I)} \leq \| f \|_{L^p_{\mu}(I)} \| g \|_{L^q_{\mu}(I)}
\]

(1.11)

(1.12)

for

\[
\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1.
\]

(1.13)

By using the theory of Fourier transformation, the \( L^p \)-boundedness result of the classical pseudo-differential operators was introduced by Wong [3] and others.

Motivated from these results, Pathak and Upadhyay [1] studied an \( L^p_{\mu} \)-boundedness pseudo-differential operator associated with Bessel operator \( S_{\mu} = \frac{d^2}{d\xi^2} + \frac{1-4\mu^2}{4\xi^2} \). It was shown, under certain condition, that the pseudo-differential operator \( h_{\mu, a} : L^p_{\mu}(I) \to L^p_{\mu}(I) \) is a bounded linear operator for \( 1 < p < \infty \). Recently an \( L^p \)-boundedness result for the localization operators associated to left regular representation of locally compact and Hausdorff group for \( 1 \leq p < \infty \) has been investigated by Wong [4] and an application to wavelet multipliers was given. Using the properties of Hankel transform and Hankel convolution transform our main aim in this paper is to expose the localization operators associated to left Bessel regular representations of locally and compact Hausdorff group and give an application to Bessel wavelet multipliers. For this we restate Theorems 3.1 and 3.2 from [1] which are useful for our further investigations.

**Theorem 1.1.** Let \( \theta \in C^k(I), \ k \geq 1, \) be such that there is a positive constant \( B_\alpha \) for which

\[
\left| \left( \xi^{-1} \frac{d}{d\xi} \right)^\alpha \theta(\xi) \right| \leq B_\alpha (1 + \xi)^{-2\alpha}, \quad \alpha \leq k/2.
\]

(1.14)
If
\[ f(x) = \int_{0}^{\infty} (x\xi)^{\mu} J_{\mu}(x\xi) \xi^{\frac{\mu}{2} + \frac{1}{2}} \theta(\xi) \, d\xi, \] (1.15)

then \( f \in L_{\mu}^{p}(I) \) for \( 1 \leq p < \infty \) and \( \mu \geq -\frac{1}{2} \).

**Theorem 1.2.** Let \( \mu \) be the same as in Theorem 1.1; then for \( 1 \leq p < \infty \) there exists a positive constant \( C = C(p, \mu) \) such that
\[ \| h_{\mu, \theta} \phi \|_{L_{\mu}^{p}(I)} \leq C \| \phi \|_{L_{\mu}^{p}(I)}, \quad \phi \in H_{\mu}(I). \] (1.16)

From [4], we assume that \( G \) is a locally compact and Hausdorff group on which Haar measure is denoted by \( \nu \). Let \( X \) be an infinite dimensional, separable complex Hilbert space in which the inner product and the norm are denoted by \( \langle \cdot, \cdot \rangle \) and \( \| \cdot \| \). Let \( \pi : G \to B(X) \) be an irreducible and unitary representation of \( G \) on \( X \) such that there exists a non-zero element \( \phi \) in \( X \) for which
\[ \int_{G} |\langle \phi, \pi(g)\phi \rangle|^{2} \, d\nu(g) < \infty, \] (1.17)

where \( \pi_{g}\phi \) is defined as
\[ (\pi_{g}\phi)(x) = \int_{0}^{\infty} \phi(t) D(x, g, t) \, d\nu(t). \] (1.18)

Then the representation \( \pi \) of \( G \) on \( X \) is called left Bessel square, integrable regular representation. If \( \phi \in X, \| \phi \|_{L_{\mu}^{2}(I)} = 1 \) and (1.18) is valid, then \( \phi \) is called an admissible wavelet for the square integrable representation \( \pi : G \to B(X) \) of \( G \) on \( X \) and we define the constant \( C_{\phi} = \int_{G} |\langle \phi, \pi_{g}\phi \rangle|^{2} \, d\nu(g) \).

In this paper an \( L_{\mu}^{p} \)-boundedness of localization operators associated to Bessel left regular representation is investigated and the properties of Bessel left regular representation are obtained by using the theory of Hankel transformation.

An \( L_{\mu}^{p} \)-boundedness of Bessel wavelet transform is exposed and it is shown how the Bessel wavelet transform can be expressed in form of pseudo-differential operators with symbol \( \sigma(\xi) \) associated with Bessel operator \( S_{\mu} \). Finally we show that the Bessel wavelet multipliers can be seen as localization operators when the underlying group is taken to be an additive group \( G \) and we find that the wavelet multipliers is a bounded linear operator from \( L_{\mu}^{p}(I) \) into \( L_{\mu}^{p}(I) \) for \( 1 \leq p < \infty \).

### 2. \( L_{\mu}^{p} \)-Boundedness and \( L_{\mu}^{\infty} \)-boundedness

In this section we study \( L_{\mu}^{p} \)-boundedness and \( L_{\mu}^{\infty} \)-boundedness of localization operators associated with left Bessel regular representation.
Theorem 2.1. Let $G$ be a unimodular, locally compact Hausdorff group on which the left Haar measure is denoted by $\sigma_v$. Let $\pi : G \to B(L^p_\mu(G))$ be the Bessel left regular representation of $G$ on $L^p_\mu(G)$, $1 \leq p < \infty$, i.e., $(\pi_g \phi)(h) = \int_0^\infty \phi(t)D(h, g, t)\,d\sigma_v(t)$, $g, h \in G$ for all $\phi \in L^p_\mu(G)$. Let $\phi \in \bigcap_{1 \leq p < \infty} L^p_\mu(G)$ be such that $\|\phi\|_{L^2_\mu(G)} = 1$. Then

$$C_{\phi} = \int_G |\langle \phi, \pi_g \phi \rangle|^2 \,d\sigma_v(g) \leq \|\phi\|_{L^2_\mu(G)}^2. \quad (2.1)$$

Proof. Let $(\pi_g \phi)(x)$ be the Bessel left regular representation of $G$ on $L^p_\mu(G)$ which is defined by

$$(\pi_g \phi)(x) = \int_0^\infty \phi(t)D(x, g, t)\,d\sigma_v(t).$$

Therefore

$$\langle \phi, \pi_g \phi \rangle = \Bigg\langle \phi, \int_0^\infty \phi(t)D(x, g, t)\,d\sigma_v(t) \Bigg\rangle$$

$$= \left( \int_0^\infty \phi(x) \left( \int_0^\infty \phi(t)D(x, g, t)\,d\sigma_v(t) \right) \,d\sigma_v(x) \right)$$

$$= \int_0^\infty \phi(x, g)\phi(x)\,d\sigma_v(x)$$

$$= (\phi \# \phi)(g).$$

Hence

$$\langle \phi, \pi_g \phi \rangle = (\phi \# \phi)(g). \quad (2.2)$$

Now,

$$C_{\phi} = \int_G |\langle \phi, \pi_g \phi \rangle|^2 \,d\sigma_v(g) = \int_G |(\phi \# \phi)(g)|^2 \,d\sigma_v(g)$$

$$= \int_G \|h_\mu(\phi \# \phi)(\xi)\|^2 \,d\sigma_v(\xi) = \int_G \|h_\mu(\phi)(\xi)\| \,d\sigma_v(\xi)$$

$$= \int_G \|h_\mu(\phi)(\xi)\|^2 \,d\sigma_v(\xi) \leq \int_G \|h_\mu(\phi)(\xi)\|^2 \,d\sigma_v(\xi)$$

$$= \|\phi\|_{L^2_\mu(G)}^2 \int_G |\phi(g)|^2 \,d\sigma_v(g) \leq \|\phi\|_{L^2_\mu(G)}^2 \quad \text{for } \|\phi\|_{L^2_\mu(G)}^2 = 1. \quad \Box$$
Remark 1. In order to obtain an explicit formula for $C_\phi$, we assume that $G$ is a second countable, unimodular, type I, locally compact and Hausdorff group. Let $\hat{G}$ be the set of all equivalence classes of irreducible and unitary representations of $G$ on $L^2_{\mu}(G)$. Then by Plancherel’s theorem, we have

$$C_\phi = \int_G \left| \langle \phi, \pi_g \phi \rangle \right|^2 d\sigma_\nu(g).$$

Using (2.2), we get

$$C_\phi = \int_G \left| \langle \phi \# \phi^* \rangle(g) \right|^2 d\sigma_\nu(g) = \int_{\hat{G}} t_r \left\{ \langle h_\mu \psi \rangle(\xi)(h_\mu \psi)^*(\xi) \right\} d\sigma_\omega(\xi), \tag{2.3}$$

where $\phi^*$ is defined by

$$\phi^*(g) = \overline{\phi(g^{-1})}, \quad g \in G,$$

$$\psi(g) = \langle \phi \# \phi^* \rangle(g), \quad g \in G. \tag{2.4}$$

$t_r\{\ldots\}$ is the trace of trace class operator $\{\ldots\}$ and $\{\ldots\}^*$ denotes the adjoint of the bounded linear operator $\{\ldots\}$. Now

$$\langle h_\mu \psi \rangle(\xi) = (h_\mu \phi)^*(\xi)(h_\mu \phi)(\xi), \quad \xi \in \hat{G}. \tag{2.5}$$

From (2.3) and (2.4) we get

$$C_\phi = \int_{\hat{G}} t_r \left\{ \left| (h_\mu \phi)(\xi) \right|^4 \right\} d\sigma_\omega(\xi), \tag{2.6}$$

where

$$\left| (h_\mu \phi)(\xi) \right| = \left( (h_\mu \phi)^*(\xi)(h_\mu \phi)(\xi) \right)^{1/2}. \tag{2.7}$$

Therefore, from (2.6) and (2.7) we have

$$C_\phi = \int_{\hat{G}} \left\| (h_\mu \phi)(\xi) \right\|^4_{S_4} d\sigma_\omega(\xi), \tag{2.8}$$

where $\| \|_{S_4}$ is the norm in the Schatten–von Neumann class $S_4$. Now, we can express (2.8) in the following form:

$$C_\phi = \int_{\hat{G}} \left\| (h_\mu \phi) \right\|^4_{L^4_\mu(\hat{G}, S_4)} d\sigma_\omega(\xi), \tag{2.9}$$

where $L^4(\hat{G}, S_4)$ is the Banach space of all $S_4$-valued functions $f$ on $\hat{G}$ for which

$$\int_{\hat{G}} \| f(\omega) \|_{S_4}^4 d\sigma_\nu(\xi) < \infty.$$
and the norm \( \| f \|_{L^4(\hat{G}, S_4)} \) is given by
\[
\| f \|_{L^4(\hat{G}, S_4)} = \left\{ \int_{\hat{G}} \| f(\xi) \|_{S_4}^4 \, d\sigma_\nu(\xi) \right\}^{1/4}
\]
for all \( f \) in \( L^4(\hat{G}, S_4) \).

**Theorem 2.2.** Let \( \phi \) be an admissible wavelet for the square integrable representation \( \pi : G \to B(X) \) of \( G \) on \( X \). Then
\[
\langle u, v \rangle = \frac{1}{C_\phi} \int_G \langle u, \pi_g \phi \rangle \langle \pi_g \phi, v \rangle \, d\sigma_\nu(g)
\]
for all \( u, v \in X \).

**Proof.** From the right-hand side we have
\[
\frac{1}{C_\phi} \int_G \langle u, \pi_g \phi \rangle \langle \pi_g \phi, v \rangle \, d\sigma_\nu(g) = \frac{1}{C_\phi} \int_G (u \# \phi)(g)(\phi \# v)(g) \, d\sigma_\nu(g)
\]
\[
= \int_G (u \# \phi)(g)(\phi \# v)(g) \, d\sigma_\nu(g) \cdot \frac{\int_G |\langle \phi, \pi_g \phi \rangle|^2 \, d\sigma_\nu(g)}{\int_G |\langle \phi, \pi_g \phi \rangle|^2 \, d\sigma_\nu(g)}.
\]
Using Parseval relation, we get
\[
\frac{1}{C_\phi} \int_G \langle u, \pi_g \phi \rangle \langle \pi_g \phi, v \rangle \, d\sigma_\nu(g)
\]
\[
= \int_G (h_\mu u)(\xi)(h_\mu \phi)(\xi)(h_\mu \phi)(\xi)(h_\mu v)(\xi) \, d\sigma_\nu(\xi)
\]
\[
= \int_G (h_\mu u)(\xi)(h_\mu \phi)(\xi) \, d\sigma_\nu(\xi)
\]
\[
= \int_G (u \# \phi)(g) \, d\sigma_\nu(g).
\]
From Parseval relation we obtain
\[
\frac{1}{C_\phi} \int_G \langle u, \pi_g \phi \rangle \langle \pi_g \phi, v \rangle \, d\sigma_\nu(g) = \int_G (u(g) \overline{v(g)}) \, d\sigma_\nu(g).
\]
Let $F \in L^1_\mu(G) \cap L^\infty_\mu(G)$. Then from Theorem 2.2 the localization operators $A_{\mu,\phi}: L^p_\mu(G) \to L^p_\mu(G)$ associated with symbol $F$ and the admissible wavelet $\phi$ are defined by

$$\langle A_{\mu,\phi}, Fu, v \rangle = \frac{1}{C_{\phi}} \int_G F(g) \langle u, \pi_g \phi \rangle \langle \pi_g \phi, v \rangle d\sigma_\nu(g)$$

(2.10)

for all $u \in L^p_\mu(G)$ and $v \in L^{p'}_\mu(G)$, where

$$\langle u, v \rangle = \int_G u(g) \overline{v(g)} d\sigma_\nu(g)$$

(2.11)

for $1 \leq p < \infty$. □

**Theorem 2.3.** Let $p, q, r \geq 1$ and $f \in L^p_\mu(I)$, $g \in L^q_\mu(I)$ and $h \in L^r_\mu(I)$. Then for $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 2$ we have

$$\left| \int_0^\infty f(x)(g \# h)(x) d\sigma_\nu(x) \right| \leq \| f \|_{L^p_\mu(I)} \| g \|_{L^q_\mu(I)} \| h \|_{L^r_\mu(I)}.$$

**Proof.** Let $p, q, r \geq 1$; then we have

$$\left| \int_0^\infty f(x)(g \# h)(x) d\sigma_\nu(x) \right| \leq \| f \|_{L^p_\mu(I)} \| g \# h \|_{L^{p'}_\mu(I)} \quad \text{for} \quad \frac{1}{p} + \frac{1}{p'} = 1. \quad \square$$

Using (1.12), we get

$$\left| \int_0^\infty f(x)(g \# h)(x) d\sigma_\nu(x) \right| \leq \| f \|_{L^p_\mu(I)} \| g \|_{L^q_\mu(I)} \| h \|_{L^r_\mu(I)}$$

for $\frac{1}{p'} = \frac{1}{q} + \frac{1}{r} = 1$.

**Theorem 2.4.** Let $f \in L^1_\mu(I)$, $g \in L^p_\mu(I)$ and $h \in L^q_\mu(I)$. Then for $\frac{1}{p} + \frac{1}{q} = 1$ we have the following relation:

$$\left| \int_0^\infty f(x)(g \# h)(x) d\sigma_\nu(x) \right| \leq \| f \|_{L^1_\mu(I)} \| g \|_{L^p_\mu(I)} \| h \|_{L^q_\mu(I)}.$$

**Proof.** This theorem is a particular case of Theorem 2.4. □
Theorem 2.5. Let $F \in L^1_\mu(G)$. Then for $1 \leq p < \infty$, the localization operator $A_{\mu,F,\phi} : L^p_\mu(G) \to L^p_\mu(G)$ is a bounded linear operator and satisfies the following norm inequality:

$$\| A_{\mu,F,\phi} \|_{B(L^p_\mu(G))} \leq \frac{1}{C_\phi} \| \phi \|_{L^p_\mu(G)} \| \phi \|_{L^p_\mu(G)} \| F \|_{L^1_\mu(G)}$$

for all $u \in L^p_\mu(G)$ and $v \in L^{p'}_\mu(G)$.

Proof. In view of (2.4) we have

$$\left| \langle A_{\mu,F,\phi} u, v \rangle \right| \leq \frac{1}{C_\phi} \int_G |F(g)| \left| \langle u, \pi_g \phi \rangle \right| \left| \langle \pi_g \phi, v \rangle \right| d\sigma_\nu(g).$$

Now, we use (2.4) and obtain

$$\left| \langle A_{\mu,F,\phi} u, v \rangle \right| \leq \frac{1}{C_\phi} \int_G |F(g)| \left| (u \# \phi)(g) \right| \left| (\phi \# v)(g) \right| d\sigma_\nu(g).$$

From Theorem 2.4 and assuming $U(g) = (F(u \# \phi))(g)$, we have

$$\left| \langle A_{\mu,F,\phi} u, v \rangle \right| \leq \frac{1}{C_\phi} \| U \|_{L^1_\mu(G)} \| \phi \|_{L^p_\mu(I)} \| u \|_{L^{p'}_\mu(I)} \| v \|_{L^{p'}_\mu(I)} \| \phi \|_{L^{p'}_\mu(I)}.$$ 

Now, applying the arguments of Theorem 2.3, we get

$$\left| \langle A_{\mu,F,\phi} u, v \rangle \right| \leq \frac{1}{C_\phi} \| F \|_{L^\infty_\mu(G)} \| u \|_{L^p_\mu(G)} \| \phi \|_{L^p_\mu(G)} \| v \|_{L^{p'}_\mu(G)} \| \phi \|_{L^{p'}_\mu(G)}. \quad \square$$

Theorem 2.6. Let $F \in L^\infty_\mu(G)$. Then for $1 \leq p < \infty$ the localization operator $A_{\mu,F,\phi} : L^p_\mu(G) \to L^p_\mu(G)$ is bounded linear operator and

$$\| A_{\mu,F,\phi} \|_{B(L^p_\mu(G))} \leq \frac{1}{C_\phi} \| \phi \|_{L^1_\mu(G)}^2 \| F \|_{L^\infty_\mu(G)}.$$

Proof. By (2.4), we have

$$\left| \langle A_{\mu,F,\phi} u, v \rangle \right| \leq \frac{1}{C_\phi} \int_G |F(g)| \left| \langle u, \pi_g \phi \rangle \right| \left| \langle \pi_g \phi, v \rangle \right| d\sigma_\nu(g)$$

$$\leq \frac{1}{C_\phi} \sup_{g \in G} |F(g)| \int_G \left| \langle u, \pi_g \phi \rangle \right| \left| \langle \pi_g \phi, v \rangle \right| d\sigma_\nu(g)$$

$$\leq \frac{1}{C_\phi} \| F \|_{L^\infty_\mu(G)} \int_G \left| (u \# \phi)(g) \right| \left| (\phi \# v)(g) \right| d\sigma_\nu(g)$$

$$\leq \frac{1}{C_\phi} \| F \|_{L^\infty_\mu(G)} \left( \int_G \left| (u \# \phi)(g) \right|^p d\sigma_\nu(g) \right)^{1/p}$$
\begin{equation*}
\times \left( \int_{G} |(\phi \# v)(g)|^{p'} \, d\sigma_v(g) \right)^{1/p'}.
\end{equation*}

From (1.10) we get

\begin{equation*}
|\langle A_{\mu,F,\phi,u,v} \rangle| \leq \frac{1}{C_{\phi}} \|F\|_{L^\infty_{\mu}(G)} \|\phi\|_{L^1_{\mu}(G)} \|u\|_{L^p_{\mu}(G)} \|\phi\|_{L^1_{\mu}(G)} \|v\|_{L^{p'}_{\mu}(G)}
\end{equation*}

for all \( u \in L^p_{\mu}(G) \) and \( v \in L^{p'}_{\mu}(G) \).

**Theorem 2.7.** Let \( \phi \in L^2_{\mu}(I) \cap L^\infty_{\mu}(I) \) and \( \|\phi\|_{L^2_{\mu}(I)} = 1 \). Then

\begin{equation*}
\langle \phi u, \phi v \rangle = \int_{0}^{\infty} \langle u, \pi(\xi)\phi \rangle \langle \pi(\xi)\phi, v \rangle \, d\sigma_v(\xi)
\end{equation*}

for all \( u, v \in H_{\mu}(I) \) and \( \langle \cdot \rangle \) is the inner product in \( L^2_{\mu}(I) \).

**Proof.** From right-hand side we have

\begin{equation*}
\int_{0}^{\infty} \langle u, \pi(\xi)v \rangle \langle \pi(\xi)\phi, v \rangle \, d\sigma_v(\xi) = \int_{0}^{\infty} (u \# \phi)(\xi)(\phi \# v)(\xi) \, d\sigma_v(\xi).
\end{equation*}

Applying Parseval relation, we have

\begin{equation*}
\int_{0}^{\infty} \langle u, \pi(\xi)\phi \rangle \langle \pi(\xi)\phi, v \rangle \, d\sigma_v(\xi)
\end{equation*}

\begin{equation*}
= \int_{0}^{\infty} h_{\mu}(u \# \phi)(g)h_{\mu}(\phi \# v)(g) \, d\sigma_v(g)
\end{equation*}

\begin{equation*}
= \int_{0}^{\infty} (h_{\mu}u)(g)(h_{\mu}\phi)(g)(h_{\mu}v)(g) \, d\sigma_v(g)
\end{equation*}

\begin{equation*}
= \int_{0}^{\infty} (h_{\mu}u)(g)(h_{\mu}v)(g) \left[ (h_{\mu}\phi)(g) \right]^2 \, d\sigma_v(g)
\end{equation*}

\begin{equation*}
= \int_{0}^{\infty} u(\xi)v(\xi) \left[ \phi(\xi) \right]^2 \, d\sigma_v(\xi) = \int_{0}^{\infty} \phi(\xi)u(\xi)v(\xi) \, d\sigma_v(\xi) = \langle \phi u, \phi v \rangle.
\end{equation*}
Theorem 2.8. Let \( \sigma \in L^1_{\mu}(I) \cap L^\infty_{\mu}(I) \). Then
\[
\langle P_{\sigma, \phi} u, v \rangle = \int_0^\infty \sigma(\xi) \langle u, \pi(\xi) \phi \rangle \langle \pi(\xi) \phi, v \rangle d\sigma \nu(\xi)
\]
for \( u, v \in L^2_{\mu}(I) \) and is a bounded linear operator on \( L^2_{\mu}(I) \) and satisfies the following norm inequality:
\[
\| P_{\sigma, \phi} \|_{L^2_{\mu}(I)} \leq \| \phi \|_{L^2_{\mu}(I)}^{2/r'} \| \phi \|_{L^2_{\mu}(I)}.
\]

Proof. The proof of the above theorem can be easily obtained by using Theorems 2.1, 2.5 and 2.6. \( \square \)

3. \( L^p_{\mu} \)-Boundedness

In this section we study the \( L^p_{\mu} \)-boundedness of Bessel’s localization operators associated to left regular representation. In this connection we recall Riesz–Thorin Theorem from [3].

Theorem 3.1. Let \( (X, \nu) \) be a measurable space and \( (Y, \alpha) \) a \( \sigma \)-finite measurable space. Let \( T \) be a linear transformation with domain \( D \) consisting of all \( \nu \)-simple functions on \( X \) such that
\[
\nu \{ x \in X : \ |f(x)| \neq 0 \} < \infty
\]
and such that the range of \( T \) is contained in the set of all \( \alpha \)-measurable functions on \( Y \). Suppose that \( x_1, x_2 \) and \( y_1, y_2 \) are real numbers in \([0, 1]\) and there exist positive constants \( M_1 \) and \( M_2 \) such that
\[
\| Tf \|_{L^{1/r}_\mu(Y)} \leq M_j \| f \|_{L^{1/r}_\mu(X)}, \quad f \in D \text{ and } j = 1, 2, 3, \ldots.
\]
Then for \( 0 < \theta < 1 \),
\[
x = (1 - \theta)x_1 + \theta x_2, \quad y = (1 - \theta)y_1 + \theta y_2.
\]
We have
\[
\| Tf \|_{L^{1/r}_\mu(Y)} \leq M_1^{1-\theta} M_2^\theta \| f \|_{L^{1/r}_\mu(X)}, \quad f \in D.
\]

Theorem 3.2. Let \( F \in L^r_{\mu}(G) \), \( 1 \leq r < \infty \). Then for \( 1 \leq p < \infty \) there exists a unique bounded linear operator \( A_{\mu, F, \phi} : L^p_{\mu}(G) \to L^p_{\mu}(G) \) such that the formula (2.6) is valid for all \( u \in L^p_{\mu} \), \( v \in L^p_{\mu} \) and \( \nu \)-simple functions \( F \) and \( G \) for which
\[
\nu \{ g \in G : F(g) \neq 0 \} < \infty.
\]
Moreover,
\[
\| A_{\mu, F, \phi} \|_{B(L^p_{\mu}(G))} \leq \frac{1}{C_\Phi} \| \phi \|_{L^r_{\mu}(G)} \| \phi \|_{L^r_{\mu}(G)} \| \phi \|_{L^r_{\mu}(G)} \| F \|_{L^p_{\mu}(G)}.
\]
Proof. We need to prove that the theorem for $1 \leq r < \infty$. Let $T_u$ be the linear transformation with domain $D$ consisting of all $\nu$-functions (simple) on $G$ with property that $\nu\{g \in G : F(g) \neq 0\} < \infty$,

$$T_u F = A_{\mu, F, \phi} u, \quad u \in D.$$  

By Theorems 2.5 and 2.6, we get

$$\|T_u F\|_{L^p_\mu(G)} = \|A_{\mu, F, \phi} u\|_{L^p_\mu(G)}$$  

$$\leq \frac{1}{C_\phi} \|\phi\|_{L^p_\mu(G)} \|\phi\|_{L^p_\mu'(G)} \|u\|_{L^p_\mu'(G)} \|F\|_{L^1_\mu(G)} \tag{3.1}$$  

and

$$\|T_u F\|_{L^p_\mu(G)} = \|A_{\mu, F, \phi} u\|_{L^p_\mu'(G)} \leq \frac{1}{C_\phi} \|\phi\|_{L^1_\mu(G)} \|u\|_{L^p_\mu(G)} \|F\|_{L^{p'}_\mu(G)}$$  

for $F \in D$.

In order to apply Riesz–Thorin Theorem 3.1, we may assume $x_1 = 1$, $x_2 = 0$ and $y_1 = y_2 = 1/p$. Let $x = 1/r$, then $\theta = 1/r'$ where $r'$ is the conjugate index of $r$. Now, we may put $x = 1/r$, $y = 1/p'$ and from (3.1), (3.2) and Riesz–Thorin theorem:

$$\|A_{\mu, F, \phi} u\|_{L^p_\mu(G)} = \|T_u F\|_{L^p_\mu(G)}$$  

$$\leq \frac{1}{C_\phi} \|\phi\|_{L^1_\mu(G)}^{1/r} \|\phi\|_{L^p_\mu'(G)}^{1/r'} \|\phi\|_{L^1_\mu'(G)}^{2/r'} \|u\|_{L^p_\mu'(G)} \|F\|_{L^p_\mu(G)}$$  

for all $F \in D$. Since $D$ is dense in $L^{r'}_\mu(G)$. Therefore by using density argument the above proof is complete. □

4. Bessel wavelet transform

In this section we shall study the Bessel wavelet transform and shall try to show that how the Bessel wavelet transform can be expressed in form of pseudo-differential operators with symbol $\psi(a\xi)$.

Definition 4.1. If $\psi(t) \in L^2_\mu(I)$ satisfy the admissibility condition

$$C_\psi := \int_0^\infty \frac{|(h_\mu \psi(a\omega))|^2}{\omega} d\omega < \infty,$$

then $\psi$ is called basic wavelet.

The Bessel wavelet transform relative to the Basic wavelet $\psi$ on $L^2_\mu(I)$ is defined by

$$(W_{\mu, \psi} f)(b, a) = \int_0^\infty f(t)\psi(t/a, b/a) dt \quad \text{for all } a, b \in I \text{ and } a \neq 0.$$
Theorem 4.2. The Bessel wavelet transform can be represented in the following form:

\[
(W_{\mu, \psi} f)(b, a) = a^\mu \int_{0}^{\infty} (b\xi)^{\frac{1}{2}} J_{\mu}(b\xi) (h_{\mu, \psi})(\xi) \xi^{\mu + \frac{1}{2}} (h_{\mu} f)(\xi) \, d\xi
\]

for \( \psi \in L^2_{\mu}(I) \) and \( \mu \geq -\frac{1}{2} \).

Proof. The Bessel wavelet transform is defined by

\[
(W_{\mu, \psi} f)(b, a) = \int_{0}^{\infty} f(t) \psi(t/a, b/a) \, dt
\]

\[
= a^{-\frac{1}{2}} \left( \int_{0}^{\infty} \left( \int_{0}^{\infty} (t\xi)^{\frac{1}{2}} J_{\mu}(t\xi) (h_{\mu} f)(\xi) \, d\xi \right) \psi(t/a, b/a) \, dt \right)
\]

\[
= a^{-\frac{1}{2}} \left( \int_{0}^{\infty} \left( \int_{0}^{\infty} (t\xi)^{\frac{1}{2}} J_{\mu}(t\xi) \psi(t/a, b/a) \, dt \right) (h_{\mu} f)(\xi) \, d\xi \right).
\]

Using [1, p. 142], we have

\[
(W_{\mu, \psi} f)(b, a)
\]

\[
= a^{-\frac{1}{2}} \left( \int_{0}^{\infty} \left( \int_{0}^{\infty} (a\xi t/a)^{\frac{1}{2}} J_{\mu}(a\xi t/a) D_{\mu}(x/a, b/a, z/a) \, dt \right) \psi(z/a) \, dz \right)
\]

\[
\times (h_{\mu} f)(\xi) \, d\xi
\]

\[
= a^{-\frac{1}{2}} \left( \int_{0}^{\infty} \left( \int_{0}^{\infty} (a\xi z/a)^{\frac{1}{2}} J_{\mu}(a\xi z/a) (a\xi)^{\mu + \frac{1}{2}} \psi(z/a) \, dz \right)
\]

\[
\times (h_{\mu} f)(\xi) \, d\xi
\]

\[
= a^{\mu} \left( \int_{0}^{\infty} (b\xi)^{\frac{1}{2}} J_{\mu}(b\xi) (h_{\mu} f)(\xi) \left( \int_{0}^{\infty} (a\xi z/a)^{\frac{1}{2}} J_{\mu}(a\xi z/a) \psi(z/a) \, dz \right) \xi^{\mu + \frac{1}{2}} \, d\xi \right)
\]
\[ a^\mu \int_0^\infty \left( \frac{1}{2} J_\mu(b\xi)(h_\mu f)(\xi)\xi^{\mu+\frac{1}{2}} + \frac{1}{2} (h_\mu \psi)(a\xi) \right) d\xi = a^\mu \int_0^\infty \left( \frac{1}{2} J_\mu(b\xi)(h_\mu f)(\xi)\xi^{\mu+\frac{1}{2}} (h_\mu \psi)(a\xi) \right) d\xi. \]

**Theorem 4.3.** For \( \mu \geq -\frac{1}{2} \) and \( \psi \in L^2_\mu(I) \), the Bessel wavelet transform can be expressed in form of pseudo-differential operator with symbol \( \psi(a\xi) \) and \( f \in H_\mu(I) \).

**Proof.** The Bessel wavelet transform is defined by

\[
(W_{\mu, \psi} f)(b, a) = a^\mu \int_0^\infty \left( \frac{1}{2} J_\mu(b\xi)(h_\mu f)(\xi)\xi^{\mu+\frac{1}{2}} + \frac{1}{2} (h_\mu \psi)(a\xi) \right) d\xi.
\]

If \( h_\mu \psi \in L^2_\mu(I) \), then by Plancherel’s relation this implies \( \psi \in L^2_\mu(I) \). Then the above expression can be expressed in the following form:

\[
(W_{\mu, \psi} f)(b, a) = a^\mu \int_0^\infty \left( \frac{1}{2} J_\mu(b\xi)(h_\mu f)(\xi)\xi^{\mu+\frac{1}{2}} \psi(a\xi) \right) d\xi
\]

\[
= a^\mu h_\mu^{-1} \left[ (h_\mu f)(\xi)\xi^{\mu+\frac{1}{2}} \psi(a\xi) \right](b). \]

**Theorem 4.4.** For \( \mu \geq -\frac{1}{2} \) and \( 1 \leq p < \infty \), the Bessel wavelet transform of the function \( f \) with respect to wavelet \( \psi \) satisfies the following inequality:

\[
\|(W_{\mu, \psi} f)(b, a)\|_{L^p_\mu(I)} \leq C_{\mu, p} \|f\|_{L^p_\mu(I)}.
\]

**Proof.** The Bessel wavelet transform is defined by

\[
(W_{\mu, \psi} f)(b, a) = a^\mu h_\mu^{-1} \left[ (h_\mu f)(\xi)\xi^{\mu+\frac{1}{2}} \psi(a\xi) \right](b).
\]

Then from [1, p. 147] we can write

\[
(W_{\mu, \psi} f)(b, a) = a^\mu (h_{\mu, \psi} f)(b).
\]

Hence

\[
\|(W_{\mu, \psi} f)(b, a)\|_{L^p_\mu(I)} = \|a^\mu (h_{\mu, \psi} f)(b)\|_{L^p_\mu(I)} \leq C_{\mu, p} \|f\|_{L^p_\mu(I)}.
\]

5. Wavelet multipliers

Let \( \pi : I \rightarrow B(L^p_\mu(I)) \) be the Bessel left regular representation of the additive group \( I \) on \( L^p_\mu(I) \), i.e.,

\[
(\pi(y) f)(x) = \int_0^\infty f(z) D(x, y, z) d\sigma(z), \quad x, y \in I.
\]
for $1 \leq p < \infty$. Let $\phi \in \bigcap_{1 \leq p < \infty} L^p_{\mu}(I)$ be such that $\|\phi\|_{L^p_{\mu}(I)} = 1$. Then by Plancherel’s theorem, we have

$$C_{\phi} = \int_0^\infty \left| \left| \int_0^\infty \phi(x)D(x, y, z) \, d\sigma(z) \right|^2 \, d\sigma(y) \right.$$ 

$$= \int_0^\infty \left| \left( \int_0^\infty \phi(z) \left( \int_0^\infty J(xt)J(yt)J(zt) \, d\sigma(t) \right) \, d\sigma(z) \right|^2 \, d\sigma(y) \right.$$ 

$$= \int_0^\infty \left| \left( \int_0^\infty \phi(z) \left( \int_0^\infty (h_{\mu}\phi)(t)J(yt)J(xt) \, d\sigma(t) \right) \, d\sigma(x) \right|^2 \, d\sigma(y) \right.$$ 

$$= \int_0^\infty \left| \left( \int_0^\infty (h_{\mu}\phi)(t)J(yt)(h_{\mu}\phi)(t) \, d\sigma(t) \right|^2 \, d\sigma(y) \right.$$ 

$$= \int_0^\infty \left| \left( \int_0^\infty (h_{\mu}\phi)(t)J(yt)(h_{\mu}\phi)(t) \, d\sigma(t) \right|^2 \, d\sigma(y) \right.$$ 

Therefore,

$$C_{\phi} = \int_0^\infty \int_0^\infty J(yt) \left| (h_{\mu}\phi)(t) \right|^2 \, d\sigma(t) \, d\sigma(y)$$

$$= \int_0^\infty \left| (h_{\mu}\phi)(y) \right|^4 \, d\sigma(y) = \| h_{\mu}\phi \|^4_{L^4_{\mu}(I)}. \quad (5.2)$$

**Theorem 5.1.** Let $\sigma \in L^r_{\mu}(I), 1 \leq r < \infty$ and $\phi \in \bigcap_{1 \leq p < \infty} L^p_{\mu}(I)$ be such that $\|\phi\|_{L^p_{\mu}(G)} = 1$. Then there exists a unique localization operator $L_{\sigma, \phi, \mu} : L^p_{\mu}(I) \to L^p_{\mu}(I)$ such that

$$\langle L_{\mu, \sigma, \phi}u, v \rangle = \| h_{\mu}\phi \|_{L^4_{\mu}(I)}^4 \int_0^\infty \sigma(y)\left| u(y)\phi(y) \right| \left| \pi(y)\phi \right| d\sigma(y)$$
for all \( u, v \in H^\mu(I) \). Moreover,
\[
\left\| L^\mu,\sigma,\phi \right\|_{B(L^p_\mu(I))} \leq \left\| h^\mu \phi \right\|_{L^4_\mu(I)} \left\| \phi \right\|_{L^p_\mu(I)} \left\| \phi \right\|_{L^{p'}_\mu(I)} \left\| \phi \right\|_{L^2_\mu(I)} \left\| \sigma \right\|_{L^p_\mu(I)}.
\]

**Proof.** Now for \( \sigma \) in \( L^r_\mu(I) \), \( 1 \leq r < \infty \) and \( u, v \in H^\mu(I) \), we get by (2.2), (5.2) and Plancherel’s theorem,
\[
\langle L^\mu,\sigma,\phi u, v \rangle = \frac{1}{C_\phi} \int_0^\infty \sigma(y) \langle u, \pi(y)\phi \rangle \langle \pi(y)\phi, v \rangle d\sigma(y)
\]
\[
= \left\| h^\mu \phi \right\|_{L^4_\mu(I)} \int_0^\infty \sigma(y) \langle (h^\mu u)(t), J(yt)(h^\mu \phi)(t) \rangle \times \langle J(yt)(h^\mu \phi)(t), (h^\mu v)(t) \rangle d\sigma(y)
\]
\[
= \left\| h^\mu \phi \right\|_{L^4_\mu(I)} \int_0^\infty \sigma(y) h^\mu (h^\mu uh^\mu \phi)(y) h^\mu (h^\mu vh^\mu \phi)(y) d\sigma(y)
\]
\[
= \left\| (h^\mu \phi) \right\|_{L^4_\mu(I)} \langle h^{-1}(\sigma(y)h^\mu (h^\mu uh^\mu \phi)), h^\mu vh^\mu \phi \rangle.
\]

By Theorem (1.1), we have
\[
\langle L^\mu,\sigma,\phi u, v \rangle = \left\| h^\mu \phi \right\|_{L^4_\mu(I)} \left\{ h^\mu,\phi (h^\mu \phi h^\mu u), h^\mu \phi h^\mu v \right\}.
\]

Thus, when \( p = 2 \), (5.3) tells us that the localization operators \( L^\mu,\sigma,\phi : L^2_\mu(I) \to L^2_\mu(I) \) is unitary equivalent to wavelet multiplier \( P_{\sigma,\phi} : L^2_\mu(I) \to L^2_\mu(I) \). \( \square \)

**Acknowledgments**

The author is thankful to the referee and Prof. R.S. Pathak, for their valuable comments and suggestions.

**References**