JOURNAL OF ALGEBRA 165, 550–565 (1994)

Completely Separating Algebras

PETER DRÄXLER

SFB 343, Universität Bielefeld, POB 100131, Bielefeld 33501, Germany

Communicated by Peter A. Fröhlich

Received April 3, 1992

INTRODUCTION

It is conjectured in [Pe2] that a so-called good algebra A is of tame representation type if and only if its Tits form q_A is weakly non-negative. In [Ke, MP, Pe1, PT] the validity of this conjecture is proved for several classes of algebras. The problem arises naturally, since for good algebras A weak positivity of q_A is equivalent to A being of finite representation type by [Bo1].

Another criterion for checking finiteness of type in a combinatorial way was given in [D2]. It was shown that algebras corresponding to simply connected ray-categories without infinite chains are of finite representation type if and only if the finite partially ordered sets of thin start modules are representation finite for every point of the algebra. The interest in these ray-categories comes from the fundamental paper [BGRS] on representation finite algebras.

Good algebras are introduced in [Pe2] as schurian $\tilde{\mathbb{A}}$-free algebras satisfying the (s)-condition (see [BL]). It is known (see [Bo3]) that algebras corresponding to simply connected ray-categories without infinite chains are good. Being interested in extending our criterion via thin start modules to the tame situation, it seemed to be natural to consider also the class of good algebras. On the other hand our criterion obviously works well for hereditary tree algebras which are not always $\tilde{\mathbb{A}}$-free (see Section 3 for an example). Analyzing the proofs about good algebras, we realized that the property really needed was that every convex subalgebra also satisfies the (s)-condition. So we decided to work rather with algebras having this property which we now call completely separating algebras. The aim of this paper is to present the basic properties and characterizations of completely separating algebras. We briefly sketch the content of each of the sections.

The first two sections lead to the theorem that a completely separating algebra A can be written as a factor algebra of the incidence algebra $k[S]$ of a finite partially ordered set S by an admissible ideal I of $k[S]$. This is

0021-8693/94 $6.00

proved by applying topological techniques for schurian algebras introduced in [BrG] which also are applied to obtain a useful characterization of indecomposable thin (i.e., locally one-dimensional) A-modules. Moreover it turns out that for a completely separating algebra the opposite algebra has this property as well.

The third section discusses the question of how to check in practice whether a given algebra is completely separating. It is shown that certain subalgebras which are themselves of the form $k[S]$ play an important role. Therefore a handy criterion for these algebras to be completely separating is presented.

The last section is devoted to adapting the results about critical (see [Bo1, Bo2, HV]) and hypercritical (see [Pe2]) good algebras to the completely separating case. The results of this section will be used heavily in the subsequent paper [DN] for clarifying the relations between the representation type of an algebra and its posets of thin start modules. Nevertheless they seemed to be interesting also for other questions, so that we decided to present them here.

As the expression partially ordered set occurs frequently in this paper, we abbreviate it to poset.

1. Definitions and Preliminaries

Let us fix an algebraically closed field k. Suppose A is a finite dimensional basic algebra over k. We choose a decomposition $1_A = \sum_{x \in A_0} e(x)$ of the unit element 1_A of A into primitive orthogonal idempotents $e(x)$. The notations introduced in this section will formally depend on this decomposition. However, it will always be evident that these properties of A hold for every such decomposition if and only if they hold for the chosen one.

1.1. Let us define a relation $\leqslant_A$ on A_0 by setting $y \leqslant_A x$ if there is a sequence $y = x_0, ..., x_n = x$ in A_0 such that $e(x_{i-1}) Ae(x_i) \neq 0$ for all $i = 1, ..., n$. This relation is clearly reflexive and transitive. We will frequently use the intervals $[y, x]_A$, $]y, x[_A$, $[y, x[_A$, $]y, x]_A$ with respect to $\leqslant_A$. Moreover we set $]\infty, x]_A := \{y \in A_0 : y \leqslant_A x\}$ and define $]\infty, x[_A$, $[x, \infty[_A$, $]x, \infty[_A$ in the analogous way. If no confusion is possible we sometimes omit the index A in the just introduced notations.

A is called schurian if $\dim_k e(y) Ae(x) \leqslant 1$ for all $x, y \in A_0$. A is said to be directed provided $\dim_k e(x) Ae(x) = 1$ for all $x \in A_0$ and the relation $\leqslant_A$ is antisymmetric (i.e., a partial order on A_0).

A basic algebra over an algebraically closed field k is often written as a factor algebra $A = k[\Delta]/I$ of the path algebra $k[\Delta]$ of a finite quiver Δ by an admissible ideal I of $k[\Delta]$ (see [Ga]). We may assume that $A_0 = \Delta_0$

and that the idempotents induced by the points of Δ coincide with the $e(x)$. In this case there exists an oriented path from x to y in Δ if and only if $y \leqslant_A x$. This shows in particular that A is directed if and only if the quiver Δ is directed (i.e. has no oriented cycles) which is the common definition of directedness.

1.2. Suppose A is schurian and directed. For $s \in A_0$ we define another partial order $\leqslant_s$ on $]A, s] := \{x \in A_0 : e(x) Ae(s) \neq 0\}$ by setting $y \leqslant_s x$ provided $e(y) Ae(x) Ae(s) \neq 0$ (compare [BrG]). Furthermore we use the notation $]A, s[:=]A, s] \backslash s$. The sets $[s, A[$, $]s, A[$ and the partial order $\leqslant^s$ are defined dually. Obviously $]A, s[\subseteq A_0 \backslash [s, \infty[_A$ holds and the relation $\leqslant_A$ is stronger than $\leqslant_s$.

The point $s \in A_0$ is called *separating* if any two different connected components of $(]A, s[, \leqslant_s)$ lie in different connected components of $(A_0 \backslash [s, \infty[_A, \leqslant_A)$. We call A *separating* if all points s of A_0 are separating.

Of course the connected components of $(]A, s[, \leqslant_s)$ are exactly the supports $\operatorname{supp} M := \{x \in A_0 : e(x) M \neq 0\}$ of the indecomposable direct summands M of the radical of the indecomposable projective module $P_s := Ae(s)$. Hence s is separating if and only if P_s has separated radical, and A is separating if and only if A satisfies the (s)-condition in the notations introduced in [BL].

Let us recall from [Bo3] that the Auslander–Reiten quiver Γ_A of a separating algebra A has a preprojective component $\mathscr{P}$ such that its orbit quiver $\mathcal{O}(\mathscr{P})$ is a tree. Conversely it is remarked in [Bo4] that, if Γ_A has a complete (i.e., contains all indecomposable projective A-modules) preprojective component $\mathscr{P}$ such that its orbit quiver $\mathcal{O}(\mathscr{P})$ is a tree, then A is separating. It is easy to see that the proof of [BLS, 2.2] works also in this more general situation.

1.3. For a subset $T \subseteq A_0$ we define $e(T) := \sum_{t \in T} e(t)$ and $A(T) := e(T) Ae(T)$. $A(T)$ is a k-algebra with decomposition $\sum_{t \in T} e(t)$ of its unit element $e(T)$ into primitive orthogonal idempotents. For $x, y \in T$ we have $y \leqslant_A x$ provided $y \leqslant_{A(T)} x$.

$T \subseteq A_0$ is called a *convex* subset of A if it is convex with respect to the partial order $\leqslant_A$ on A_0. For convex T the relations $\leqslant_A$ and $\leqslant_{A(T)}$ coincide on this set. Moreover the canonical map $A(T) \to A/Ae(A_0 \backslash T)A$ is a k-algebra isomorphism. This allows to consider $A(T)$ as factor algebra of A. An algebra of the form $A(T)$ for some convex subset T of A is called a *convex subalgebra* of A.

We say that the algebra A is *completely separating* if $A(T)$ is separating for every convex subset T of A_0. Obvious examples of completely separating algebras are algebras of the form $k[\Delta]/I$ such that the quiver Δ is a tree. We will meet further well-known examples of completely separating algebras in Section 2.

1.4. We call a schurian and directed algebra A *weakly transitive* if for all $x, y, z \in A_0$ satisfying $z \leqslant_A y \leqslant_A x$ and $e(z)\,Ae(x) \neq 0$ also $e(z)\,Ae(y)\,Ae(x) \neq 0$ (hence $e(z)\,Ae(y)\,Ae(x) = e(z)\,Ae(x)$) holds. We use the term weakly transitive rather than semicommutative, since these algebras are the natural generalization of the transitive square-free algebras introduced in [An] (transitive square-free algebras are exactly the schurian directed algebras A satisfying $e(z)\,Ae(y)\,Ae(x) \neq 0$ for all $z \leqslant_A y \leqslant_A x$ in A_0).

It is easily checked that for a weakly transitive algebra A the sets $]A, s]$ are convex subsets of A. Moreover the relations $\leqslant_s$ and $\leqslant_A$ coincide on these sets.

The following lemma was proved in [D2, 2.1] for the algebras corresponding to simply connected ray-categories without infinite chains. Nevertheless the proof only used that these algebras are completely separating. So we simply restate this result.

LEMMA. *Completely separating algebras are weakly transitive.*

1.5. Let us present our favourite class of examples for weakly transitive algebras. If $S = (S_0, \leqslant)$ is a finite poset we denote by $k[S]$ the k-vectorspace which has as basis all pairs $(y, x) \in S_0^2$ such that $y \leqslant x$. Recall that with the multiplication

$$(y', x')(y, x) := \begin{cases} (y', x) & \text{if } x' = y \\ 0 & \text{else} \end{cases}$$

the space $k[S]$ becomes a finite dimensional k-algebra which we call the *poset algebra of* S. The sum $\sum_{x \in S_0} (x, x)$ is a decomposition of the unit element of $k[S]$ into primitive orthogonal idempotents. The radical of $k[S]$ is generated as k-space by the pairs (y, x) satisfying $y < x$.

We say that an ideal I of $k[S]$ is *admissible* if $I \subseteq (\operatorname{Rad} k[S])^2$. Consider the algebra $A := k[S]/I$ for an admissible ideal I of $k[S]$. Obviously $\sum_{x \in S_0} ((x, x) + I)$ is a decomposition of the unit element of A into primitive orthogonal idempotents. Hence A is a schurian, directed, weakly transitive algebra. Moreover the relations $\leqslant_A$ and $\leqslant$ coincide.

1.6. Let us examine in more detail the admissible ideals of $k[S]$ for a given poset S. A subset $R \subseteq S_0^2$ consisting of pairs (y, x) such that $[y, x]$ contains at least three elements is called a set of relation pairs. Clearly every set of relation pairs generates an admissible ideal of $k[S]$. It is also not hard to see that conversely every admissible ideal of $k[S]$ is generated by a set of relation pairs. Among all sets of relation pairs generating a given admissible ideal I there is a unique minimal set R characterized by the fact that for all $(y, x), (y', x') \in R$ the inclusion $\{x', y'\} \subseteq [y, x]$

implies $x' = x$ and $y' = y$. This set is called the *minimal set of relation pairs* for I.

Given an admissible ideal I of $k[S]$ the elements induced by the pairs (x, x) are the unique primitive idempotents of the radical factor algebra of $k[S]/I$. From this we easily obtain the following lemma which is helpful for checking whether two given algebras of the form $k[S]/I$ are isomorphic.

Remark. Let S, S' be two posets and I, I' be admissible ideals of $k[S]$, resp. $k[S']$ with minimal sets of relation pairs R, resp. R'. The algebras $k[S]/I$ and $k[S']/I'$ are isomorphic if and only if there is an isomorphism of partially ordered set $\varphi: S \to S'$ such that $R' = \{(\varphi(y), \varphi(x)): (y, x) \in R\}$.

1.7. For a given schurian, directed, weakly transitive algebra A we consider the algebra $k[S]$ induced by the poset $S := (A_0, \leqslant_A)$ and the admissible ideal I of $k[S]$ generated by the pairs (y, x) such that $y \leqslant_A x$ but $e(y)\, Ae(x) = 0$. The algebra $A' := k[S]/I$ behaves very much like A in the sense that for $y \leqslant_A x$ in A_0 we have $e(y)\, Ae(x) = 0$ if and only if $((y, y) + I)\, A'((x, x) + I) = 0$.

An even stronger relation holds in the following special case. The proof can be adapted easily from [D1, 2.3].

LEMMA. *Suppose A is a schurian, directed, weakly transitive k-algebra and s is an element of A_0 satisfying $e(x)\, Ae(s) \neq 0$ for all $x \in A_0$. Then s is the unique maximal element of the poset $S := (A_0, \leqslant_A)$ and $A \cong k[S]$.*

1.8. We close the introductory section by giving a necessary condition for an algebra to be completely separating which will be helpful later. Suppose A is a schurian, directed algebra and $(x_0, ..., x_n)$ is a sequence in A_0. For all $i \in \mathbb{Z}$ we write $\bar{i}$ for the coset of i in $\mathbb{Z}/(n+1)\mathbb{Z}$. The sequence $(x_0, ..., x_n)$ is called *cyclic path sequence* if the following conditions are satisfied:

(i) n is odd and $n \geqslant 3$.

(ii) $x_j \leqslant_A x_i$ for all $i, j \in \mathbb{N}_0$ satisfying $0 \leqslant i, j \leqslant n$, $\bar{i} - \bar{j} \in \{\bar{0}, \bar{1}, -\bar{1}\}$ and i odd.

LEMMA. *Suppose A is a schurian, directed, weakly transitive k-algebra. A is not completely separating provided there exists a cyclic path sequence $(x_0, ..., x_n)$ in A_0 satisfying the following three conditions:*

(α) $[x_{n-1}, x_n]_A \cap [x_0, x_n]_A = \{x_n\}$.

(β) $x_{n-1} \neq x_n \neq x_0$.

(γ) *For all $i = 1, ..., n-2$ the points x_i and x_n are incomparable with respect to $\leqslant_A$.*

Proof. We show that x_n is not separating in $A(T)$ with T defined as the convex hull of the set $\{x_0, ..., x_n\}$. From (γ) follows that x_n is maximal in

T and $]\infty, x_n]_{A(T)} = [x_{n-1}, x_n]_A \cup [x_0, x_n]_A$. Consequently $]A(T), x_n[=]\infty, x_n]_{A(T)} \cap]A(T), x_n[= D_0 \cup D_{n-1}$ setting $D_j := [x_j, x_n]_A \cap]A(T), x_n[$. From ($\alpha$) we obtain that the D_j are closed subsets of $]A(T), x_n[$ with respect to $\leqslant_A$. Hence there exist connected components $D_j' \subseteq D_j$ of $(]A(T), x_n[, \leqslant_A)$ with $D_{n-1}' \neq D_0'$. On the other hand the set $(T \backslash [x_n, \infty[_{A(T)}, \leqslant_A) = (T \backslash x_n, \leqslant_A)$ is apparently connected. Thus x_n is not separating in $A(T)$.

2. Some Homology Groups

In this section we will use extensively the topological notations and methods introduced in [BrG, 2] to study schurian algebras. Let us suppose that A is a schurian, directed, basic k-algebra with decomposition $\sum_{x \in A_0} e(x)$ of its unit element into primitive orthogonal idempotents.

2.1. For every $n \in \mathbb{N}_0$ the free $\mathbb{Z}$-module with basis $S_n A := \{(x_0, ..., x_n) \in A_0^{n+1}: e(x_0)\, Ae(x_1) \cdots e(x_{n-1})\, Ae(x_n) \neq 0\}$ is denoted by $C_n A$. Given $n \in \mathbb{N}$ we have a $\mathbb{Z}$-homomorphism $d_n: C_n A \to C_{n-1} A$, $x = (x_0, ..., x_n) \mapsto \sum_{i=0}^{n} (-1)^i \hat{x}_i$ where $\hat{x}_i := (x_0, ..., x_{i-1}, x_{i+1}, ..., x_n)$ for all $i = 0, ..., n$. The complex

$$\cdots \xrightarrow{d_3} C_2 A \xrightarrow{d_2} C_1 A \xrightarrow{d_1} C_0 A \xrightarrow{d_0} 0$$

is denoted by $C_{\bullet} A$. Its homology groups $H_n A := \operatorname{Ker} d_n / \operatorname{Im} d_{n+1}$ are called the *simplicial homology groups* of A. For an arbitrary abelian group Z the *simplicial cohomology groups* of A with values in Z are given by $H^n(A, Z) := \operatorname{Ker} \operatorname{Hom}_{\mathbb{Z}}(d_{n+1}, Z) / \operatorname{Im} \operatorname{Hom}_{\mathbb{Z}}(d_n, Z)$.

If $C(A)$ is the set of all connected components C of $(A_0, \leqslant_A)$, then the $\mathbb{Z}$-homomorphism $H_0 A \to \mathbb{Z}^{C(A)}$ mapping the coset of $x \in A_0$ in $H_0 A$ to the component C of $(A_0, \leqslant_A)$ containing x is bijective.

We observe that for weakly transitive A the sets $S_n A$ can be written as $\{(x_0, ..., x_n) \in A_0^{n+1}: x_0 \leqslant_A \cdots \leqslant_A x_n, e(x_0) A e(x_n) \neq 0\}$.

2.2. Let $s \in A_0$. We define the k-subspace $\bar{B}^s(A)$ of A as the sum of all $e(y)\, Ae(x)$ such that $x, y \in]A, s]$ and $y \leqslant_s x$. For $x, y \in]A, s]$ therefore $e(y)\, \bar{B}^s(A)\, e(x) \neq 0$ is equivalent to $y \leqslant_s x$. Hence $\bar{B}^s(A)$ is a subalgebra of A. The unit element of this subalgebra is $e(]A, s])$ which has the decomposition $\sum_{x \in]A, s]} e(x)$ into primitive orthogonal idempotents. Moreover the $\leqslant$-relation with respect to $\bar{B}^s(A)$ is just $\leqslant_s$ and $\bar{B}^s(A)$ is weakly transitive. As $\bar{B}^s(A)$ is a direct complement of the annihilator of the module P_s in A, we may consider $\bar{B}^s(A)$ as factor algebra of A by this ideal.

We set $B^s(A) := e(]A, s[)\, \bar{B}^s(A)\, e(]A, s[)$ and define $\underline{B}_s(A)$, $B_s(A)$ in the analogous way. From 1.7 follows $\bar{B}^s(A) \cong k(]A, s], \leqslant_s)$ and $B^s(A) \cong$

$k(]A, s[, \leqslant_s)$. If A is in addition weakly transitive, then $\bar{B}^s(A) = A(]A, s])$ by 1.4.

Let T be a convex subset of A_0 with respect to $\leqslant_A$ and suppose $s \in T$. It is easy to verify that $T \cap]A, s]$ is a convex subset of $]A, s]$ with respect to $\leqslant_s$ and $\bar{B}^s(A(T))$ coincides with $\bar{B}^s(A)(T \cap]A, s])$.

2.3. Let $s \in A_0$. We introduce $\bar{A}^s := A(A_0 \backslash]s, \infty[_A)$, $A^s := A(A_0 \backslash [s, \infty[_A)$ and define $\underline{A}_s$, A_s in the analogous way. The canonical inclusions induce morphisms of complexes u, v, i, j such that the sequence

$$0 \to C_\bullet B^s(A) \xrightarrow{(u,\,-v)^t} C_\bullet \bar{B}^s(A) \oplus C_\bullet A^s \xrightarrow{(i,j)} C_\bullet \bar{A}^s \to 0$$

is exact (see [BrG, p. 31]). This short exact sequence yields the following long exact sequence of homology groups:

$$\cdots \to H_n B^s(A) \to H_n \bar{B}^s(A) \oplus H_n A^s \to H_n \bar{A}^s \xrightarrow{\partial} H_{n-1} B^s(A) \to \cdots$$

The occurring homomorphism ι: $H_0 B^s(A) \to H_0 \bar{B}^s(A) \oplus H_0 A^s$ is injective if and only if s is separating (see [BrG, p. 34]).

THEOREM. *A is completely separating if and only if $H_1 A(T) = 0$ for all convex subsets T of A_0 with respect to $\leqslant_A$.*

Proof. We start by assuming $H_1 A(T) = 0$ for all convex subsets T and consider such a set T. The homology sequence in 2.3 furnishes an exact sequence:

$$H_1 \overline{A(T)}^s \to H_0 B^s(A(T)) \xrightarrow{\iota} H_0 \bar{B}^s(A(T)) \oplus H_0 A(T)^s$$

$\overline{A(T)}^s_0$ is a convex subset of T and hence also of A_0. Thus $H_1 \overline{A(T)}^s = 0$ by assumption. Consequently ι is injective and s is separating in $A(T)$ by 2.3.

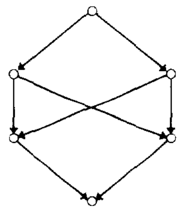
For the converse we apply induction on the cardinality of T. If T has only one element, nothing has to be proved. If T has more than one element, we choose a maximal element s of T with respect to $\leqslant_A$ and obtain $\overline{A(T)}^s = A(T)$. 2.3 yields the exact sequence:

$$H_1 \bar{B}^s(A(T)) \oplus H_1 A(T)^s \to H_1 A(T) \to H_0 B^s(A(T)) \xrightarrow{\iota} H_0 \bar{B}^s(A(T)) \oplus H_0 A(T)^s$$

Observing that $A(T)^s = A(T \backslash s)$ and $T \backslash s$ is a convex subset of A_0 with one element less than T, we derive $H_1 A(T)^s = 0$ from the induction hypothesis. 2.2 shows $\bar{B}^s(A(T)) \cong k[S]$ for some poset S with an unique maximal element. Hence $H_1 \bar{B}^s(A(T)) = 0$ follows from [BrG, p. 31]. Finally we recall that s lies separating in $A(T)$. Thus by 2.3 the homomorphism ι is injective.

2.5. Corollary. *A is completely separating if and only if A^{op} is completely separating.*

As illustration for 2.4 let us present an example of an algebra A which is not completely separating but A and A^{op} are separating. Namely consider the poset algebra of the poset given by the following diagram:



2.6. Corollary. *If A is completely separating, the following assertions hold:*

(a) $H_n A = 0$ *for all* $n \in \mathbb{N}$.

(b) $H^n(A, Z) = 0$ *for all* $n \in \mathbb{N}$ *and all abelian groups* Z.

Proof. (a) We apply induction on the cardinality of A_0. The case that A_0 has only one element is obvious. Otherwise we choose a maximal element s of A_0 with respect to $\leqslant_A$. We obtain $\bar{A}^s = A$ and hence by 2.3 an exact sequence

$$H_n \bar{B}^s(A) \oplus H_n A^s \to H_n A \to H_{n-1} B^s(A)$$

for all $n \geqslant 2$. $H_1 A = 0$ follows immediately from 2.4. For $n \geqslant 2$ we first observe that with A also A^s is completely separating and therefore $H_n A^s = 0$ by induction. As in the proof of 2.4 we apply the fact that $\bar{B}^s(A) \cong k[S]$ for some poset S with an unique maximal element. [BrG, p. 31] shows also $H_n \bar{B}^s(A) = 0$ for all $n \geqslant 2$.

Once again we use that A is completely separating in order to derive from 1.4 the weakly transitivity of A. By 1.4 $]A, s[$ is a convex subset of A_0 and $B^s(A) = A(]A, s[)$. Because $]A, s[$ has certainly less elements than A_0, by induction we obtain $H_{n-1} B^s(A) = 0$.

(b) follows from (a) as

$$\cdots \to C_3 A \to C_2 A \to C_1 A \to C_0 A \to H_0 A \to 0$$

is a split exact sequence (see [BrG, p. 36]).

2.7. Corollary. *If A is completely separating, $S := (A_0, \leqslant_A)$, and I is the admissible ideal of $k[S]$ introduced in 1.7, then $A \cong k[S]/I$.*

Proof. A is weakly transitive by 1.4. Using 1.7, $S_{\bullet}A = S_{\bullet}(k[S]/I)$ follows. We apply [BrG, Lemma 2.2] and obtain $A \cong k[S]/I$ because of $H^2(A, k^*) = 0$.

2.8. The end of this section is devoted to adapting the description of indecomposable thin modules over simply connected ray-categories without infinite chains given in [D2, 3.1] to completely separating algebras.

We start with a k-algebra $A = k[S]/I$ where $S = (S_0, \leqslant)$ is a finite poset and I is an admissible ideal of $k[S]$ generated by a set R of relation pairs (see 1.6). Of course a $k[S]$-module M is given by a k-vectorspace $M(x)$ for each $x \in S_0$ and a k-linear map $M(y, x)$: $M(x) \to M(y)$ for all $y \leqslant x$ in S_0, satisfying the conditions $M(x, x) = \mathrm{id}_{M(x)}$ for all $x \in S_0$, $M(z, y)\, M(y, x) = M(z, x)$ for all $z \leqslant y \leqslant x$, and $M(y, x) = 0$ for all $(y, x) \in R$.

An A-module M is called *thin* if $\dim_k M(x) \leqslant 1$ for all $x \in S_0$. Let us present our favourite example for thin modules. We say that a subset T of S_0 is strictly convex if it is convex and for all $(y, x) \in R$ the set $\{x, y\}$ is not contained in T. For a strictly convex subset T the A-module M_T is defined by $M_T(x) := k$ for all $x \in T$ and $M_T(x) := 0$ else. Moreover $M_T(y, x) := \mathrm{id}_k$ for all $y \leqslant x$ such that $\{x, y\} \subseteq T$ and $M_T(y, x) := 0$ in all other cases. Clearly M_T is indecomposable if and only if T is connected.

2.9. THEOREM. *If $A = k[S]/I$ is completely separating and M is an indecomposable, thin A-module with support T, then $M \cong M_T$.*

Proof. Recalling $H^1(A(T), k^*) = 0$ from 2.6, we may use the proof of [D2, 3.1].

3. CHARACTERIZATIONS OF COMPLETELY SEPARATING ALGEBRAS

Let us suppose again that A is a schurian, directed, basic k-algebra with decomposition $\sum_{x \in A_0} e(x)$ of its unit element into primitive orthogonal idempotents.

3.1. THEOREM. *The following three assertions about A are equivalent.*

(a) *A is completely separating.*

(b) *A is separating and the algebras $\bar{B}^s(A)$ and $\underline{B}_s(A)$ are completely separating for all $s \in A_0$.*

(c) *$H_1 A = 0$ and the algebras $\bar{B}^s(A)$ and $\underline{B}(A)$ are completely separating for all $s \in A_0$.*

Proof. (a) $\Rightarrow$ (b): A is weakly transitive by 1.4. Using 2.2 the algebras $\bar{B}^s(A) \cong A(]A, s])$ are also completely separating. For $\underline{B}_s(A)$ we may use the same argument.

(b) ⇒ (c): Let us apply induction on the cardinality of A_0. If A_0 has only one element, of course nothing has to be proved. If A_0 has more than one element, we choose a maximal element s with respect to $\leqslant_A$ and obtain $A = \bar{A}^s$. 2.3 furnishes the following exact sequence:

$$H_1 \bar{B}^s(A) \oplus H_1 A^s \to H_1 A \to H_0 B^s(A) \xrightarrow{\iota} H_0 \bar{B}^s(A) \oplus H_0 A^s$$

A is separating and thus ι injective. For the completely separating algebra $\bar{B}^s(A)$ by 2.4 the equation $H_1 \bar{B}^s(A) = 0$ holds.

To finish this part of the proof, we want to show $H_1 A^s = 0$. As $T := A_0 \backslash s$ has one element less than A_0, we may apply the induction hypothesis, as soon as we proved the following: $A^s = A(T)$ is separating and for all $t \in T$ the algebras $\bar{B}^t(A(T))$ and $\underline{B}_t(A(T))$ are completely separating.

That A^s is separating, can be derived directly from this property for A. For all $t \in T$ the algebras $\bar{B}^t(A(T)) = \bar{B}^t(A)$ are completely separating by assumption. $T \cap [t, A[$ by 2.2 is a convex subset of $([t, A[, \leqslant^t)$ satisfying $\underline{B}_t(A(T)) = \underline{B}_t(A)(T \cap [t, A[)$. Thus also $\underline{B}_t(A(T))$ is completely separating.

(c) ⇒ (a): Using 2.4 it is enough to show $H_1 A(T) = 0$ for every convex subset T of A_0. For $T = A_0$ this is obvious by assumption. If T is a proper subset of A_0, then there exists a convex subset T' of A_0 and an extremal element s of T' such that $T = T' \backslash s$. By induction we may assume $H_1 A(T') = 0$. 2.2 shows again that the algebras $\bar{B}^s(A(T'))$ and $\underline{B}_s(A(T'))$ are also completely separating. We assume without loss of generality that s is maximal in T'. Consequently $A(T')^s = A(T)$ and we obtain the exact sequence:

$$H_1 B^s A(T') \to H_1 \bar{B}^s(A(T')) \oplus H_1 A(T) \to H_1 A(T')$$

Thus $H_1 A(T) = 0$ follows from $H_1 B^s A(T') = 0$ which is true by 2.4.

3.2. (a) A cyclic path sequence $(x_0, \ldots, x_n)$ in A_0 is called a *cyclic chain sequence* if the following conditions are satisfied:

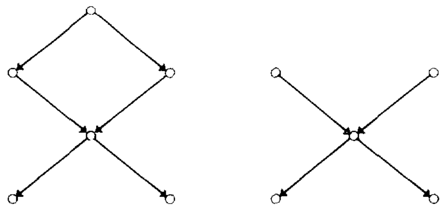
(i) $e(x_j)\, Ae(x_i) \neq 0$ for all $0 \leqslant i, j \leqslant n$ such that $\bar{i} - \bar{j} \in \{\bar{0}, \bar{1}, -\bar{1}\}$ and i odd.

(ii) $e(x_j)\, Ae(x_l)\, Ae(x_i) \neq e(x_j)\, Ae(x_i)$ and $e(x_l)\, Ae(x_j)\, Ae(x_i) \neq e(x_l)\, Ae(x_i)$ for all $0 \leqslant i, j, l \leqslant n$ such that $\bar{j} = \bar{i} - \bar{1}$, $\bar{l} = \bar{i} + \bar{1}$ and i odd.

(iii) $e(x_i)\, Ae(x_l)\, Ae(x_j) \neq e(x_i)\, Ae(x_j)$ and $e(x_i)\, Ae(x_j)\, Ae(x_l) \neq e(x_i)\, Ae(x_l)$ for all $0 \leqslant i, j, l \leqslant n$ such that $\bar{j} = \bar{i} - \bar{1}$, $\bar{l} = \bar{i} + \bar{1}$ and i even.

It is clear that A admits no infinite chains (see [Fi]) if and only if there is no cyclic chain sequence in A_0. If A has no cyclic chain sequence, then this is true for the algebras $\bar{B}^s(A)$ and $\underline{B}_s(A)$ as well. In particular in this case the posets $(]A, s], \leqslant_s)$ and $([s, A[, \leqslant^s)$ are $\tilde{\mathbb{A}}$-free, i.e. do not contain a subposet with diagram of type $\tilde{\mathbb{A}}_n$. In [Bo1, 2.3] it is proved that for a finite, $\tilde{\mathbb{A}}$-free posets the algebra $k[S]$ is completely separating.

Using 3.1, we see that the class of completely separating algebras contains the class of algebras for which each of the posets $(]A, s], \leqslant_s)$ and $([s, A[, \leqslant^s)$ is $\tilde{\mathbb{A}}$-free. This class in turn contains the class of separating algebras without infinite chains which seems to be the class of good algebras. That both inclusions are proper, can be seen using the poset algebras to the two posets given by the following diagrams:



3.3. As the algebras $\bar{B}^s(A)$ and $\underline{B}_s(A)$ are poset algebras, a practicable characterization of completely separating poset algebras together with Theorem 3.1 would furnish a practicable criterion for a given algebra to be completely separating. The rest of the section is devoted to this aim.

Let $S = (S_0, \leqslant)$ be a finite poset and $A := k[S]$. A cyclic chain sequence $(x_0, ..., x_n)$ in S_0 is called a *crown* if the following conditions are satisfied:

(i) For all $0 \leqslant i, j \leqslant n$ the elements x_i, x_j are only comparable, provided $\bar{i} - \bar{j} \in \{\bar{0}, \bar{1}, -\bar{1}\}$.

(ii) For all $0 \leqslant i, j, l \leqslant n$ such that $\bar{j} = \bar{i} - \bar{1}$ and $\bar{l} = \bar{i} + \bar{1}$ the intersection of the convex hull of $\{x_i, x_j\}$ and the convex hull of $\{x_i, x_l\}$ is exactly $\{x_i\}$.

THEOREM. *A poset algebra $A = k[S]$ is completely separating if and only if A has no crowns.*

Proof. Using 1.8, we see that a completely separating poset algebra does not admit a crown. To prove the converse, we assume that A is not completely separating and construct a crown in S. As first step we see that there is a convex subset T of S and an element $s \in T$ such that s is not separating in T. Hence s is not separating in $T \setminus]s, \infty[_A$ as well. As $T \setminus]s, \infty[$ again is a convex subset of S, without loss of generality we may assume $S_0 = T \setminus]s, \infty[$.

The fact that s is not separating yields the existence of two distinct connected components D_1, D_2 of $]\infty, s[$ lying in the same component C of $S_0 \setminus s$. Hence there is a cyclic path sequence $(x_0, ..., x_n)$ in S with the properties:

(i) $x_n = s$.

(ii) $x_0, ..., x_{n-1} \in S_0 \backslash s$.

(iv′) $x_0 \in D_1$, $x_{n-1} \in D_2$.

Choosing D_1, D_2, $(x_0, ..., x_n)$ with n minimal, we may assume furthermore:

(iii) $x_1, ..., x_{n-2} \notin]\infty, s[$.

From (iv′) we obtain:

(iv) $[x_{n-1}, x_n] \cap [x_0, x_n] = \{x_n\}$.

Now we choose a cyclic path sequence $(x_0, ..., x_n)$ satisfying the properties (i) to (iv) for which the pair $(n, l+l')$ is lexicographically minimal where l denotes the sum of the cardinalities of the convex hulls of $\{x_{i-1}, x_i\}$ for $i = 1, ..., n$ and l' denotes the cardinality of the convex hull of $\{x_0, x_n\}$. A simple, but tedious case by case inspection shows that $(x_0, ..., x_n)$ is a crown.

3.4. COROLLARY. *If A is a separating algebra such that A/AeA is representation finite for every idempotent $e \neq 1_A$ of A, then A is completely separating.*

Proof. In order to apply 3.1 and 3.3, we want to show that none of the algebras $\bar{B}^s(A)$ and $\underline{B}_s(A)$ admits a crown. By symmetry we only consider $\bar{B}^s(A)$. If we assume that $\bar{B}^s(A)$ contains a crown, then this algebra is of infinite representation type. On the other hand, we observed in 2.2 that $\bar{B}^s(A)$ is a factor algebra of A. Hence $B^s(A)$ is a factor algebra of $A/Ae(s)A$ which furnishes the contradiction.

3.5. COROLLARY. *Suppose A is a separating algebra. Then A is completely separating if and only if there are only finitely many isomorphism classes of indecomposable thin A-modules.*

Proof. By 2.9 it is clear that a completely separating algebra has only finitely many isomorphism classes of indecomposable thin modules. If conversely A is not completely separating, then by 3.1 there is $s \in A_0$ such that $\bar{B}^s(A)$ or $\underline{B}_s(A)$ contains a crown. But obviously a crown gives rise to an infinite family of pairwise non-isomorphic indecomposable thin modules over these algebras. Using extension by zero we can lift this family to a family of A-modules.

4. CRITICAL AND HYPERCRITICAL ALGEBRAS

In the following section we want to discuss the problem of determining the representation type of a given completely separating algebra A.

4.1. Of course A is representation infinite if and only if it contains a convex subalgebra B which is *convex minimal representation infinite.* This means that B is representation infinite, but every proper convex subalgebra is representation finite. Analogously A is wild if and only if it contains a convex subalgebra B which is *convex minimal wild.* This means that B is wild, but every proper convex subalgebra is tame. Convex minimality is adapted to the definition of completely separating algebras. In general A is said to be minimal representation infinite resp. minimal wild if A is representation infinite resp. wild but the factor algebra A/AeA is representation finite resp. tame for every idempotent $e \neq 1_A$ of A.

Another popular way to get information about the representation type of a given algebra is to consider its Tits form q_A. Assuming again that our given algebra is basic and has a decomposition $\sum_{x \in A_0} e(x)$ of its unit element into primitive orthogonal idempotents, we recall that the quadratic form $q_A: \mathbb{Z}^{A_0} \to \mathbb{Z}$ is defined by $q_A(d) := \sum_{x,y \in A_0} (\sum_{i=0}^{2} (-1)^i \dim_k \operatorname{Ext}_A^i(E_x, E_y))\, d_x d_y$ for all $d = (d_x)_{x \in A_0} \in \mathbb{Z}^{A_0}$ where the E_x are the simple modules $P_x/\operatorname{Rad} P_x$. Every subset I of A_0 gives rise to a restricted quadratic form $q_A^I: \mathbb{Z}^I \to \mathbb{Z}$. The algebra A is called critical resp. hypercritical if q_A is not weakly positive resp. weakly indefinite but for every proper subset I of A_0 the form q_A^I is weakly positive resp. non-negative.

4.2. The following theorem shows that, considering completely separating algebras and the step from representation finite to infinite, all the notions considered above coincide. The result is well-known (see [Bo1, Bo2, HV]) with the exception that all schurian algebras which are concealments of a tame hereditary tree algebra are actually completely separating. In contrast not all these algebras are good. This seems to be another indication that completely separating is a more satisfactory notion than good. Let us also remark that the list of all algebras which are concealed of tame hereditary algebras can be found in [HV].

THEOREM. *For a given algebra A the following assertions are equivalent*:

(a) *A is completely separating and convex minimal representation infinite.*

(b) *A is schurian concealment of a connected tame hereditary tree algebra.*

(c) *A is completely separating and minimal representation infinite.*

(d) *A is completely separating and critical.*

Proof. (a) $\Rightarrow$ (b): It is easy to see that the Auslander-Reiten quiver of A has a complete preprojective component $\mathscr{P}$ without injectives whose orbit quiver $\mathscr{O}(\mathscr{P})$ is a tree with underlying graph T. Analogously there is a complete preinjective component without projectives. By [Ha, 7.2] A is a concealment of a hereditary tree algebra of type T which is clearly connected. Using the notations introduced in [BoG], we know $A = A_g$ for some admissible grading g of T. That A is convex minimal representation infinite means exactly that the grading g is critical. Hence by [Bo2, Theorem 1] T is a Euclidean tree.

(b)$\Rightarrow$(c): As A is a concealment of a hereditary tree algebra, there is a complete preprojective component whose orbit quiver is a tree. By 1.2 A is separating. In [HV, Theorem 2] it is shown that A is minimal representation infinite. Thus by 3.4 A is actually completely separating.

(c)$\Rightarrow$(d): First we apply [Bo1, Theorem 3.3] to derive that q_A is not weakly positive from the assumption that A is representation infinite. We use [Bo1, Theorem 3.3] again to see that the Tits forms $q_{A/Ae(s)A}$ of the representation finite algebras $A/Ae(s)A$ are weakly positive for all $s \in A_0$. Hence for a vector $d \in \mathbb{Z}^{A_0 \setminus s}$ with non-negative coefficients we obtain $0 \leqslant q_{A/Ae(s)A}(d) \leqslant q_A^{A_0 \setminus s}(d)$.

(d)$\Rightarrow$(a): As q_A is not weakly positive, we know by [Bo1, Theorem 3.3] that A is representation infinite. On the other hand for every proper convex subset T of A_0 the quadratic form $q_{A(T)} = q_A^T$ is weakly positive and thus $A(T)$ is representation finite.

4.3. The following result shows that the hypercritical algebras for completely separating algebras with weakly indefinite Tits form play the same role as the convex minimal wild algebras do for wild completely separating algebras. It is proved in [Pe2, 2.2] for good algebras, but the proof carries over easily to the completely separating case.

THEOREM. *Let A be a completely separating algebra. The Tits form of A is weakly indefinite if and only if A has a hypercritical convex subalgebra.*

We want to state another lemma which is presented in [Pe2, 2.6] for the good case, but the proof can be used verbatim.

LEMMA. *If A is a hypercritical completely separating algebra then there is an algebra B which is a concealment of a tame hereditary tree algebra and has the following property: A is a one point extension of B by an indecomposable preprojective B-module or A is a one point coextension of B by an indecomposable preinjective B-module.*

4.4. Now we come to the analogous result to 4.2 for the wild case which is proved in [Pe2, 3.6] for good algebras. As it is open whether the assumption that a completely separating algebra A has a weakly non-negative Tits form implies that A is tame, the analogy is not complete. Let us remark here that the list of all algebras which are concealments of minimal wild hereditary algebras is given in [U].

THEOREM. *Let A be a basic k-algebra with decomposition $\sum_{x \in A_0} e(x)$ of its unit element into primitive orthogonal idempotents. If A is completely separating and hypercritical, then A is convex minimal wild.*

Moreover the following assertions about A are equivalent:

(a) *A is a schurian concealment of a minimal wild hereditary tree algebra.*

(b) *A is completely separating and minimal wild. If $x \in A_0$ and $A/Ae(x)A$ is representation infinite, then x is an extremal element of $(A_0, \leqslant_A)$ and $A/Ae(x)A$ is a concealment of a connected tame hereditary algebra.*

(c) *A is completely separating and hypercritical.*

Proof. It is obvious that minimal wild algebras are convex minimal wild. So only the equivalences have to be proved. In view of 4.3 for (b)$\Rightarrow$(c) and (c)$\Rightarrow$(a) the arguments in [Pe2, 3.6] can be used. This applies also for (a)$\Rightarrow$(b) with the exception that we still have to show that schurian algebras which are concealments of a minimal wild hereditary tree algebra are completely separating.

In order to use 3.1, our first observation is that the Auslander-Reiten quiver of A has a complete preprojective component $\mathscr{P}$ whose orbit quiver is a tree. We denote the underlying graph of this tree by T. From 1.2 follows that the A is separating. So it remains to show that the algebras $\bar{B}^s(A)$ and $\underline{B}_s(A)$ are completely separating. By symmetry we only consider $\bar{B}^s(A)$.

Let us choose $u \in A_0$ such that P_u is not a predecessor in $\mathscr{P}$ of P_t for any $t \neq u$. By [Ri, 4.3(6)] (see also [Pe2, 3.3]) A^u is a concealed algebra whose orbit graph is just the subgraph of T induced by all points different from u. Thus A^u is also a separating algebra. Because we already know that A is minimal wild, we obtain that A^u is a product of indecomposable tame concealed or representation finite algebras $C_1, ..., C_r$. By 4.2 and 3.4 the algebras C_i are completely separating.

For $s \in A_0$ with $s \neq u$ we have $\bar{B}^s(A) \cong \bar{B}^s(C_i)$ for some $i = 1, ..., r$ which is completely separating. As A is separating, the radical R of P_u has a decomposition $R = \bigoplus_{i=1}^r R_i$ into indecomposable thin A-modules R_i which are actually C_i-modules. By 2.9 R is of the form M_V for some strictly convex subset V of A_0^u and therefore A is weakly transitive. Thus $B^u(A)$ is a convex subalgebra of A^u and consequently completely separating. As $\bar{B}^u(A)$ is the poset algebra of a poset with a unique maximal element, we derive from 3.4 that $\bar{B}^u(A)$ is also completely separating.

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