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# $C^*$-crossed products and shift spaces

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**Abstract**

We use Exel's $C^*$-crossed products associated to non-invertible dynamical systems to associate a $C^*$-algebra to arbitrary shift space. We show that this $C^*$-algebra is canonically isomorphic to the $C^*$-algebra associated to a shift space given by Carlsen [Cuntz–Pimsner $C^*$-algebras associated with subshifts, Internat. J. Math. (2004) 28, to appear, available at arXiv:math.OA/0505503], has the $C^*$-algebra defined by Carlsen and Matsumoto [Some remarks on the $C^*$-algebras associated with subshifts, Math. Scand. 95 (1) (2004) 145–160] as a quotient, and possesses properties indicating that it can be thought of as the universal $C^*$-algebra associated to a shift space.

We also consider its representations and its relationship to other $C^*$-algebras associated to shift spaces. We show that it can be viewed as a generalization of the universal Cuntz–Krieger algebra, discuss uniqueness and present a faithful representation, show that it is nuclear and satisfies the Universal Coefficient Theorem, provide conditions for it being simple and purely infinite, show that the constructed $C^*$-algebras and thus their $K$-theory, $K_0$ and $K_1$, are conjugacy invariants of one-sided shift spaces, present formulas for those invariants, and present a description of the structure of gauge invariant ideals.

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## 1. Introduction

When a dynamical system consisting of a homeomorphism of a topological space (or more generally, when an action of a group of invertible transformations of some space) is studied, there is a standard construction of a crossed product $C^*$-algebra. Historically this construction has its origins in foundations of quantum mechanics. The important idea behind this construction is that it encodes the action and the space within one algebra thus providing opportunities for their investigation on the same level. It is known that properties of the topological space can be considered via properties of the algebra of continuous functions defined on it. The crossed product algebra is typically generated by a copy of this algebra of functions together with additional elements which encode the action. The action is implemented by multiplication in the new algebra via covariance commutation relations between the elements in the algebra of functions and the elements used to encode the action. The crossed product construction has considerable applications in quantum mechanics and quantum field theory, and provides an important source of examples for further development of non-commutative geometry. A lot of research has been done on interplay between properties of invertible dynamical systems and properties of the corresponding crossed product $C^*$-algebras and $W^*$-algebras.

There are several ways to generalize the construction of $C^*$-crossed products to the non-invertible setting. The one we will focus on in this paper was introduced by Exel in [17]. This construction relies on a choice of transfer operator. Exel showed that for a natural choice of transfer operator, the $C^*$-algebra obtained from a one-sided topological Markov chain with finite state space is isomorphic to the Cuntz–Krieger algebra of the transition matrix of the Markov chain.

The Cuntz–Krieger algebras were introduced by Cuntz and Krieger in [16]. They can in a natural way be viewed as universal $C^*$-algebras associated with shift spaces of finite type. From the point of view of operator algebra these $C^*$-algebras were important examples of $C^*$-algebras with new properties, and from the point of view of topological dynamics these $C^*$-algebras (or rather, the $K$-theory of these $C^*$-algebras) led to new invariants of shift spaces of finite type.

In [26] Matsumoto proposed a generalization of this idea by constructing $C^*$-algebras associated to arbitrary shift spaces (he calls them *subshifts*). He studied these algebras in [27,28,31–33]. Unfortunately there is an error in [31] which invalidates many of the results in [27,28,31–33] for the $C^*$-algebra constructed in [26]. Since this error came to light, there has been some confusion about the right definition of the $C^*$-algebra associated to a shift space.

In this paper we use Exel's construction to associate a $C^*$-algebra to an arbitrary shift space, and we show that it has the properties which Matsumoto intended his algebra should have. In particular all the results of [26–28,31–33] hold for our algebra. We will also show that our algebra is canonically isomorphic to the $C^*$-algebra associated to a shift space by the first named author in [7], and that it has the $C^*$-algebra defined in [14] by the first named author and Matsumoto as a quotient. Thus it seems reasonable to think of this $C^*$-algebra as the universal $C^*$-algebra associated to a shift space.

Matsumoto's original construction associated a $C^*$-algebra to every *two-sided* shift space, but it seems more natural to work with *one-sided* shift spaces, and we do so in this paper. Since every two-sided shift space comes with a canonical one-sided shift space (see Section 4), the $C^*$-algebras we define in this paper can in a natural way also be seen as $C^*$-algebras associated to two-sided shift spaces.

The paper is organized as follows. In Section 2 we briefly recall the construction and properties of the classical $C^*$-crossed product of invertible dynamical systems, and in Section 3 we give a short description of Exel's construction of a $C^*$-crossed product of a non-invertible dynamical system. Section 4 is a short introduction to shift spaces, and in Section 5 we construct and characterize the $C^*$-algebra associated to a shift space. Section 6 is devoted to constructing a representation of our $C^*$-algebra. In Section 7 we prove that the $C^*$-algebra associated to a shift space in this paper is canonically isomorphic to the $C^*$-algebra associated to a shift space in [7], that it is a quotient of the $C^*$-algebra originally associated to a shift space by Matsumoto in [26], and that it has the $C^*$-algebra defined in [14] as a quotient. We prove in Section 8 that the class of $C^*$-algebras we obtain in this paper is a generalization of the Cuntz–Krieger algebras in the sense that the $C^*$-algebra associated to a one-sided topological Markov chain with finite state space is isomorphic the Cuntz–Krieger algebra of the transition matrix of the Markov chain. In Section 9 we present results similar to the uniqueness result for Cuntz–Krieger algebras and use this to construct a faithful representation of the $C^*$-algebra associated to a shift space. In Section 10 we prove that the $C^*$-algebra associated to a shift space is nuclear and satisfies the universal coefficient theorem, and we give conditions under which it is simple and purely infinite, and in Section 11 we prove that it is an invariant for one-sided conjugacy in the sense that if two one-sided shift spaces are conjugate, then the associated $C^*$-algebras are isomorphic. In Section 12 we present formulas for the $K$-theory of the $C^*$-algebra associated to a shift space, and in Section 13 we briefly describe the structure of the gauge invariant ideals. We end this paper in Section 14 by giving some references to papers in which the $C^*$-algebra associated to a shift space has been studied further for particular examples of shift spaces.

## 2. $C^*$-algebras of invertible dynamical systems

In this section, we review the construction and some properties of a $C^*$-crossed product of a $C^*$-algebra by the action of the discrete group of automorphisms. In particular, invertible dynamical systems generated by homeomorphisms of topological spaces correspond to crossed product $C^*$-algebras obtained from the actions of the group of integers on the $C^*$-algebra of complex-valued continuous functions.

Let $(A, G, \alpha)$ be a triple consisting of a unital $C^*$-algebra $A$, a discrete group $G$ and an action $\alpha : G \to \mathrm{Aut}(A)$ of $G$ on $A$, meaning a homomorphism from the group $G$ into the group $\mathrm{Aut}(A)$ of automorphisms of the $C^*$-algebra $A$. A pair $\{\pi, u\}$ consisting of a representation $\pi$ of $A$ and a unitary representation $u$ of $G$ on the same Hilbert space $H$ is called a covariant representation of the system $(A, G, \alpha)$ if the equation

$$u_s\pi(a)u_s^* = \pi(\alpha_s(a))$$

holds for every $a \in A$ and $s \in G$. The $C^*$-crossed product $A \rtimes_\alpha G$ is defined to be the universal $C^*$-algebra for covariant representations of $(A, G, \alpha)$.

Since a homomorphism $\phi$ of $\mathbb{Z}$ is completely determined by $\phi(1)$, it is, when $G = \mathbb{Z}$, enough to specify the defining covariance relation for $A \rtimes_\alpha G$ for $s = 1$, that is

$$u_1 \pi(a) u_1^* = \pi(\alpha_1(a)).$$

An object of special interest to us is the crossed product $C^*$-algebra for an invertible dynamical system consisting of iterations of a homeomorphism acting on a topological space.

Let $(X, \sigma)$ be a topological dynamical system consisting of a homeomorphism of a compact Hausdorff topological space $X$. The $*$-algebra of all continuous functions on $X$ will be denoted by $C(X)$. For a subset $Y$ of some given set $X$, we write $1_Y$ for the characteristic function

$$1_Y(x) = \begin{cases} 1 & \text{if } x \in Y, \\ 0 & \text{if } x \notin Y, \end{cases} \qquad x \in X.$$

In particular, $1_X$ is a unit for $C(X)$, and $C(X)$ becomes a unital $C^*$-algebra with respect to the supremum norm defined by

$$\|f\| = \|f\|_{C(X)} = \sup\{|f(x)| \mid x \in X\}.$$

The mapping $\alpha : C(X) \to C(X)$ defined by $\alpha(f)(x) = f(\sigma^{-1}(x))$ is an automorphism of the $C^*$-algebra $C(X)$, and the mapping defined by

$$j \mapsto \alpha^j(f)(x) = f(\sigma^{-j}(x))$$

is a homomorphism of $\mathbb{Z}$ into the group $\mathrm{Aut}(C(X))$ of automorphisms of $C(X)$.

The $C^*$-crossed product $C(X) \rtimes_\alpha \mathbb{Z}$ of the $C^*$-dynamical system $(C(X), \mathbb{Z}, \alpha)$ can then be characterized as the universal unital $C^*$-algebra generated by a copy of $C(X)$ and a unitary $u$ which satisfies the equation

$$u f u^* = \alpha(f) \tag{1}$$

for every $f \in C(X)$.

Using relation (1), it is not difficult to show that the set

$$\left\{ \sum_{j \in J} f_j u^j \,\middle|\, J \text{ is a finite subset of } \mathbb{Z},\ \ f_j \in C(X) \text{ for all } j \in J \right\}$$

is a dense $*$-subalgebra of $C(X) \rtimes_\alpha \mathbb{Z}$. The mapping

$$\sum_{j \in J} f_j u^j \mapsto f_0$$

(here we assume that $0 \in J$) from this $*$-subalgebra to $C(X)$ can be extended to a projection $E$ of norm 1 (a *conditional expectation*) from $C(X) \rtimes_\alpha \mathbb{Z}$ to $C(X)$ which has the following properties:

(i) $E(fxg) = fE(x)g$ for all $f, g \in C(X)$ and $x \in C(X)\rtimes_\alpha\mathbb{Z}$,
(ii) $E(u) = 0$,
(iii) $E(x^*x)\geqslant 0$ for all $x \in C(X)\rtimes_\alpha\mathbb{Z}$,
(iv) $E(x^*x) = 0$ implies that $x = 0$ for all $x \in C(X)\rtimes_\alpha\mathbb{Z}$.

## 3. $C^*$-algebras of non-invertible dynamical systems

There are several ways to generalize the $C^*$-crossed product to non-invertible dynamical systems. One of these is due to Exel. It relies on *transfer operators*.

We will in this section give a short description of Exel's construction:

**Definition 1.** A $C^*$-*dynamical system* is a pair $(A, \alpha)$ consisting of a unital $C^*$-algebra $A$ and an endomorphism $\alpha : A \to A$.

**Definition 2.** A *transfer operator* for the $C^*$-dynamical system $(A, \alpha)$ is a continuous linear map $\mathscr{L} : A \to A$ such that

1. $\mathscr{L}$ is positive in the sense that $\mathscr{L}(x)$ is positive if $x$ is positive,
2. $\mathscr{L}(\alpha(a)b) = a\mathscr{L}(b)$ for all $a, b \in A$.

**Definition 3.** Given a $C^*$-dynamical system $(A, \alpha)$ and a transfer operator $\mathscr{L}$ of $(A, \alpha)$, we let $\mathscr{T}(A, \alpha, \mathscr{L})$ be the universal unital $C^*$-algebra generated by a copy of $A$ and an element $s$ subject to the relations

1. $sa = \alpha(a)s$,
2. $s^*as = \mathscr{L}(a)$

for all $a \in A$.

Using [3], it is easy to see that relations (1) and (2) are admissible and thus that $\mathscr{T}(A, \alpha, \mathscr{L})$ exists. It is proved in [17, Corollary 3.5] that the standard embedding of $A$ into $\mathscr{T}(A, \alpha, \mathscr{L})$ is injective. We will therefore from now on view $A$ as a $C^*$-subalgebra of $\mathscr{T}(A, \alpha, \mathscr{L})$.

**Definition 4.** By a redundancy we will mean a pair $(a, k) \in A\times\overline{ASS^*A}$ such that $abS=kbS$ for all $b \in A$.

**Definition 5.** The crossed product $A\rtimes_{\alpha,\mathscr{L}}\mathbb{N}$ is the quotient of $\mathscr{T}(A, \alpha, \mathscr{L})$ by the closed two-sided ideal generated by the set of differences $a - k$, for all redundancies $(a, k)$ such that $a \in \overline{A\alpha(A)A}$.

We will denote the quotient map from $\mathscr{T}(A, \alpha, \mathscr{L})$ to $A\rtimes_{\alpha,\mathscr{L}}\mathbb{N}$ by $\rho$.

We will now show that this construction in fact is a generalization of the $C^*$-crossed product we considered in Section 2.

**Remark 6.** If $(A, \alpha)$ is an invertible $C^*$-dynamical system, meaning that $\alpha$ is an automorphism, then $\alpha^{-1}$ is a transfer operator for $(A, \alpha)$.

Let us consider $\mathscr{T}(A, \alpha, \alpha^{-1})$. It follows from (2) that $s^*s = 1_A$, where $1_A$ denotes the unit of $A$. For all $b \in A$, we have

$$ss^*bs = s\alpha^{-1}(b) = bs = 1_A bs,$$

so $(1_A, ss^*)$ is a redundancy. Thus $\rho(s)$ is a unitary which satisfies

$$\rho(s)\rho(a)\rho(s)^* = \rho(\alpha(a))$$

for all $a \in A$. In other words, $(\rho, \rho(s))$ is a covariant representation of $(A, \mathbb{Z}, \alpha)$.

On the other hand, in $A \rtimes_\alpha \mathbb{Z}$, the unitary element $u_1$ satisfies

1. $u_1 a = \alpha(a) u_1$,
2. $u_1^* a u_1 = \alpha^{-1}(a)$,

for all $a \in A$, and if $abu_1 = kbu_1$ for all $b \in A$, then we have

$$a = a1_A = a1_A u_1 u_1^* = k1_A u_1 u_1^* = k1_A = k,$$

so $a - k = 0$ for all redundancies $(a, k)$, and thus $A \rtimes_\alpha \mathbb{Z}$ is isomorphic to $A \rtimes_{\alpha,\alpha^{-1}} \mathbb{N}$.

## 4. Shift spaces

As mentioned in the introduction, the purpose of this paper is to apply Exel's construction to shift spaces. Shift spaces (also called subshifts) are a class of topological dynamical systems with obvious applications in information technology, but have also been used in the study of more complex dynamical systems.

In this section, we briefly give the basic definition of a shift space and related concepts and introduce some notation which will be used throughout the paper. We recommend [25,23] to the reader who wants to know more about shift spaces.

Let $\mathfrak{a}$ be a finite set endowed with the discrete topology. We will call this set the *alphabet* and its elements *letters*. Let $\mathfrak{a}^{\mathbb{N}}$ be the infinite product space $\prod_{n=0}^{\infty} \mathfrak{a}$ endowed with the product topology. The transformation $\sigma$ on $\mathfrak{a}^{\mathbb{N}}$ given by

$$(\sigma(x))_i = x_{i+1}, \quad i \in \mathbb{N}$$

is called the *shift*. Let $\mathsf{X}$ be a shift invariant closed subset of $\mathfrak{a}^{\mathbb{N}}$ (by shift invariant we mean that $\sigma(\mathsf{X}) \subseteq \mathsf{X}$, not necessarily $\sigma(\mathsf{X}) = \mathsf{X}$). The topological dynamical system $(\mathsf{X}, \sigma_{|\mathsf{X}})$ is called a *shift space* (or a *subshift*). We will denote $\sigma_{|\mathsf{X}}$ by $\sigma_{\mathsf{X}}$ or $\sigma$ for simplicity, and on occasion the alphabet $\mathfrak{a}$ by $\mathfrak{a}_{\mathsf{X}}$.

We denote the $n$-fold composition of $\sigma$ with itself by $\sigma^n$, and we denote the preimage of a set $X$ under $\sigma^n$ by $\sigma^{-n}(X)$.

A finite sequence $u = (u_1, \ldots, u_k)$ of elements $u_i \in \mathfrak{a}$ is called a finite *word*. The *length* of $u$ is $k$ and is denoted by $|u|$. For each $k \in \mathbb{N}$, we let $\mathfrak{a}^k$ be the set of all words with length $k$, and we let $\mathsf{L}^k(\mathsf{X})$ be the set of all words with length $k$ appearing in some

$x \in \mathsf{X}$. We set $\mathsf{L}_l(\mathsf{X}) = \bigcup_{k=0}^{l} \mathsf{L}^k(\mathsf{X})$ and $\mathsf{L}(\mathsf{X}) = \bigcup_{k=0}^{\infty} \mathsf{L}^k(\mathsf{X})$ and likewise $\mathfrak{a}_l = \bigcup_{k=0}^{l} \mathfrak{a}^k$ and $\mathfrak{a}^* = \bigcup_{k=0}^{\infty} \mathfrak{a}^k$ where $\mathsf{L}^0(\mathsf{X}) = \mathfrak{a}^0$ denotes the set $\{\varepsilon\}$ consisting of the empty word $\varepsilon$ which has length 0. The set $\mathsf{L}(\mathsf{X})$ is called the *language* of $\mathsf{X}$. Note that $\mathsf{L}(\mathsf{X}) \subseteq \mathfrak{a}^*$ for every shift space.

For a shift space $\mathsf{X}$ and a word $u \in \mathsf{L}(\mathsf{X})$, we denote by $C_{\mathsf{X}}(u)$ the *cylinder set*

$$C_{\mathsf{X}}(u) = \{x \in \mathsf{X} | (x_1, x_2, \ldots, x_{|u|}) = u\}.$$

It is easy to see that the family

$$\{C_{\mathsf{X}}(u) | u \in \mathsf{L}(\mathsf{X})\}$$

of cylinder sets, is a basis for the topology of $\mathsf{X}$, and that $C_{\mathsf{X}}(u)$ is closed and compact for every $u \in \mathsf{L}(\mathsf{X})$. We will write $C(u)$ instead of $C_{\mathsf{X}}(u)$ when it is clear which shift space we are working with.

For a shift space $\mathsf{X}$ and words $u, v \in \mathsf{L}(\mathsf{X})$, we denote by $C(u, v)$ the set

$$C(v) \cap \sigma^{-|v|}(\sigma^{|u|}(C(u))) = \{vx \in \mathsf{X} \mid ux \in \mathsf{X}\}.$$

What we have defined above is a *one-sided* shift space. A *two-sided* shift space is defined in the same way, except that we replace $\mathbb{N}$ with $\mathbb{Z}$: Let $\mathfrak{a}^{\mathbb{Z}}$ be the infinite product space $\prod_{n=-\infty}^{\infty} \mathfrak{a}$ endowed with the product topology, and let $\sigma$ be the transformation on $\mathfrak{a}^{\mathbb{Z}}$ given by

$$(\sigma(x))_i = x_{i+1}, \quad i \in \mathbb{Z}.$$

A shift invariant closed subset $\Lambda$ of $\mathfrak{a}^{\mathbb{Z}}$ (here, by shift invariant we mean $\sigma(\Lambda) = \Lambda$) is called a *two-sided shift space*. The set

$$\mathsf{X}_\Lambda = \{(x_i)_{i \in \mathbb{N}} \mid (x_i)_{i \in \mathbb{Z}} \in \Lambda\}$$

is a one-sided shift space, and it is called the one-sided shift space of $\Lambda$.

If $\mathsf{X}$ and $\mathsf{Y}$ are two shift spaces and $\phi : \mathsf{X} \to \mathsf{Y}$ is a homeomorphism such that $\phi \circ \sigma_{\mathsf{X}} = \sigma_{\mathsf{Y}} \circ \phi$, then we say that $\phi$ is a *conjugacy* and that $\mathsf{X}$ and $\mathsf{Y}$ are *conjugate* or *one-sided conjugate* if we want to emphasis that we are dealing with one-sided shift spaces. Likewise we say that two two-sided shift spaces $\Lambda$ and $\Gamma$ are *two-sided conjugate* if there exists a homeomorphism $\phi : \Lambda \to \Gamma$ such that $\phi \circ \sigma_\Lambda = \sigma_\Gamma \circ \phi$. It is an easy exercise to prove that if $\mathsf{X}_\Lambda$ and $\mathsf{X}_\Gamma$ are one-sided conjugate, then $\Lambda$ and $\Gamma$ are two-sided conjugate.

The weaker notion of *flow equivalence* among two-sided shift spaces is also of importance here. This notion is defined using the *suspension flow space* of $(\Lambda, \sigma)$ defined as $S\Lambda = (\Lambda \times \mathbb{R}) / \sim$ where the equivalence relation $\sim$ is generated by the relations $(x, t+1) \sim (\sigma(x), t)$. Equipped with the quotient topology, $S\Lambda$ is a compact space with a *continuous flow*: a family of maps $(\phi_t)$ defined by $\phi_t([x, s]) = [x, s+t]$. We say that two two-sided shift spaces $\Lambda$ and $\Gamma$ are *flow equivalent* and write $\Lambda \cong_f \Gamma$ if there exists a homeomorphism $F : S\Lambda \to S\Gamma$ such that for every $x \in S\Lambda$, there is a monotonically increasing map $f_x : \mathbb{R} \to \mathbb{R}$ satisfying

$$F(\phi_t(x)) = \phi'_{f_x(t)}(F(x)).$$

In words, $F$ takes flow orbits to flow orbits in an orientation-preserving way. It is not hard to see that two-sided conjugacy implies flow equivalence.

## 5. The $C^*$-algebra associated with a shift space

In [17], Exel proved that if $A = (A(i, j))_{i,j=1,\dots,n}$ is an $n \times n$-matrix with $A(i, j) \in \{0, 1\}$ for all $i, j \in \{1, \dots, n\}$, then the crossed product $C(\mathsf{X}_A) \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$ is isomorphic to the Cuntz–Krieger algebra $\mathcal{O}_A$ where $\mathsf{X}_A$ is the one-sided topological Markov chain

$$\mathsf{X}_A = \{(x_n)_{n\in\mathbb{N}} \in \{1, \dots, n\}^{\mathbb{N}} | \forall n \in \mathbb{N} : A(x_n, x_{n+1}) = 1\},$$

with transition matrix $A$, $\alpha$ is the endomorphism on $C(\mathsf{X}_A)$ given by

$$\alpha(f) = f \circ \sigma \tag{2}$$

(where $\sigma$ is the shift mapping on $\mathsf{X}_A$), and $\mathscr{L}$ is the transfer operator of the system $(C(\mathsf{X}_A), \alpha)$ defined by

$$\mathscr{L}(f)(x) = \begin{cases} \dfrac{1}{\#\sigma^{-1}(\{x\})} \displaystyle\sum_{y\in\sigma^{-1}\{x\}} f(y) & \text{if } x \in \sigma(\mathsf{X}_A), \\ 0 & \text{if } x \notin \sigma(\mathsf{X}_A), \end{cases}$$

where the symbol # is used for the cardinality of a set.

We want to copy this approach for an arbitrary one-sided shift space $(\mathsf{X}, \sigma)$. There is, however, a problem with this. If we define $\mathscr{L}$ by

$$\mathscr{L}(f)(x) = \begin{cases} \dfrac{1}{\#\sigma^{-1}(\{x\})} \displaystyle\sum_{y\in\sigma^{-1}\{x\}} f(y) & \text{if } x \in \sigma(\mathsf{X}), \\ 0 & \text{if } x \notin \sigma(\mathsf{X}), \end{cases} \tag{3}$$

then $\mathscr{L}$ might take us out of the class of continuous functions on $\mathsf{X}$ (in fact, it follows from [19, Theorem 1] that $\mathscr{L}$ maps $C(\mathsf{X})$ into $C(\mathsf{X})$ if and only if $\mathsf{X}$ is of finite type). We deal which this problem by enlarging $C(\mathsf{X})$ to a $C^*$-algebra $\mathscr{D}_{\mathsf{X}}$ which is closed under $\mathscr{L}$.

For a one-sided shift space $(\mathsf{X}, \sigma)$, we let $\mathscr{D}_{\mathsf{X}}$ be the smallest $C^*$-subalgebra of the $C^*$-algebra of bounded functions on $\mathsf{X}$ which contains $C(\mathsf{X})$ and is closed under $\mathscr{L}$ and $\alpha$ where $\alpha$ is defined by (2) and $\mathscr{L}$ by (3).

**Lemma 7.** *For every $n \in \mathbb{N}$, every $f \in \mathscr{D}_{\mathsf{X}}$ and every $x \in \mathsf{X}$, we have*

$$\mathscr{L}^n(f)(x) = \begin{cases} \dfrac{1}{\#\sigma^{-n}(\{x\})} \displaystyle\sum_{y\in\sigma^{-n}\{x\}} f(y) & \text{if } x \in \sigma^n(\mathsf{X}), \\ 0 & \text{if } x \notin \sigma^n(\mathsf{X}). \end{cases}$$

**Proof.** The lemma is easily proved by induction over $n$. $\square$

**Lemma 8.** *The function*

$$x \mapsto \#\sigma^{-n}\{x\}, \quad x \in \mathsf{X}$$

*belongs to* $\mathscr{D}_{\mathsf{X}}$ *for every* $n \in \mathbb{N}$.

**Proof.** Let $n \in \mathbb{N}$. Then the function

$$g = 1 - \mathscr{L}^n(1) + \sum_{u \in \mathfrak{a}^n} (\mathscr{L}^n(1_{C(u)}))^2$$

belongs to $\mathscr{D}_{\mathsf{X}}$, and since the equality

$$g(x) = \begin{cases} \dfrac{1}{\#\sigma^{-n}\{x\}} & \text{if } x \in \sigma^n(\mathsf{X}), \\ 1 & \text{if } x \notin \sigma^n(\mathsf{X}), \end{cases}$$

holds for every $x \in \mathsf{X}$, the function $g$ is invertible and $g^{-1}$ belongs to $\mathscr{D}_{\mathsf{X}}$, and so does $g^{-1} + \mathscr{L}^n(1) - 1$. Since we have

$$(g^{-1} + \mathscr{L}^n(1) - 1)(x) = \begin{cases} \#\sigma^{-n}\{x\} & \text{if } x \in \sigma^n(\mathsf{X}), \\ 0 & \text{if } x \notin \sigma^n(\mathsf{X}), \end{cases} = \#\sigma^{-n}\{x\}$$

for every $x \in \mathsf{X}$, we are done. $\square$

**Lemma 9.** *The* $C^*$*-algebra* $\mathscr{D}_{\mathsf{X}}$ *is the smallest* $C^*$*-subalgebra of the* $C^*$*-algebra of bounded functions of* $\mathsf{X}$ *which contains* $\{1_{C(u,v)} \mid u, v \in \mathfrak{a}^*\}$.

**Proof.** Let $u, v \in \mathfrak{a}^*$ and let $f(x) = \#\sigma^{-|u|}\{x\}$. Then Lemma 8 implies that $f \in \mathscr{D}_{\mathsf{X}}$. Hence the function

$$1_{C(u,v)} = 1_{C(v)}\alpha^{|v|}(f\mathscr{L}^{|u|}(1_{C(u)}))$$

belongs to $\mathscr{D}_{\mathsf{X}}$. We thus have

$$\{1_{C(u,v)} \mid u, v \in \mathfrak{a}^*\} \subseteq \mathscr{D}_{\mathsf{X}}.$$

In the other direction, since $\{C(v) \mid v \in \mathfrak{a}^*\}$ is a basis of the topology of $\mathsf{X}$ consisting of clopen sets, the family $\{1_{C(v)} \mid v \in \mathfrak{a}^*\}$ generates $C(\mathsf{X})$, and since $1_{C(v)} = 1_{C(\varepsilon,v)}$, it follows that $C(\mathsf{X})$ is contained in any $C^*$-algebra which contains $\{1_{C(u,v)} \mid u, v \in \mathfrak{a}^*\}$.

Since the equation

$$\alpha(1_{C(u,v)}) = \sum_{a \in \mathfrak{a}} 1_{C(u,av)}$$

holds for all $u, v \in \mathfrak{a}^*$, and $\alpha$ is a $*$-homomorphism, the $C^*$-algebra generated by $\{1_{C(u,v)} \mid u, v \in \mathfrak{a}^*\}$ is closed under $\alpha$. Somewhat tedious calculations show that $\mathscr{L}$ maps any product of the form $\prod_{i=1}^n 1_{C(u_i,v_i)}$ with $u_1, \ldots, u_n, v_1, \ldots, v_n \in \mathfrak{a}^*$ into the $C^*$-algebra generated by $\{1_{C(u,v)} \mid u, v \in \mathfrak{a}^*\}$, and since $\mathscr{L}$ is continuous and linear, it follows that the $C^*$-algebra generated by $\{1_{C(u,v)} \mid u, v \in \mathfrak{a}^*\}$ is also closed under $\mathscr{L}$ and thus contains $\mathscr{D}_{\mathsf{X}}$. $\square$

The operator $\mathscr{L}$ defined by (3) is a transfer operator for the $C^*$-dynamical system $(\mathscr{D}_{\mathsf{X}}, \alpha)$ with $\alpha$ defined by (2). Thus we can form the crossed product $\mathscr{D}_{\mathsf{X}} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$. It is characterized by the following theorem.

**Theorem 10.** *Let* $(\mathsf{X}, \sigma)$ *be a one-sided shift space and let* $\mathscr{D}_{\mathsf{X}}$, $\alpha$ *and* $\mathscr{L}$ *be as above. Then the crossed product* $\mathscr{D}_{\mathsf{X}} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$ *is the universal* $C^*$*-algebra generated by a family of partial isometries* $(s_u)_{u\in\mathfrak{a}^*}$ *satisfying*:

(1) $s_u s_v = s_{uv}$ *for all* $u, v \in \mathfrak{a}^*$,
(2) *the map*

$$1_{C(u,v)} \mapsto s_v s_u^* s_u s_v^*, \quad u, v \in \mathfrak{a}^*$$

*extends to a* $*$*-homomorphism from* $\mathscr{D}_{\mathsf{X}}$ *to the* $C^*$*-algebra generated by* $\{s_u \mid u \in \mathfrak{a}^*\}$.

**Proof.** We will first show that $\mathscr{D}_{\mathsf{X}} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$ is generated by a family of partial isometries $(s_u)_{u\in\mathfrak{a}^*}$ which satisfies (1) and (2), and then that if $A$ is a $C^*$-algebra generated by a family of partial isometries $(\widetilde{s}_u)_{u\in\mathfrak{a}^*}$ which satisfies (1) and (2), then there is a $*$-homomorphism from $\mathscr{D}_{\mathsf{X}} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$ to $A$ sending $s_u$ to $\widetilde{s}_u$ for all $u \in \mathfrak{a}^*$.

For each $a \in \mathfrak{a}$, we let $t_a$ be the element of $\mathscr{T}(\mathscr{D}_{\mathsf{X}}, \alpha, \mathscr{L})$ given by

$$t_a = 1_{C(a)}(\alpha(f))^{1/2} s,$$

where $f$ is the function $x \mapsto \#\sigma^{-1}\{x\}$, which belongs to $\mathscr{D}_{\mathsf{X}}$ by Lemma 8. For each $u = u_1 u_2 \cdots u_n \in \mathfrak{a}^*$, let $s_u$ be the element of $\mathscr{D}_{\mathsf{X}} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$ defined by

$$s_u = \rho(t_{u_1})\rho(t_{u_2}) \cdots \rho(t_{u_n}).$$

Then clearly the family $(s_u)_{u\in\mathfrak{a}^*}$ satisfies (1).

Let $a \in \mathfrak{a}$ and $g \in \mathscr{D}_{\mathsf{X}}$. We will show that the pair

$$(\alpha(g)1_{C(a)}, t_a g t_a^*)$$

is a redundancy (see Definition 4). So let $h \in \mathscr{D}_{\mathsf{X}}$. We have

$$\begin{aligned} t_a g t_a^* h s &= 1_{C(a)}(\alpha(f))^{1/2} s g s^* (\alpha(f))^{1/2} 1_{C(a)} h s \\ &= 1_{C(a)}(\alpha(f))^{1/2} s g \mathscr{L}((\alpha(f))^{1/2} 1_{C(a)} h) \\ &= 1_{C(a)} \alpha(f^{1/2} g \mathscr{L}((\alpha(f))^{1/2} 1_{C(a)} h)) s, \end{aligned}$$

and

$$(1_{C(a)}\alpha(f^{1/2}g\mathscr{L}((\alpha(f))^{1/2}1_{C(a)}h)))(x)=\begin{cases} g(\sigma(x))h(x) & \text{if } x\in C(a),\\ 0 & \text{if } x\notin C(a),\end{cases}$$

for every $x\in\mathsf{X}$. Thus $t_a g t_a^* h s=\alpha(g)1_{C(a)}hs$.

Since $sgs^*=\alpha(g)ss^*$, we have

$$t_a g t_a^* = 1_{C(a)}(\alpha(f))^{1/2} sgs^* (\alpha(f))^{1/2} 1_{C(a)} \in \overline{\mathscr{D}_{\mathsf{X}} ss^* \mathscr{D}_{\mathsf{X}}},$$

so $(\alpha(g)1_{C(a)}, t_a g t_a^*)$ is a redundancy. Since $\alpha(g)1_{C(a)}\in\overline{\mathscr{D}_{\mathsf{X}}\alpha(\mathscr{D}_{\mathsf{X}})\mathscr{D}_{\mathsf{X}}}$, it follows from Definition 5 that $s_a\rho(g)s_a^*$ and $\rho(\alpha(g)1_{C(a)})$ are equal in $\mathscr{D}_{\mathsf{X}}\rtimes_{\alpha,\mathscr{L}}\mathbb{N}$.

We also have

$$\begin{aligned}(t_a^* g t_a)(x) &= (s^*\alpha(f)1_{C(a)}gs)(x)\\ &= (\mathscr{L}(\alpha(f)1_{C(a)}g))(x)\\ &= (f\mathscr{L}(1_{C(a)}g))(x)\\ &= \begin{cases} g(ax) & \text{if } ax\in\mathsf{X},\\ 0 & \text{if } ax\notin\mathsf{X},\end{cases}\end{aligned}$$

for every $x\in\mathsf{X}$.

Thus $s_a^*\rho(g)s_a=\rho(\lambda_a(g))$ and $s_a\rho(g)s_a^*=\rho(\alpha(g)1_{C(a)})$ for every $a\in\mathfrak{a}$ and $g\in\mathscr{D}_{\mathsf{X}}$, where $\lambda_a(g)$ is the map given by

$$\lambda_a(g)(x)=\begin{cases} g(ax) & \text{if } ax\in\mathsf{X},\\ 0 & \text{if } ax\notin\mathsf{X},\end{cases}$$

for $x\in\mathsf{X}$. It easily follows from this that

$$\rho(1_{C(u,v)})=s_v s_u^* s_u s_v^*$$

for every $u,v\in\mathfrak{a}^*$. Hence the family $(s_u)_{u\in\mathfrak{a}^*}$ is a family of partial isometries and satisfies (2). To see that $\mathscr{D}_{\mathsf{X}}\rtimes_{\alpha,\mathscr{L}}\mathbb{N}$ is generated by $\{s_u \mid u\in\mathfrak{a}^*\}$, we first notice that $\mathscr{T}(\mathscr{D}_{\mathsf{X}},\alpha,\mathscr{L})$ is generated by $\mathscr{D}_{\mathsf{X}}$ and $s$, and that $\mathscr{D}_{\mathsf{X}}$, by Lemma 9, is generated by $\{1_{C(u,v)} \mid u,v\in\mathfrak{a}^*\}$, and then that the function $\alpha(f)$, where $f$ as before is the function $x\mapsto \#\sigma^{-1}\{x\},\ x\in\mathsf{X}$, is invertible and that $s=\sum_{a\in\mathfrak{a}}\alpha(f)^{-1/2}t_a$. Thus it follows that $\mathscr{D}_{\mathsf{X}}\rtimes_{\alpha,\mathscr{L}}\mathbb{N}$ is generated by $\{s_u \mid u\in\mathfrak{a}^*\}$.

Assume now that $A$ is a $C^*$-algebra generated by a family $(\widetilde{s}_u)_{u\in\mathfrak{a}^*}$ of partial isometries which satisfies (1) and (2). We let $\widetilde{s}=\sum_{a\in\mathfrak{a}}\phi(\alpha(f)^{-1/2})\widetilde{s}_a$, where $\phi$ is the $*$-homomorphism from $\mathscr{D}_{\mathsf{X}}$ to $A$ which extends the map

$$1_{C(u,v)}\mapsto \widetilde{s}_v\widetilde{s}_u^*\widetilde{s}_u\widetilde{s}_v^*,\quad u,v\in\mathfrak{a}^*.$$

We will show that the following two equalities:

$$\widetilde{s}\phi(g)=\phi(\alpha(g))\widetilde{s}, \tag{4}$$

$$\widetilde{s}^* \phi(g) \widetilde{s} = \phi(\mathscr{L}(g)), \tag{5}$$

hold for all $g \in \mathscr{D}_{\mathsf{X}}$.

Observe first that if $a, b \in \mathfrak{a}$ and $a \neq b$, then $\widetilde{s}_a^* \widetilde{s}_b = 0$, because we have

$$\widetilde{s}_a^* \widetilde{s}_b = \widetilde{s}_a^* \widetilde{s}_a \widetilde{s}_a^* \widetilde{s}_b \widetilde{s}_b^* \widetilde{s}_b = \widetilde{s}_a^* \phi(1_{C(a)} 1_{C(b)}) \widetilde{s}_b.$$

Let $a \in \mathfrak{a}$ and $u, v \in \mathfrak{a}^*$. If $v \neq \varepsilon$, then the following equalities:

$$\begin{aligned}
\widetilde{s}_a^* \phi(1_{C(u,v)}) \widetilde{s}_a &= \widetilde{s}_a^* \widetilde{s}_v \widetilde{s}_u^* \widetilde{s}_u \widetilde{s}_v^* \widetilde{s}_a \\
&= \widetilde{s}_a^* \widetilde{s}_{v_1} \widetilde{s}_{v_2 v_3 \cdots v_{|v|}} \widetilde{s}_u^* \widetilde{s}_u \widetilde{s}_{v_2 v_3 \cdots v_{|v|}}^* \widetilde{s}_{v_1}^* \widetilde{s}_a \\
&= \begin{cases} \phi(1_{C(v_1,\varepsilon)} 1_{C(u, v_2 v_3 \cdots v_{|v|})}) & \text{if } a = v_1, \\ 0 & \text{if } a \neq v_1 \end{cases} \\
&= \phi(\lambda_a(1_{C(u,v)}))
\end{aligned}$$

hold, and if $v = \varepsilon$, then the following equalities:

$$\begin{aligned}
\widetilde{s}_a^* \phi(1_{C(u,v)}) \widetilde{s}_a &= \widetilde{s}_a^* \widetilde{s}_u^* \widetilde{s}_u \widetilde{s}_a \\
&= \widetilde{s}_{ua}^* \widetilde{s}_{ua} \\
&= \phi(1_{C(ua,\varepsilon)}) \\
&= \phi(\lambda_a(1_{C(u,v)}))
\end{aligned}$$

hold. Since $\mathscr{D}_{\mathsf{X}}$ is generated by $\{1_{C(u,v)} \mid u, v \in \mathfrak{a}^*\}$, it follows that $\widetilde{s}_a^* \phi(g) \widetilde{s}_a = \phi(\lambda_a(g))$ for each $a \in \mathfrak{a}$ and every $g \in \mathscr{D}_{\mathsf{X}}$. Therefore, we have

$$\begin{aligned}
\widetilde{s}^* \phi(g) \widetilde{s} &= \sum_{a \in \mathfrak{a}} \sum_{b \in \mathfrak{a}} \widetilde{s}_a^* \phi(\alpha(f)^{-1/2} g \alpha(f)^{-1/2}) \widetilde{s}_b \\
&= \sum_{a \in \mathfrak{a}} \sum_{b \in \mathfrak{a}} \widetilde{s}_a^* \phi(\alpha(f)^{-1} 1_{C(a)} 1_{C(b)} g) \widetilde{s}_b \\
&= \sum_{a \in \mathfrak{a}} \widetilde{s}_a^* \phi(\alpha(f)^{-1} 1_{C(a)} g) \widetilde{s}_a \\
&= \sum_{a \in \mathfrak{a}} \phi(\lambda_a(\alpha(f)^{-1} 1_{C(a)} g)) \\
&= \phi(\mathscr{L}(g))
\end{aligned}$$

for every $g \in \mathscr{D}_{\mathsf{X}}$. Thus (5) holds.

Let $u, v \in \mathfrak{a}^*$. We then have that the following series of equalities:

$$\begin{aligned}
\phi(\alpha(1_{C(u,v)}))\widetilde{s} &= \sum_{a\in\mathfrak{a}} \phi(1_{C(u,av)})\widetilde{s} \\
&= \sum_{a\in\mathfrak{a}} \phi(1_{C(u,av)}) \sum_{b\in\mathfrak{a}} \phi(\alpha(f)^{-1/2})\widetilde{s}_b \\
&= \sum_{a\in\mathfrak{a}} \sum_{b\in\mathfrak{a}} \phi(\alpha(f)^{-1/2} 1_{C(u,av)})\widetilde{s}_b\widetilde{s}_b^*\widetilde{s}_b \\
&= \sum_{a\in\mathfrak{a}} \sum_{b\in\mathfrak{a}} \phi(\alpha(f)^{-1/2} 1_{C(u,av)} 1_{C(b)})\widetilde{s}_b \\
&= \sum_{a\in\mathfrak{a}} \phi(\alpha(f)^{-1/2} 1_{C(u,av)})\widetilde{s}_a \\
&= \sum_{a\in\mathfrak{a}} \phi(\alpha(f)^{-1/2})\widetilde{s}_{av}\widetilde{s}_u^*\widetilde{s}_u\widetilde{s}_{av}^*\widetilde{s}_a \\
&= \sum_{a\in\mathfrak{a}} \phi(\alpha(f)^{-1/2})\widetilde{s}_a\widetilde{s}_v\widetilde{s}_u^*\widetilde{s}_u\widetilde{s}_v^*\widetilde{s}_a^*\widetilde{s}_a \\
&= \sum_{a\in\mathfrak{a}} \phi(\alpha(f)^{-1/2})\widetilde{s}_a\phi(1_{C(u,v)}1_{C(a,\varepsilon)}) \\
&= \sum_{a\in\mathfrak{a}} \phi(\alpha(f)^{-1/2})\widetilde{s}_a\phi(1_{C(a,\varepsilon)}1_{C(u,v)}) \\
&= \sum_{a\in\mathfrak{a}} \phi(\alpha(f)^{-1/2})\widetilde{s}_a\widetilde{s}_a^*\widetilde{s}_a\phi(1_{C(u,v)}) \\
&= \sum_{a\in\mathfrak{a}} \phi(\alpha(f)^{-1/2})\widetilde{s}_a\phi(1_{C(u,v)}) \\
&= \widetilde{s}\phi(1_{C(u,v)})
\end{aligned}$$

holds, and since $\mathscr{D}_\mathsf{X}$ is generated by $\{1_{C(u,v)} \mid u, v \in \mathfrak{a}^*\}$, this shows that $\phi(\alpha(g))\widetilde{s} = \widetilde{s}\phi(g)$ for every $g \in \mathscr{D}_\mathsf{X}$. Hence (4) holds.

Thus (4) and (5) hold, so it follows from the universal property of $\mathscr{T}(\mathscr{D}_\mathsf{X}, \alpha, \mathscr{L})$, that there exists a $*$-homomorphism $\psi$ from $\mathscr{T}(\mathscr{D}_\mathsf{X}, \alpha, \mathscr{L})$ to $A$ which maps $g$ to $\phi(g)$ for $g \in \mathscr{D}_\mathsf{X}$, and $s$ to $\widetilde{s}$. It suffices to show that $\psi$ vanishes on the closed two-sided ideal generated by the set of differences $g - k$, for all redundancies $(g, k)$ such that $g \in \overline{\mathscr{D}_\mathsf{X}\alpha(\mathscr{D}_\mathsf{X})\mathscr{D}_\mathsf{X}}$. For if so, then $\psi$ factors through the quotient and yields a $*$-homomorphism $\widetilde{\psi}:\mathscr{D}_{\mathsf{X}}\rtimes_{\alpha,\mathscr{L}}\mathbb{N}\to A$ such that $\widetilde{\psi}(\rho(g))=\phi(g)$ and $\widetilde{\psi}(\rho(s))=\widetilde{s}$, and hence

$$\begin{aligned}
\widetilde{\psi}(s_a) &= \widetilde{\psi}(\rho(t_a)) \\
&= \widetilde{\psi}(\rho(1_{C(a)}(\alpha(f)^{1/2})s)) \\
&= \phi(1_{C(a)}(\alpha(f))^{1/2})\widetilde{s} \\
&= \phi(1_{C(a)}(\alpha(f))^{1/2})\sum_{b\in\mathfrak{a}}\phi((\alpha(f))^{-1/2})\widetilde{s}_b \\
&= \sum_{b\in\mathfrak{a}}\phi(1_{C(a)}1_{C(b)})\widetilde{s}_b \\
&= \phi(1_{C(a)})\widetilde{s}_a \\
&= \widetilde{s}_a\widetilde{s}_a^*\widetilde{s}_a \\
&= \widetilde{s}_a
\end{aligned}$$

for all $a\in\mathfrak{a}$, and thus $\widetilde{\psi}(s_u)=\widetilde{s}_u$ for every $u\in\mathfrak{a}^*$.

So assume that $g\in\overline{\mathscr{D}_{\mathsf{X}}\alpha(\mathscr{D}_{\mathsf{X}})\mathscr{D}_{\mathsf{X}}}$, that $k\in\overline{\mathscr{D}_{\mathsf{X}}SS^*\mathscr{D}_{\mathsf{X}}}$ and that $ghS=khS$ for every $h\in\mathscr{D}_{\mathsf{X}}$. We then have

$$\begin{aligned}
\psi(g) &= \psi\left(g\sum_{a\in\mathfrak{a}}1_{C(a)}\right)=\psi(g)\sum_{a\in\mathfrak{a}}\widetilde{s}_a\widetilde{s}_a^* \\
&= \psi(g)\sum_{a\in\mathfrak{a}}\phi(1_{C(a)}(\alpha(f))^{1/2})\widetilde{s}\widetilde{s}_a^* \\
&= \sum_{a\in\mathfrak{a}}\psi(g1_{C(a)}(\alpha(f))^{1/2}s)\widetilde{s}_a^* \\
&= \sum_{a\in\mathfrak{a}}\psi(k1_{C(a)}(\alpha(f))^{1/2}s)\widetilde{s}_a^* \\
&= \psi(k)\sum_{a\in\mathfrak{a}}\phi(1_{C(a)}(\alpha(f))^{1/2})\widetilde{s}\widetilde{s}_a^* \\
&= \psi(k)\sum_{a\in\mathfrak{a}}\widetilde{s}_a\widetilde{s}_a^*=\psi\left(k\sum_{a\in\mathfrak{a}}1_{C(a)}\right) \\
&= \psi(k),
\end{aligned}$$

so $\psi$ vanishes on the closed two-sided ideal generated by the set of differences $g-k$, for all redundancies $(g,k)$ such that $g\in\overline{\mathscr{D}_{\mathsf{X}}\alpha(\mathscr{D}_{\mathsf{X}})\mathscr{D}_{\mathsf{X}}}$. $\square$

## 6. A representation of $\mathcal{D}_{\mathsf{X}} \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$

In this section, we present a representation of $\mathcal{D}_{\mathsf{X}} \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$ which we will use in the following sections. This representation is not in general faithful. In Section 9, we will construct a representation of $\mathcal{D}_{\mathsf{X}} \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$ which is faithful for every shift space $\mathsf{X}$.

Let $\mathsf{X}$ be a shift space, and let $\mathsf{H}_{\mathsf{X}}$ be a Hilbert space with an orthonormal basis $\{e_x\}_{x\in\mathsf{X}}$ indexed by $\mathsf{X}$. For every $u \in \mathfrak{a}^*$, let $S_u$ be the operator on $\mathsf{H}_{\mathsf{X}}$ defined by

$$S_u(e_x) = \begin{cases} e_{ux} & \text{if } ux \in \mathsf{X}, \\ 0 & \text{if } ux \notin \mathsf{X}. \end{cases}$$

We leave it to the reader to check that the family $(S_u)_{u\in\mathfrak{a}^*}$ is a family of partial isometries on $\mathsf{H}_{\mathsf{X}}$ which satisfies conditions (1) and (2) of Theorem 10. Thus there exists a $*$-homomorphism $\phi$ from $\mathcal{D}_{\mathsf{X}} \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$ to the $C^*$-algebra of bounded operators on $\mathsf{H}_{\mathsf{X}}$ such that $\phi(s_u) = S_u$ for every $u \in \mathfrak{a}^*$. In other words, $s_u \mapsto S_u$ is a representation of $\mathcal{D}_{\mathsf{X}} \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$ on the Hilbert space $\mathsf{H}_{\mathsf{X}}$.

This representation is in general not faithful. If for example $\mathsf{X}$ only consists of one element, then $\mathcal{D}_{\mathsf{X}} \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$ is isomorphic to $C(\mathbb{T})$, whereas $C^*(S_u \mid u \in \mathfrak{a}^*)$ is isomorphic to $\mathbb{C}$. In Section 9, we will see that if the shift space $\mathsf{X}$ satisfies a certain condition $(I)$, then the representation $\phi$ is injective. In Section 9, we will also construct a representation of $\mathcal{D}_{\mathsf{X}} \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$ which is faithful for every shift space $\mathsf{X}$.

**Remark 11.** Although the $*$-homomorphism $\phi : \mathcal{D}_{\mathsf{X}} \rtimes_{\alpha,\mathcal{L}} \mathbb{N} \to C^*(S_u \mid u \in \mathfrak{a}^*)$ is not in general injective, the restriction of $\phi$ to $\mathcal{D}_{\mathsf{X}}$ is, and so it follows from the universal property of $\mathcal{D}_{\mathsf{X}} \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$, that the restriction of $\rho : \mathcal{T}(\mathcal{D}_{\mathsf{X}}, \alpha, \mathcal{L}) \to \mathcal{D}_{\mathsf{X}} \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$ to $\mathcal{D}_{\mathsf{X}}$ is also injective. Thus we will allow ourselves to view $\mathcal{D}_{\mathsf{X}}$ as a sub-algebra of $\mathcal{D}_{\mathsf{X}} \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$. We then have

$$1_{C(u,v)} = s_v s_u^* s_u s_v^*$$

for all $u, v \in \mathfrak{a}^*$.

## 7. The relationship of $\mathcal{D}_{\mathsf{X}} \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$ with other $C^*$-algebras associated to shift spaces

As mentioned in the introduction, other $C^*$-algebras have been associated to shift spaces. We will in this section look at the relation between these $C^*$-algebras and $\mathcal{D}_{\mathsf{X}} \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$.

As far as the authors know, three different constructions of $C^*$-algebras associated to shift spaces have appeared in the literature. These are:

- The $C^*$-algebra $\mathcal{O}_\Lambda$ defined in [26],
- the $C^*$-algebra $\mathcal{O}_\Lambda$ defined in [14],
- the $C^*$-algebra $\mathcal{O}_{\mathsf{X}}$ defined in [7].

These are all $C^*$-algebras generated by partial isometries $\{s_a\}_{a\in\mathfrak{a}}$, where $\mathfrak{a}$ is the alphabet of the shift space in question. The two first $C^*$-algebras are defined for every two-sided shift space $\Lambda$, whereas the last one is defined for every one-sided shift space $\mathsf{X}$.

We will see in this section that for every one-sided shift space $\mathsf{X}$, there exists a $*$-isomorphism between $\mathscr{D}_{\mathsf{X}} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$ and the $C^*$-algebra $\mathcal{O}_{\mathsf{X}}$ defined in [7] which maps $s_a$ to $s_a$ for every $a \in \mathfrak{a}$, and that for every two-sided shift space $\Lambda$ there exist a surjective $*$-homomorphism from the $C^*$-algebra $\mathcal{O}_\Lambda$ defined in [26] to $\mathscr{D}_{\mathsf{X}_\Lambda} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$ which maps $s_a$ to $s_a$ for every $a \in \mathfrak{a}$, and a surjective $*$-homomorphism from $\mathscr{D}_{\mathsf{X}_\Lambda} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$ to the $C^*$-algebra $\mathcal{O}_\Lambda$ defined in [14] which maps $s_a$ to $s_a$ for every $a \in \mathfrak{a}$. The first of these surjective $*$-homomorphisms is injective if $\Lambda$ satisfies the condition $(*)$ defined in [14], and the second surjective $*$-homomorphism is injective if $\Lambda$ satisfies the condition $(I)$ in Section 9.

**Remark 12.** In [7], a $C^*$-algebra $\mathcal{O}_{\mathsf{X}}$ has been constructed by using $C^*$-correspondences and Cuntz–Pimsner algebras for every shift space $\mathsf{X}$. It follows from Theorem 10 and [7, Remark 7.4] that for every one-sided shift space $\mathsf{X}$, there exists a $*$-isomorphism between $\mathcal{O}_{\mathsf{X}}$ and $\mathscr{D}_{\mathsf{X}} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$ which maps $s_a$ to $s_a$ for every $a \in \mathfrak{a}$. Thus it follows from [7, Remark 7.4] that for every two-sided shift space $\Lambda$, the algebra $\mathscr{D}_{\mathsf{X}_\Lambda} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$ satisfy all of the results the algebra $\mathcal{O}_\Lambda$ is claimed to satisfy in [26–28,31–36].

**Remark 13.** In [14], a $C^*$-algebra $\mathcal{O}_\Lambda$ has been defined for every two-sided shift space by defining operators on a Hilbert space with an orthonormal basis indexed by $\mathsf{X}_\Lambda$. These operators are identical to the operators $S_u$ defined in Section 6 for $\mathsf{X}$ equal to the one-sided shift space $\mathsf{X}_\Lambda$ associated to $\Lambda$. Thus for every two-sided shift space $\Lambda$, we have a surjective $*$-homomorphism from $\mathscr{D}_{\mathsf{X}_\Lambda} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$ to $\mathcal{O}_\Lambda$ which maps $s_a$ to $s_a$ for every $a \in \mathfrak{a}$. This $*$-homomorphism is injective if $\Lambda$ satisfies condition $(I)$. We also know that there are examples of two-sided shift spaces (for instance the shift only consisting of one element) for which the $*$-homomorphism is not injective.

As mentioned in Remark 12, the $C^*$-algebra $\mathscr{D}_{\mathsf{X}_\Lambda} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$ satisfies all of the results that the algebra $\mathcal{O}_\Lambda$ is claimed to satisfy [26–28,31–36], whereas the $C^*$-algebra $\mathcal{O}_\Lambda$ originally defined in [33], does not. The latter $C^*$-algebra has been properly characterized in [14] (where it is called $\mathcal{O}^*_\Lambda$). We will now use this characterization to show that for every two-sided shift space $\Lambda$, there exists a surjective $*$-homomorphism from $\mathcal{O}_\Lambda$ to $\mathscr{D}_{\mathsf{X}_\Lambda} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$ which maps $s_a$ to $s_a$ for every $a \in \mathfrak{a}$.

For every $l \in \mathbb{N}$, let $\mathscr{A}^*_l$ be the $C^*$-subalgebra of $\mathcal{O}_\Lambda$ generated by $\{s_u^* s_u \mid u \in \mathfrak{a}_l\}$, and let $\mathscr{A}^*_\Lambda$ be the $C^*$-subalgebra of $\mathcal{O}_\Lambda$ generated by $\{s_u^* s_u \mid u \in \mathfrak{a}\}$. Notice that the following identity holds:

$$\mathscr{A}^*_\Lambda = \overline{\bigcup_{l\in\mathbb{N}} \mathscr{A}^*_l}.$$

The key to characterizing $\mathcal{O}_\Lambda$ is to describe $\mathscr{A}^*_l$ and $\mathscr{A}^*_\Lambda$, and that will be done now.

For every $l \in \mathbb{N}$ and every $u \in \mathsf{L}(\Lambda)$ ($\mathsf{L}(\Lambda)$ is short for $\mathsf{L}(\mathsf{X}_\Lambda)$, cf. Section 4), let $\mathscr{P}_l(u)$ be the set defined by

$$\mathscr{P}_l(u) = \{v \in \mathfrak{a}_l \mid vu \in \mathsf{L}(\Lambda)\}.$$

We then define an equivalence relation $\sim_l$ on $\mathsf{L}(\Lambda)$ called *l-past equivalence* by

$$u \sim_l v \iff \mathscr{P}_l(u) = \mathscr{P}_l(v).$$

We denote the $l$-past equivalence class containing $u$ by $[u]_l$, and we let $\mathsf{L}_l^*(\Lambda)$ be the set defined by

$$\mathsf{L}_l^*(\Lambda) = \{u \in \mathfrak{a}_l^* \mid \text{the cardinality of } [u]_l \text{ is infinite}\},$$

and let $\Omega_l^* = \mathsf{L}_l^*/\sim_l$. Since $\mathfrak{a}_l^*$ is finite, so is $\Omega_l^*$. We equip $\Omega_l^*$ with the discrete topology (so $C(\Omega_l^*) \cong \mathbb{C}^{m^*(l)}$, where $m^*(l)$ is the number of elements of $l$-past equivalence classes).

**Lemma 14** (*cf. Carlsen and Matsumoto [14, Lemma 2.9]*). *The map*

$$1_{\{[u]_l\}} \mapsto 1_{[u]_l}, \quad u \in \mathsf{L}_l^*(\Lambda)$$

*extends to a $*$-isomorphism between $C(\Omega_l^*)$ and $\mathscr{A}_l^*$.*

We will now make the corresponding characterization of $\mathscr{D}_{\mathsf{X}} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$: Let $\mathsf{X}$ be a one-sided shift space. For every $l \in \mathbb{N}$, let $\mathscr{A}_l$ be the $C^*$-subalgebra of $\mathscr{D}_{\mathsf{X}}$ generated by $\{1_{C(v,\varepsilon)} \mid v \in \mathfrak{a}_l^*\}$, and let $\mathscr{A}_{\mathsf{X}}$ be the $C^*$-subalgebra of $\mathscr{D}_{\mathsf{X}}$ generated by $\{1_{C(v,\varepsilon)} \mid v \in \mathfrak{a}^*\}$. Notice that we then have

$$\mathscr{A}_{\mathsf{X}} = \overline{\bigcup_{l \in \mathbb{N}} \mathscr{A}_l}.$$

Following Matsumoto (cf. [28]), for every $l \in \mathbb{N}$ and every $x \in \mathsf{X}$, define $\mathscr{P}_l(x)$ by

$$\mathscr{P}_l(x) = \{u \in \mathfrak{a}_l^* \mid ux \in \mathsf{X}\}.$$

We then define an equivalence relation $\sim_l$ on $\mathsf{X}$ called *l-past equivalence* by

$$x \sim_l y \iff \mathscr{P}_l(x) = \mathscr{P}_l(y).$$

We let $\Omega_l = \mathsf{X}/\sim_l$, and denote the $l$-past equivalence class containing $x$ by $[x]_l$. Since $\mathfrak{a}_l^*$ is finite, so is $\Omega_l$. We equip $\Omega_l$ with the discrete topology (so $C(\Omega_l) \cong \mathbb{C}^{m(l)}$, where $m(l)$ is the number of elements of $l$-past equivalence classes). Let $x \in \mathsf{X}$ and $l \in \mathbb{N}$. Since we have

$$[x]_l = \left( \bigcap_{u \in \mathscr{P}_l(x)} C(u,\varepsilon) \right) \cap \left( \bigcap_{v \in \mathfrak{a}_l^* \setminus \mathscr{P}_l(x)} \mathsf{X} \setminus C(v,\varepsilon) \right),$$

the function $1_{[x]_l}$ belongs to $\mathscr{A}_l$, and $\{1_{[x]_l} \mid x \in \mathsf{X}\}$ generates $\mathscr{A}_l$. Thus the function

$$1_{\{[x]_l\}} \mapsto 1_{[x]_l}$$

is a $*$-isomorphism between $C(\Omega_l)$ and $\mathscr{A}_l$, which extends to an isomorphism between $C(\Omega_{\mathsf{X}})$ and $\mathscr{A}_{\mathsf{X}}$.

Consider the condition:

$(*)$ For each $l \in \mathbb{N}$ and each infinite sequence of admissible words $(u_i)_{i\in\mathbb{N}}$ satisfying $\mathscr{P}_l(u_i) = \mathscr{P}_l(u_j)$ for all $i, j \in \mathbb{N}$, there exists an $x \in \mathsf{X}_\Lambda$ such that for all $i \in \mathbb{N}$, the identity

$$\mathscr{P}_l(x) = \mathscr{P}_l(u_i)$$

holds.

It follows from [14, Corollary 3.3] that there is a surjective $*$-homomorphism from $\mathscr{A}^*_\Lambda$ to $\mathscr{A}_{\mathsf{X}_\Lambda}$, and that this $*$-homomorphism is injective if and only if $\Lambda$ satisfies the condition $(*)$. As a consequence of this, for every two-sided shift space $\Lambda$, we get a surjective $*$-homomorphism from $\mathcal{O}_\Lambda$ to $\mathscr{D}_{\mathsf{X}_\Lambda} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$ which maps $s_a$ to $s_a$ for every $a \in \mathfrak{a}$, and this $*$-homomorphism is injective if $\Lambda$ satisfies the condition $(*)$.

In [14], there is an example of a sofic shift space $\Lambda$ for which $\mathcal{O}_\Lambda$ and $\mathscr{D}_{\mathsf{X}_\Lambda} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$ are not isomorphic.

## 8. Generalization of the Cuntz–Krieger algebras

We are now able to show that $\mathscr{D}_{\mathsf{X}} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$ is in fact a generalization of the Cuntz–Krieger algebras. Actually, we will prove that $\mathscr{D}_{\mathsf{X}} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$ is a generalization of the universal Cuntz–Krieger algebra $\mathscr{A}\mathcal{O}_A$ that An Huef and Raeburn have constructed in [1].

**Theorem 15.** *Let $A = (A(i,j))_{i,j\in\{1,2,\ldots,n\}}$ be a $n \times n$-matrix with entries in $\{0,1\}$ and no zero rows, and let $\mathsf{X}_A$ be the one-sided shift space*

$$\{(x_i)_{i\in\mathbb{N}} \in \{1,2,\ldots,n\}^{\mathbb{N}} \mid \forall i \in \mathbb{N} : A(x_i, x_{i+1}) = 1\}.$$

*Then $\mathscr{D}_{\mathsf{X}_A} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$ is generated by a family $(s_i)_{i\in\{1,2,\ldots n\}}$ of partial isometries that satisfies*

$$\sum_{j=1}^{n} s_j s_j^* = 1$$

*and*

$$s_i^* s_i = \sum_{j=1}^{n} A(i,j) s_j s_j^*$$

*for every $i \in \{1,2,\ldots,n\}$.*

*Suppose $X$ is a unital $C^*$-algebra such that there exists a family $(\widetilde{s}_i)_{i\in\{1,2,\ldots,n\}}$ of partial isometries in $X$ that satisfy*

$$\sum_{j=1}^{n} \widetilde{s}_j \widetilde{s}_j^* = 1,$$

*and*

$$\widetilde{s}_i^* \widetilde{s}_i = \sum_{j=1}^{n} A(i,j) \widetilde{s}_j \widetilde{s}_j^*$$

*for every* $i \in \{1, 2, \dots, n\}$. *Then there exists a* $*$*-homomorphism from* $\mathscr{D}_{\mathsf{X}_A} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$ *to* $X$ *sending* $s_i$ *to* $\widetilde{s}_i$ *for every* $i \in \{1, 2, \dots, n\}$.

**Proof.** Since $\mathsf{X}_A$ is the disjoint union of $(C(j))_{j\in\{1,2,\dots,n\}}$, the identity

$$\sum_{j=1}^{n} \widetilde{s}_j \widetilde{s}_j^* = 1$$

holds. For every $i \in \{1, 2, \dots, n\}$, we have that $C(i, \varepsilon)$ is the disjoint union of those $C(j)$'s where $A(i, j) = 1$. So, it follows that

$$\widetilde{s}_i^* \widetilde{s}_i = 1_{C(i,\varepsilon)} = \sum_{j=1}^{n} A(i, j) 1_{C(j)} = \sum_{j=1}^{n} A(i, j) \widetilde{s}_j \widetilde{s}_j^*.$$

The $C^*$-algebra $\mathscr{D}_{\mathsf{X}_A} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$ is generated by $\{s_u \mid u \in \{1, 2, \dots, n\}^*\}$, but since we have $s_u s_v = s_{uv}$ for all $u, v \in \{1, 2, \dots, n\}^*$, the family $(s_i)_{i\in\{1,2,\dots,n\}}$ generates $\mathscr{D}_{\mathsf{X}_A} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$.

Let $X$ be a unital $C^*$-algebra with a family $(\widetilde{s}_i)_{i\in\{1,2,\dots,n\}}$ of partial isometries that satisfies

$$\sum_{j=1}^{n} \widetilde{s}_j \widetilde{s}_j^* = 1_X$$

and

$$\widetilde{s}_i^* \widetilde{s}_i = \sum_{j=1}^{n} A(i, j) \widetilde{s}_j \widetilde{s}_j^*$$

for every $i \in \{1, 2, \dots, n\}$. Let $\widetilde{s}_\varepsilon = 1_X$ and let $\widetilde{s}_u = \widetilde{s}_{u_1} \widetilde{s}_{u_2} \cdots \widetilde{s}_{u_n}$ for every $u = u_1 u_2 \cdots u_n \in \{1, 2, \dots, n\}^* \setminus \{\varepsilon\}$. We will show that the following two conditions are satisfied:

(1) $\widetilde{s}_u \widetilde{s}_v = \widetilde{s}_{uv}$ for all $u, v \in \{1, 2, \dots, n\}^*$,

(2) the map

$$1_{C(u,v)} \mapsto \widetilde{s}_v \widetilde{s}_u^* \widetilde{s}_u \widetilde{s}_v^*, \quad u, v \in \mathfrak{a}^*$$

extends to a $*$-homomorphism from $\mathscr{D}_{\mathsf{X}}$ to $X$.

It will then follow from this and Theorem 10 that there exists a $*$-homomorphism form $\mathscr{D}_{\mathsf{X}_A} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$ to $X$ sending $s_u$ to $\widetilde{s}_u$ for every $u \in \{1, 2, \dots, n\}^*$, and in particular $s_i$ to $\widetilde{s}_i$ for every $i \in \{1, 2, \dots, n\}$.

It is clear from the way we defined $\widetilde{s}_u$ that condition (1) is satisfied. Let $m \in \mathbb{N}$, and denote by $\mathscr{D}_m$ the $C^*$-subalgebra of $\mathscr{D}_{\mathsf{X}_A}$ generated by $\{1_{C(u)} \mid u \in \{1, 2, \dots, n\}^m\}$. If $u, v \in \{1, 2, \dots, n\}^m$ and $u \neq v$, then we have

$$\widetilde{s}_u \widetilde{s}_u^* + \widetilde{s}_v \widetilde{s}_v^* \leqslant \sum_{w\in\{1,2,\dots,n\}^m} \widetilde{s}_w \widetilde{s}_w^* = 1_x$$

and thus

$$\widetilde{s}_u^*\widetilde{s}_u + \widetilde{s}_u^*\widetilde{s}_v\widetilde{s}_v^*\widetilde{s}_u = \widetilde{s}_u^*(\widetilde{s}_u\widetilde{s}_u^* + \widetilde{s}_v\widetilde{s}_v^*)\widetilde{s}_u \leqslant \widetilde{s}_u^* 1_X \widetilde{s}_u = \widetilde{s}_u^*\widetilde{s}_u,$$

which implies that $\widetilde{s}_u\widetilde{s}_u^*\widetilde{s}_v\widetilde{s}_v^* = \widetilde{s}_u\widetilde{s}_u^*\widetilde{s}_v\widetilde{s}_v^*\widetilde{s}_u\widetilde{s}_u^* = 0$.

Thus $(\widetilde{s}_u\widetilde{s}_u^*)_{u\in\{1,2,\ldots,n\}^m}$ is a family of mutually orthogonal projections. Notice that $(1_{C(u)})_{u\in\{1,2,\ldots,n\}^m}$ also is a family of mutually orthogonal projections, and that

$$\begin{aligned}
1_{C(u)} = 0 &\Rightarrow C(u) = \emptyset \\
&\Rightarrow u \notin \mathsf{L}(\mathsf{X}_A) \\
&\Rightarrow \exists i \in \{1, 2, \ldots, m-1\} : A(u_i, u_{i+1}) = 0 \\
&\Rightarrow \widetilde{s}_{u_i}\widetilde{s}_{u_{i+1}} = \widetilde{s}_{u_i}\widetilde{s}_{u_i}^*\widetilde{s}_{u_i}\widetilde{s}_{u_{i+1}}\widetilde{s}_{u_{i+1}}^*\widetilde{s}_{u_{i+1}} \\
&= \widetilde{s}_{u_i}\sum_{k=1}^{n} A(U_i, k)\widetilde{s}_k\widetilde{s}_k^*\widetilde{s}_{u_{i+1}}\widetilde{s}_{u_{i+1}}^*\widetilde{s}_{u_{i+1}} = 0 \\
&\Rightarrow \widetilde{s}_u\widetilde{s}_u^* = 0
\end{aligned}$$

for every $u \in \{1, 2, \ldots, n\}^m$. It follows that there is a unital $*$-homomorphism $\psi_m$ from $\mathscr{D}_m$ to $X$ obeying $\psi_m\left(1_{C(u)}\right) = \widetilde{s}_u\widetilde{s}_u^*$ for every $u \in \{1, 2, \ldots, n\}^m$.

Since $C(u)$ is the disjoint union of $(C(ui))_{i\in\{1,2,\ldots,n\}}$, we have

$$1_{C(u)} = \sum_{i=1}^{n} 1_{C(ui)} \in D_{m+1}$$

for every $u \in \{1, 2, \ldots, n\}^m$, so $\mathscr{D}_m \subseteq \mathscr{D}_{m+1}$. Let us denote the inclusion of $\mathscr{D}_m$ into $\mathscr{D}_{m+1}$ by $\imath_m$. We then have

$$\begin{aligned}
\psi_{m+1}(1_{C(u)}) &= \psi_{m+1}\left(\sum_{i=1}^{n} 1_{C(ui)}\right) \\
&= \sum_{i=1}^{n} \widetilde{s}_{ui}\widetilde{s}_{ui}^* \\
&= \widetilde{s}_u\left(\sum_{i=1}^{n} \widetilde{s}_i\widetilde{s}_i^*\right)\widetilde{s}_u^* \\
&= \widetilde{s}_u\widetilde{s}_u^* = \psi_m(1_{C(u)}),
\end{aligned}$$

from which it follows that $\psi_{m+1} \circ \imath_m = \psi_m$. Thus the family $\{\psi_m\}_{m\in\mathbb{N}}$ extends to a $*$-homomorphism $\psi$ from $\overline{\bigcup_{m\in\mathbb{N}}\mathscr{D}_m}$ to $X$ which maps $1_{C(u)}$ to $\widetilde{s}_u\widetilde{s}_u^*$ for every $u \in \{1, 2, \ldots, n\}^m$.

Let $u, v \in \{1, 2, \ldots, n\}^m$. It is easy to check that the following equation

$$1_{C(u,\varepsilon)} = \begin{cases} \sum_{j=1}^{n} A(u_{|u|}, j) 1_{C(j)} & \text{if } u \in \mathsf{L}(\mathsf{X}_A) \\ 0 & \text{if } u \notin \mathsf{L}(\mathsf{X}_A), \end{cases}$$

holds. It is also easy to check that if $v \neq \varepsilon$, then the following equation

$$1_{C(u,v)} = \begin{cases} 1_{C(v)} & \text{if } A(u_1, u_2) = A(u_2, u_3) = \cdots = A(u_{|u|-1}, u_{|u|}) \\ & \qquad = A(u_{|u|}, v_1) = 1, \\ 0 & \text{else} \end{cases}$$

holds. It is equally easy to check that the following equation

$$\widetilde{s}_u^* \widetilde{s}_u = \begin{cases} \sum_{j=1}^{n} A(u_{|u|}, j) \widetilde{s}_j \widetilde{s}_j^* & \text{if } u \in \mathsf{L}(\mathsf{X}_A) \\ 0 & \text{if } u \notin \mathsf{L}(\mathsf{X}_A), \end{cases}$$

holds, and that if $v \neq \varepsilon$, then the following equation

$$\widetilde{s}_v \widetilde{s}_u^* \widetilde{s}_u \widetilde{s}_v^* = \begin{cases} \widetilde{s}_v \widetilde{s}_v^* & \text{if } A(u_1, u_2) = A(u_2, u_3) = \cdots = A(u_{|u|-1}, u_{|u|}) \\ & \qquad = A(u_{|u|}, v_1) = 1, \\ 0 & \text{else} \end{cases}$$

holds. Thus, $\mathscr{D}_{\mathsf{X}_A}$ is contained in $\overline{\bigcup_{m \in \mathbb{N}} \mathscr{D}_m}$, and $\psi\left(1_{C(u,v)}\right) = \widetilde{s}_v \widetilde{s}_u^* \widetilde{s}_u \widetilde{s}_v^*$ for all $u, v \in \{1, 2, \ldots, n\}^*$.

Consequently, the family $(\widetilde{s}_u)_{u \in \{1,2,\ldots,n\}}$ satisfies condition (2). $\square$

This result is generalized in [6], where it is shown that $\mathscr{D}_{\mathsf{X}} \rtimes_{\alpha, \mathscr{L}} \mathbb{N}$ is isomorphic to a universal Cuntz–Krieger algebra when $\mathsf{X}$ is a sofic shift.

If $A(i, j) = 1$ for every $i, j \in \{1, 2, \ldots, n\}$, then $\mathscr{O}_A$, and hence $\mathscr{D}_{\mathsf{X}_A} \rtimes_{\alpha, \mathscr{L}} \mathbb{N}$, is the Cuntz algebra $\mathscr{O}_n$ which was originally defined in [15]. The Cuntz algebras have proved to be very important examples in the theory of $C^*$-algebras, for example in classification of $C^*$-algebras; see for example [41], and in the study of wavelets, see for example [4].

## 9. Uniqueness and a faithful representation

As we have just seen, the $C^*$-algebra $\mathscr{D}_{\mathsf{X}} \rtimes_{\alpha, \mathscr{L}} \mathbb{N}$ is a generalization of the Cuntz–Krieger algebras. One of the many things that make Cuntz–Krieger algebras interesting is the fact that a representation of a Cuntz–Krieger algebra, under conditions which are often easy to verify, is faithful. In this section, we will present similar results for $\mathscr{D}_{\mathsf{X}} \rtimes_{\alpha, \mathscr{L}} \mathbb{N}$. This will put us in a position where we can construct a representation of $\mathscr{D}_{\mathsf{X}} \rtimes_{\alpha, \mathscr{L}} \mathbb{N}$ which is faithful for every one-sided shift space $\mathsf{X}$.

Let $\mathsf{X}$ be a one-sided shift space. It follows from the universal property of $\mathscr{D}_{\mathsf{X}} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$ that there exists an action $\gamma : \mathbb{T} \to \mathrm{Aut}(\mathscr{D}_{\mathsf{X}} \rtimes_{\alpha,\mathscr{L}} \mathbb{N})$ defined by $\gamma_z(s_u) = z^{|u|} s_u$ for every $z \in \mathbb{T}$. This action is known as the *gauge action*.

Let $\mathscr{F}_{\mathsf{X}}$ denote the $C^*$-subalgebra of $\mathscr{D}_{\mathsf{X}} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$ generated by $\{s_v s_u^* s_u s_w^* \mid u, v, w \in \mathfrak{a}^*, |v| = |w|\}$. It is not difficult to check that

$$\left\{ \sum_{v \in J_-} x_v s_v^* + x_0 + \sum_{u \in J_+} s_u x_u \,\middle|\, J_- \text{ and } J_+ \text{ are finite subsets of } \mathfrak{a}^* \right.$$
$$\left. \text{and } x_0, x_v, x_u \in \mathscr{F}_{\mathsf{X}} \text{ for all } v \in J_-, u \in J_+ \right\}$$

is a dense $*$-subalgebra of $\mathscr{D}_{\mathsf{X}} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$. It easily follows from this that $\mathscr{F}_{\mathsf{X}}$ is equal to the fixed point algebra

$$\{x \in \mathscr{D}_{\mathsf{X}} \rtimes_{\alpha,\mathscr{L}} \mathbb{N} \mid \forall z \in \mathbb{T} : \gamma_g(x) = x\}$$

for the gauge action.

If we let

$$E(x) = \int_{\mathbb{T}} \alpha_z(x)\,\mathrm{d}z$$

for every $x \in \mathscr{D}_{\mathsf{X}} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$, then $E$ is a projection of norm one (a *conditional expectation*) from $\mathscr{D}_{\mathsf{X}} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$ onto $\mathscr{F}_{\mathsf{X}}$ which has the following properties:

(i) $E(abc) = aE(b)c$ for all $a, c \in \mathscr{F}_{\mathsf{X}}$ and $b \in \mathscr{D}_{\mathsf{X}} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$,
(ii) $E(s_u) = 0$ for all $u \in \mathfrak{a}^* \setminus \{\varepsilon\}$,
(iii) $E(x^*x) \geqslant 0$ for all $x \in \mathscr{D}_{\mathsf{X}} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$,
(iv) $E(x^*x) = 0$ implies that $x = 0$ for all $x \in \mathscr{D}_{\mathsf{X}} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$.

Building on the work done by Matsumoto in [26], the following theorem will be proved in [9] ($\mathscr{A}_{\mathsf{X}}$ is the $C^*$-subalgebra of $\mathscr{D}_{\mathsf{X}} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$ defined in Section 12):

**Theorem 16.** *Let* $\mathsf{X}$ *be a one-sided shift space, $X$ a $C^*$-algebra generated by a family* $(\widetilde{s}_u)_{u \in \mathfrak{a}^*}$ *of partial isometries, and* $\phi : \mathscr{D}_{\mathsf{X}} \rtimes_{\alpha,\mathscr{L}} \mathbb{N} \to X$ *a $*$-homomorphism such that* $\phi(\widetilde{s}_u) = \widetilde{s}_u$ *for every* $u \in \mathfrak{a}^*$. *Then the following three statements are equivalent*:

1. *the $*$-homomorphism* $\phi : \mathscr{D}_{\mathsf{X}} \rtimes_{\alpha,\mathscr{L}} \mathbb{N} \to X$ *is injective*,
2. *the restriction of $\phi$ to $\mathscr{A}_{\mathsf{X}}$ is injective, and there exists an action* $\widetilde{\gamma} : \mathbb{T} \to \mathrm{Aut}(X)$ *such that* $\widetilde{\gamma}_z(\widetilde{s}_u) = z^{|u|}\widetilde{s}_u$ *for every* $z \in \mathbb{T}$ *and every* $u \in \mathfrak{a}^*$,
3. *the restriction of $\phi$ to $\mathscr{A}_{\mathsf{X}}$ is injective, and there exists a projection $\widetilde{E}$ of norm one from $X$ onto* $C^*(\widetilde{s}_v \widetilde{s}_u^* \widetilde{s}_u \widetilde{s}_w^* \mid u, v, w \in \mathfrak{a}^*,\ |v| = |w|)$ *satisfying*
   (i) $\widetilde{E}(abc) = a\widetilde{E}(b)c$ *for all* $a, c \in C^*(s_v s_u^* s_u s_w^* \mid u, v, w \in \mathfrak{a}^*,\ |v| = |w|)$ *and* $b \in X$,
   (ii) $\widetilde{E}(\widetilde{s}_u) = 0$ *for all* $u \in \mathfrak{a}^* \setminus \{\varepsilon\}$.

As a corollary to this theorem we get the following result.

**Corollary 17.** *Let* $\mathsf{X}$ *be a one-sided shift space and let* $X$ *be a* $C^*$*-algebra which is generated by a family* $(\widetilde{s}_u)_{u\in\mathfrak{a}^*}$ *of partial isometries satisfying*:

1. $\widetilde{s}_u\widetilde{s}_v = \widetilde{s}_{uv}$ *for all* $u, v \in \mathfrak{a}^*$,
2. *the map*

$$1_{C(u,v)} \mapsto \widetilde{s}_v\widetilde{s}_u^*\widetilde{s}_u\widetilde{s}_v^*, \quad u, v \in \mathfrak{a}^*$$

*extends to an injective* $*$*-homomorphism from* $\mathscr{D}_{\mathsf{X}}$ *to* $X$,

3. *there exists an action* $\widetilde{\gamma} : \mathbb{T} \to \mathrm{Aut}(X)$ *defined by* $\widetilde{\gamma}_z(\widetilde{s}_u) = z^{|u|}\widetilde{s}_u$ *for every* $z \in \mathbb{T}$.

*Then* $X$ *and* $\mathscr{D}_{\mathsf{X}} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$ *are isomorphic by an isomorphism which maps* $s_u$ *to* $\widetilde{s}_u$ *for every* $u \in \mathfrak{a}^*$.

**Remark 18.** As a consequence of the previous corollary, we are now able to construct a representation of $\mathscr{D}_{\mathsf{X}} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$ which is faithful for every one-sided shift space $\mathsf{X}$:

Let $\mathfrak{H}_{\mathsf{X}}$ be a Hilbert space with an orthonormal basis $(e_{(x,n)})_{(x,n)\in\mathsf{X}\times\mathbb{Z}}$ indexed by $\mathsf{X}\times\mathbb{Z}$, and for every $u \in \mathfrak{a}^*$, let $S_u$ be the operator on $\mathfrak{H}_{\mathsf{X}}$ defined by

$$S_u(e_{(x,n)}) = \begin{cases} e_{(ux,n+|u|)} & \text{if } ux \in \mathsf{X}, \\ 0 & \text{if } ux \notin \mathsf{X}. \end{cases}$$

It is easy to check that $S_uS_v = S_{uv}$ and that the following equality holds for all $u, v \in \mathfrak{a}^*$ and $(x, n) \in \mathsf{X}\times\mathbb{Z}$:

$$S_vS_u^*S_uS_v^*(e_{(x,n)}) = \begin{cases} e_{(x,n)} & \text{if } x \in C(u, v), \\ 0 & \text{if } x \notin C(u, v). \end{cases}$$

Thus $(S_u)_{u\in\mathfrak{a}^*}$ is a family of partial isometries which satisfies (1) and (2) of Corollary 17.

If, for every $z \in \mathbb{T}$, we let $U_z$ be the operator on $\mathfrak{H}_{\mathsf{X}}$ defined by

$$U_z(e_{(x,n)}) = z^n(e_{(x,n)}),$$

then $U_z$ is a unitary operator on $\mathfrak{H}_{\mathsf{X}}$, and $U_zS_uU_z^* = z^{|u|}S_u$ for every $u \in \mathfrak{a}^*$. Thus $(S_u)_{u\in\mathfrak{a}^*}$ also satisfies (3) of Corollary 17, and therefore $s_u \mapsto S_u$ defines a faithful representation of $\mathscr{D}_{\mathsf{X}} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$.

We will now briefly discuss a condition on the shift space $\mathsf{X}$ which implies that condition 3 of Corollary 17 automatically follows from conditions 1 and 2 of the same corollary.

**Definition 19.** We say that a one-sided shift space $\mathsf{X}$ satisfies condition (I) if for every $x \in \mathsf{X}$ and every $l \in \mathbb{N}$, there exists a $y \in \mathsf{X}$ such that $\mathscr{P}_l(x) = \mathscr{P}_l(y)$ and $x \neq y$.

One can show that if $\mathsf{X}$ satisfies condition (I), and $X$ is a $C^*$-algebra generated by a family $(\widetilde{s}_u)_{u\in\mathfrak{a}^*}$ of partial isometries which satisfies:

1. $\widetilde{s}_u\widetilde{s}_v=\widetilde{s}_{uv}$ for all $u,v\in\mathfrak{a}^*$,
2. the map

$$1_{C(u,v)}\mapsto \widetilde{s}_v\widetilde{s}_u^*\widetilde{s}_u\widetilde{s}_v^*,\quad u,v\in\mathfrak{a}^*$$

extends to an injective $*$-homomorphism from $\mathscr{D}_{\mathsf{X}}$ to $X$,

then there exists an action $\widetilde{\gamma}:\mathbb{T}\to\mathrm{Aut}(X)$ such that $\widetilde{\gamma}_z(\widetilde{s}_u)=z^{|u|}\widetilde{s}_u$ for every $z\in\mathbb{T}$. This was first proved by Matsumoto in the case where $\mathsf{X}$ is of the form $\mathsf{X}_\Lambda$ for some two-sided shift space $\Lambda$ in [28], where he also discuss several conditions which are equivalent to condition (I), and this has been generalized to arbitrary one-sided shift spaces $\mathsf{X}$ by the first author in [5].

The following theorem follows from this result and Corollary 17.

**Theorem 20.** *Let $\mathsf{X}$ be a one-sided shift space which satisfies condition* (I). *If $X$ is a $C^*$-algebra generated by a family $(\widetilde{s}_u)_{u\in\mathfrak{a}^*}$ of partial isometries which satisfies*:

1. $\widetilde{s}_u\widetilde{s}_v=\widetilde{s}_{uv}$ *for all* $u,v\in\mathfrak{a}^*$,
2. *the map*

$$1_{C(u,v)}\mapsto \widetilde{s}_v\widetilde{s}_u^*\widetilde{s}_u\widetilde{s}_v^*,\quad u,v\in\mathfrak{a}^*$$

*extends to an injective $*$-homomorphism from $\mathscr{D}_{\mathsf{X}}$ to $X$*,

*then $X$ and $\mathscr{D}_{\mathsf{X}}\rtimes_{\alpha,\mathscr{L}}\mathbb{N}$ are isomorphic by an isomorphism which maps $s_u$ to $\widetilde{s}_u$ for every $u\in\mathfrak{a}^*$.*

## 10. Properties of $\mathscr{D}_{\mathsf{X}}\rtimes_{\alpha,\mathscr{L}}\mathbb{N}$

In this section, we will briefly describe some of the properties of the $C^*$-algebra $\mathscr{D}_{\mathsf{X}}\rtimes_{\alpha,\mathscr{L}}\mathbb{N}$ which in some sense make it a "nice" $C^*$-algebra. We will see that $\mathscr{D}_{\mathsf{X}}\rtimes_{\alpha,\mathscr{L}}\mathbb{N}$ is always *nuclear* and satisfies *the Universal Coefficient Theorem* (the UCT), and that it is *simple and purely infinite*, if $\mathsf{X}$ satisfies certain conditions.

**Theorem 21.** *Let $\mathsf{X}$ be a one-sided shift space. Then the $C^*$-algebra $\mathscr{D}_{\mathsf{X}}\rtimes_{\alpha,\mathscr{L}}\mathbb{N}$ is nuclear and satisfies the UCT.*

**Proof.** As mentioned in Remark 12, $\mathscr{D}_{\mathsf{X}}\rtimes_{\alpha,\mathscr{L}}\mathbb{N}$ is isomorphic to the $C^*$-algebra $\mathscr{O}_{\mathsf{X}}$ defined in [7], and since $\mathscr{O}_{\mathsf{X}}$ is the $C^*$-algebra of a separable $C^*$-correspondence over $\mathscr{D}_{\mathsf{X}}$ which is separable and commutative and hence nuclear and satisfies the UCT, the same is the case for the $C^*$-algebra $J_X$ mentioned in [22, Proposition 8.8], and thus it follows from [22, Corollary 7.4, Proposition 8.8] that $\mathscr{O}_{\mathsf{X}}$ and hence $\mathscr{D}_{\mathsf{X}}\rtimes_{\alpha,\mathscr{L}}\mathbb{N}$ is nuclear and satisfies the UCT. $\square$

Furthermore, Matsumoto proved the following theorem in [28].

**Theorem 22.** *Let $\Lambda$ be a two-sided shift space.*

1. *If $\mathsf{X}_\Lambda$ is irreducible in past equivalence (i.e., for every $l \in \mathbb{N}$, every $y \in \mathsf{X}_\Lambda$ and every sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathsf{X}_\Lambda$ such that $\mathscr{P}_n(x_n) = \mathscr{P}_n(x_{n+1})$ for every $n \in \mathbb{N}$, there exist an $N \in \mathbb{N}$ and a $u \in \mathsf{L}(\Lambda)$ such that $\mathscr{P}_l(y) = \mathscr{P}_l(ux_{l+N})$), then the $C^*$-algebra $\mathscr{D}_{\mathsf{X}_\Lambda} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$ is simple.*
2. *If $\mathsf{X}_\Lambda$ is aperiodic in past equivalence (i.e., for every $l \in \mathbb{N}$, there exists an $N \in \mathbb{N}$ such that for any pair $x, y \in \mathsf{X}_\Lambda$, there exists a $u \in \mathsf{L}_N(\Lambda)$ such that $\mathscr{P}_l(y) = \mathscr{P}_l(ux)$), then the $C^*$-algebra $\mathscr{D}_{\mathsf{X}_\Lambda} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$ is simple and purely infinite.*

## 11. $\mathscr{D}_{\mathsf{X}} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$ as an invariant

In this section, we will see that $\mathscr{D}_{\mathsf{X}} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$ is an invariant for one-sided conjugacy in the sense that if two one-sided shift spaces $\mathsf{X}$ and $\mathsf{Y}$ are conjugate, then $\mathscr{D}_{\mathsf{X}} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$ and $\mathscr{D}_{\mathsf{Y}} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$ are isomorphic.

This was first proved by Matsumoto in [26] for the special case where $\mathsf{X} = \mathsf{X}_\Lambda$ and $\mathsf{Y} = \mathsf{X}_\Gamma$ for two two-sided shift spaces $\Lambda$ and $\Gamma$ satisfying condition (I), and generalized to the general case in [7]. Because of the way we have constructed $\mathscr{D}_{\mathsf{X}} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$ in this paper, we can very easily prove this result and even improve it a little bit.

Remember that in $\mathscr{T}(\mathscr{D}_{\mathsf{X}}, \alpha, \mathscr{L})$, $s^*as = \mathscr{L}(f)$ for every $f \in \mathscr{D}_{\mathsf{X}}$, so in $\mathscr{D}_{\mathsf{X}} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$ $\rho(s)^* f \rho(s) = \mathscr{L}(f)$ for every $f \in \mathscr{D}_{\mathsf{X}}$. We will therefore denote the map

$$x \mapsto \rho(s)^* x \rho(s)$$

from $\mathscr{D}_{\mathsf{X}} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$ to $\mathscr{D}_{\mathsf{X}} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$ by $\mathscr{L}$. We will by $\lambda_{\mathsf{X}}$ denote the map

$$x \mapsto \left(\sum_{a\in\mathfrak{a}} s_a^*\right) x \left(\sum_{b\in\mathfrak{a}} s_b\right)$$

from $\mathscr{F}_{\mathsf{X}}$ to $\mathscr{F}_{\mathsf{X}}$.

**Theorem 23.** *If $\mathsf{X}$ and $\mathsf{Y}$ are two one-sided shift spaces which are conjugate, then there exists a $*$-isomorphism $\Phi$ from $\mathscr{D}_{\mathsf{X}} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$ to $\mathscr{D}_{\mathsf{Y}} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$ such that:*

1. $\Phi(C(\mathsf{X})) = C(\mathsf{Y})$,
2. $\Phi(\mathscr{D}_{\mathsf{X}}) = \mathscr{D}_{\mathsf{Y}}$,
3. $\Phi(\mathscr{F}_{\mathsf{X}}) = \mathscr{F}_{\mathsf{Y}}$,
4. $\Phi \circ \alpha_{\mathsf{X}} = \alpha_{\mathsf{Y}}$,
5. $\Phi \circ \gamma_z = \gamma_z$ *for every* $z \in \mathbb{T}$,
6. $\Phi \circ \mathscr{L}_{\mathsf{X}} = \mathscr{L}_{\mathsf{Y}}$,
7. $\Phi \circ \lambda_{\mathsf{X}} = \lambda_{\mathsf{Y}}$.

**Proof.** Let $\phi$ be a conjugacy between $\mathsf{Y}$ and $\mathsf{X}$, and let $\Phi$ be the map between the bounded functions on $\mathsf{X}$ and the bounded functions on $\mathsf{Y}$ defined by

$$f \mapsto f \circ \phi.$$

Then $\Phi(C(\mathsf{X}))=C(\mathsf{Y})$, $\Phi\circ\alpha_{\mathsf{X}}=\alpha_{\mathsf{Y}}\circ\Phi$ and $\Phi\circ\mathscr{L}_{\mathsf{Y}}=\mathscr{L}_{\mathsf{X}}\circ\Phi$, and hence $\Phi(\mathscr{D}_{\mathsf{Y}})=\mathscr{D}_{\mathsf{X}}$. Thus it follows from the construction of $\mathscr{D}_{\mathsf{X}}\rtimes_{\alpha,\mathscr{L}}\mathbb{N}$ and $\mathscr{D}_{\mathsf{Y}}\rtimes_{\alpha,\mathscr{L}}\mathbb{N}$ that there is a $*$-isomorphism from $\mathscr{D}_{\mathsf{X}}\rtimes_{\alpha,\mathscr{L}}\mathbb{N}$ to $\mathscr{D}_{\mathsf{Y}}\rtimes_{\alpha,\mathscr{L}}\mathbb{N}$ which extends $\Phi$, maps $\rho(s)$ to $\rho(s)$ and satisfies $\Phi\circ\alpha_{\mathsf{X}}=\alpha_{\mathsf{Y}}$. We will also denote this $*$-isomorphism by $\Phi$.

Since the gauge action of $\mathscr{D}_{\mathsf{X}}\rtimes_{\alpha,\mathscr{L}}\mathbb{N}$ is characterized by $\gamma_z(f)=f$ for all $f\in\mathscr{D}_{\mathsf{X}}$ and $\gamma_z(\rho(s))=z\rho(s)$, and the gauge action of $\mathscr{D}_{\mathsf{Y}}\rtimes_{\alpha,\mathscr{L}}\mathbb{N}$ is characterized in a similar way, we see that $\Phi\circ\gamma_z=\gamma_z$ for every $z\in\mathbb{T}$.

Since $\mathscr{F}_{\mathsf{X}}$ is the fixed point algebra for the gauge action of $\mathscr{D}_{\mathsf{X}}\rtimes_{\alpha,\mathscr{L}}\mathbb{N}$, and $\mathscr{F}_{\mathsf{Y}}$ is the fixed point algebra for the gauge action of $\mathscr{D}_{\mathsf{Y}}\rtimes_{\alpha,\mathscr{L}}\mathbb{N}$, we have $\Phi(\mathscr{F}_{\mathsf{X}})=\mathscr{F}_{\mathsf{Y}}$.

Since $\Phi$ maps $\rho(s)$ to $\rho(s)$, we have $\Phi\circ\mathscr{L}_{\mathsf{X}}=\mathscr{L}_{\mathsf{Y}}$.

Let us denote the function

$$x\mapsto \#\sigma^{-1}\{x\},\quad x\in\mathsf{X}$$

by $f_{\mathsf{X}}$, and the function

$$x\mapsto \#\sigma^{-1}\{x\},\quad x\in\mathsf{Y}$$

by $f_{\mathsf{Y}}$. We then have that the following

$$\lambda_{\mathsf{X}}(x)=\left(\sum_{a\in\mathfrak{a}_{\mathsf{X}}} s_a^*\right)x\left(\sum_{b\in\mathfrak{a}_{\mathsf{X}}} s_b\right)=\rho(s)^*\alpha(f_{\mathsf{X}})^{1/2}x\alpha(f_{\mathsf{X}})^{1/2}\rho(s)$$

holds for all $x\in\mathsf{X}$, and similarly that

$$\lambda_{\mathsf{Y}}(y)=\left(\sum_{a\in\mathfrak{a}_{\mathsf{Y}}} s_a^*\right)y\left(\sum_{b\in\mathfrak{a}_{\mathsf{Y}}} s_b\right)=\rho(s)^*\alpha(f_{\mathsf{Y}})^{1/2}y\alpha(f_{\mathsf{Y}})^{1/2}\rho(s)$$

for all $y\in\mathsf{Y}$. Since $\Phi(f_{\mathsf{X}})=f_{\mathsf{Y}}$, this implies that $\Phi\circ\lambda_{\mathsf{X}}=\lambda_{\mathsf{Y}}$. $\square$

If two two-sided shift spaces $\Lambda$ and $\Gamma$ are flow equivalent, then the corresponding one-sided shift spaces $\mathsf{X}_\Lambda$ and $\mathsf{X}_\Gamma$ are not necessarily conjugate, so we cannot expect that $\mathscr{D}_{\mathsf{X}_\Lambda}\rtimes_{\alpha,\mathscr{L}}\mathbb{N}$ and $\mathscr{D}_{\mathsf{X}_\Gamma}\rtimes_{\alpha,\mathscr{L}}\mathbb{N}$ are isomorphic (and there are examples of pairs $\Lambda$ and $\Gamma$ of two-sided shift spaces, such that $\Lambda$ and $\Gamma$ are conjugate and hence flow equivalent, but $\mathscr{D}_{\mathsf{X}_\Lambda}\rtimes_{\alpha,\mathscr{L}}\mathbb{N}$ and $\mathscr{D}_{\mathsf{X}_\Gamma}\rtimes_{\alpha,\mathscr{L}}\mathbb{N}$ are not isomorphic), but it turns out that $\mathscr{D}_{\mathsf{X}_\Lambda}\rtimes_{\alpha,\mathscr{L}}\mathbb{N}\otimes\mathscr{K}$ and $\mathscr{D}_{\mathsf{X}_\Gamma}\rtimes_{\alpha,\mathscr{L}}\mathbb{N}\otimes\mathscr{K}$, (where $\mathscr{K}$ is the $C^*$-algebra of compact operators on a separable Hilbert space) are $*$-isomorphic in this case. This has been proved by Matsumoto in [32] for $\Lambda$ and $\Gamma$ satisfying condition (I), and will be proved in full generality in [9].

## 12. The $K$-theory of $\mathscr{D}_{\mathsf{X}} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$

Since $K_0(X)$ and $K_1(X)$ are invariants of a $C^*$-algebra $X$, it follows from the previous section that $K_0(\mathscr{D}_{\mathsf{X}} \rtimes_{\alpha,\mathscr{L}} \mathbb{N})$, $K_1(\mathscr{D}_{\mathsf{X}} \rtimes_{\alpha,\mathscr{L}} \mathbb{N})$ and $K_0(\mathscr{F}_{\mathsf{X}})$ are invariants of $\mathsf{X}$. In this section, we will present formulas based on $l$-past equivalence for these invariants. This was done in [27,28,35] for the case of one-sided shift spaces of the form $\mathsf{X}_\Lambda$, where $\Lambda$ is a two-sided shift space and generalized to the general case in [5]. In this paper, we will not prove the formulas for $K_0(\mathscr{D}_{\mathsf{X}} \rtimes_{\alpha,\mathscr{L}} \mathbb{N})$, $K_1(\mathscr{D}_{\mathsf{X}} \rtimes_{\alpha,\mathscr{L}} \mathbb{N})$ and $K_0(\mathscr{F}_{\mathsf{X}})$, but only establish the necessary setup and state the theorems which give the formulas. The interested reader can find proofs of these theorems in the before mentioned references.

From these formulas, one can directly prove that $K_0(\mathscr{D}_{\mathsf{X}} \rtimes_{\alpha,\mathscr{L}} \mathbb{N})$, $K_1(\mathscr{D}_{\mathsf{X}} \rtimes_{\alpha,\mathscr{L}} \mathbb{N})$ and $K_0(\mathscr{F}_{\mathsf{X}})$ are invariants of $\mathsf{X}$ without involving $C^*$-algebras. This is done (for one-sided shift spaces of the form $\mathsf{X}_\Lambda$, where $\Lambda$ is a two-sided shift space) in Matsumoto's paper [29], where also other invariants of shift spaces are presented.

Let $\mathsf{X}$ be a one-sided shift space. For each $l \in \mathbb{N}$, we let $m(l)$ be the number of $l$-past equivalence classes, and we denote the $l$-past equivalence classes by $\mathscr{E}^l_1, \mathscr{E}^l_2, \ldots, \mathscr{E}^l_{m(l)}$. For $0 \leqslant k \leqslant l$ and $i \in \{1, 2, \ldots, m(l)\}$, the set $\mathscr{P}_k(x)$ does not depend on the choice of $x$ as long as $x \in \mathscr{E}^l_i$. We will denote this set by $\mathscr{P}_k(\mathscr{E}^l_i)$. For each $l \in \mathbb{N}$, $j \in \{1, 2, \ldots, m(l)\}$ and $i \in \{1, 2, \ldots, m(l+1)\}$, let

$$I_l(i,j) = \begin{cases} 1 & \text{if } \mathscr{E}^{l+1}_i \subseteq \mathscr{E}^l_j \\ 0 & \text{else.} \end{cases}$$

Let $F$ be a finite set and $i_0 \in F$. Then we denote by $e_{i_0}$ the element in $\mathbb{Z}^F$ for which

$$e_{i_0}(i) = \begin{cases} 1 & \text{if } i = i_0 \\ 0 & \text{else.} \end{cases}$$

For $0 \leqslant k \leqslant l$, let $M^l_k$ be the set

$$M^l_k = \{i \in \{1, 2, \ldots, m(l)\} \mid \mathscr{P}_k(\mathscr{E}^l_i) \neq \emptyset\}.$$

Since $i \in M^{l+1}_k$, if $j \in M^l_k$ and $I_l(i,j) = 1$, there exists a positive linear map from $\mathbb{Z}^{M^l_k}$ to $\mathbb{Z}^{M^{l+1}_k}$ given by

$$e_j \mapsto \sum_{i \in M^{l+1}_k} I_l(i,j) e_i.$$

We denotes this map by $I^l_k$.

For a subset $\mathscr{E}$ of $\mathsf{X}$ and a $u \in \mathfrak{a}^*$, let $u\mathscr{E} = \{ux \in \mathsf{X} \mid x \in \mathscr{E}\}$. For each $l \in \mathbb{N}$, $j \in \{1, 2, \ldots, m(l)\}$, $i \in \{1, 2, \ldots, m(l+1)\}$ and $a \in \mathfrak{a}$, let

$$A_l(i,j,a) = \begin{cases} 1 & \text{if } \emptyset \neq a\mathscr{E}^{l+1}_i \subseteq \mathscr{E}^l_j \\ 0 & \text{else.} \end{cases}$$

Let $0 \leqslant k \leqslant l$. If $j \in M_k^l$ and there exists an $a \in \mathfrak{a}$ such that $A_l(i, j, a) = 1$, then $i \in M_{k+1}^{l+1}$. Hence there exists a positive linear map from $\mathbb{Z}^{M_k^l}$ to $\mathbb{Z}^{M_{k+1}^{l+1}}$ given by

$$e_j \mapsto \sum_{i \in M_{k+1}^{l+1}} \sum_{a \in \mathfrak{a}} A_l(i, j, a) e_i.$$

We denote this map by $A_k^l$.

**Lemma 24.** *Let $0 \leqslant k \leqslant l$. Then the following diagram commutes*:

$$\begin{tikzcd}
\mathbb{Z}^{M_k^l} \arrow[r, "I_k^l"] \arrow[d, "A_k^1"'] & \mathbb{Z}^{M_k^{l+1}} \arrow[d, "A_k^{l+1}"] \\
\mathbb{Z}^{M_{k+1}^{l+1}} \arrow[r, "I_{k+1}^{l+1}"] & \mathbb{Z}^{M_{k+1}^{l+2}}.
\end{tikzcd}$$

**Proof.** Let $j \in M_k^l$, $h \in M_{k+1}^{l+2}$ and $a \in \mathfrak{a}$. If $\emptyset \neq a\mathscr{E}_h^{l+2} \subseteq \mathscr{E}_j^l$, then there exists exactly one $i \in M_k^{l+1}$ such that $\mathscr{E}_i^{l+1} \subseteq \mathscr{E}_j^l$ and $\emptyset \neq a\mathscr{E}_h^{l+2} \subseteq \mathscr{E}_i^{l+1}$; and there exists exactly one $i' \in M_{k+1}^{l+1}$ such that $\mathscr{E}_h^{l+2} \subseteq \mathscr{E}_{i'}^{l+1}$ and $\emptyset \neq a\mathscr{E}_{i'}^{l+1} \subseteq \mathscr{E}_j^l$. If $a\mathscr{E}_h^{l+2} = \emptyset$ or $a\mathscr{E}_h^{l+2} \nsubseteq \mathscr{E}_j^l$, then there does not exists an $i \in M_k^{l+1}$ such that $\mathscr{E}_i^{l+1} \subseteq \mathscr{E}_j^l$ and $\emptyset \neq a\mathscr{E}_h^{l+2} \subseteq \mathscr{E}_i^{l+1}$; and there does not exists an $i' \in M_{k+1}^{l+1}$ such that $\mathscr{E}_h^{l+2} \subseteq \mathscr{E}_{i'}^{l+1}$ and $\emptyset \neq a\mathscr{E}_{i'}^{l+1} \subseteq \mathscr{E}_j^l$. Hence we have

$$\sum_{i \in M_k^{l+1}} A_{l+1}(h, i, a) I_l(i, j) = \sum_{i \in M_{k+1}^{l+1}} I_{l+1}(h, i) A_l(i, j, a).$$

It follows from this that

$$\begin{aligned}
A_k^{l+1}(I_k^l(e_j)) &= A_k^{l+1}\left(\sum_{i \in M_k^{l+1}} I_l(i, j) e_i\right) \\
&= \sum_{h \in M_{k+1}^{l+2}} \sum_{a \in \mathfrak{a}} A_{l+1}(h, i, a) \sum_{i \in M_k^{l+1}} I_l(i, j) e_h \\
&= \sum_{h \in M_{k+1}^{l+2}} \sum_{i \in M_{k+1}^{l+1}} \sum_{a \in \mathfrak{a}} I_{l+1}(h, i) A_l(i, j, a) e_h \\
&= I_{k+1}^{l+1}\left(\sum_{i \in M_{k+1}^{l+1}} \sum_{a \in \mathfrak{a}} A_l(i, j, a) e_i\right) \\
&= I_{k+1}^{l+1}(A_k^l(e_j))
\end{aligned}$$

for every $j \in M_k^l$. Thus the diagram commutes. $\square$

For $k \in \mathbb{N}$, the inductive limit $\varinjlim(\mathbb{Z}^{M_k^l}, (\mathbb{Z}^+)^{M_k^l}, I_k^l)$ will be denoted by $(\mathbb{Z}_{\mathsf{X}_k}, \mathbb{Z}_{\mathsf{X}_k}^+)$. It follows from Lemma 24 that the family $\{A_k^l\}_{l \geqslant k}$ induces a positive, linear map $A_k$ from $\mathbb{Z}_{\mathsf{X}_k}$ to $\mathbb{Z}_{\mathsf{X}_{k+1}}$.

Let $0 \leqslant k < l$. Denote by $\delta_k^l$ the linear map from $\mathbb{Z}^{M_k^l}$ to $\mathbb{Z}^{M_{k+1}^l}$ given by

$$e_j \mapsto \begin{cases} e_j & \text{if } j \in M_{k+1}^l, \\ 0 & \text{if } j \notin M_{k+1}^l, \end{cases}$$

for $j \in M_k^l$. It is easy to check that the following diagram

$$\begin{tikzcd} \mathbb{Z}^{M_k^l} \arrow[r, "\delta_k^l"] \arrow[d, "I_k^l"'] & \mathbb{Z}^{M_{k+1}^l} \arrow[d, "I_{k+1}^l"] \\ \mathbb{Z}^{M_k^{l+1}} \arrow[r, "\delta_k^{l+1}"] & \mathbb{Z}^{M_{k+1}^{l+1}} \end{tikzcd}$$

commutes.

Thus the family $\{\delta_k^l\}_{l \geqslant k}$ induces a positive, linear map from $\mathbb{Z}_{\mathsf{X}_k}$ to $\mathbb{Z}_{\mathsf{X}_{k+1}}$ which we denote by $\delta_k$. Since the diagram

$$\begin{tikzcd} \mathbb{Z}^{M_k^l} \arrow[r, "\delta_k^l"] \arrow[d, "A_k^l"'] & \mathbb{Z}^{M_{k+1}^l} \arrow[d, "A_{k+1}^l"] \\ \mathbb{Z}^{M_{k+1}^{l+1}} \arrow[r, "\delta_{k+1}^{l+1}"] & \mathbb{Z}^{M_{k+2}^{l+1}} \end{tikzcd}$$

commutes for every $0 \leqslant k < l$, the diagram

$$\begin{tikzcd} \mathbb{Z}_{\mathsf{X}_k} \arrow[r, "\delta_k"] \arrow[d, "A_k"'] & \mathbb{Z}_{\mathsf{X}_{k+1}} \arrow[d, "A_{k+1}"] \\ \mathbb{Z}_{\mathsf{X}_{k+1}} \arrow[r, "\delta_{k+1}"] & \mathbb{Z}_{\mathsf{X}_{k+2}} \end{tikzcd}$$

commutes.

We denote the inductive limit $\varinjlim(\mathbb{Z}_{\mathsf{X}_k}, \mathbb{Z}_{\mathsf{X}_k}^+, A_k)$ by $(\Delta_{\mathsf{X}}, \Delta_{\mathsf{X}}^+)$. Since the previous diagram commutes, the family $\{\delta_k\}_{k \in \mathbb{N}}$ induces a positive, linear map from $\Delta_{\mathsf{X}}$ to $\Delta_{\mathsf{X}}$ which we denote by $\delta_{\mathsf{X}}$.

**Theorem 25.** *For every one-sided shift space* $\mathsf{X}$,

$$(K_0(\mathscr{F}_\mathsf{X}), K_0^+(\mathscr{F}_\mathsf{X}), (\lambda_\mathsf{X})_*) \cong (\Delta_\mathsf{X}, \Delta_\mathsf{X}^+, \delta_\mathsf{X}).$$

*More precisely, the map* $[s_u 1_{\mathscr{E}_i^l} s_v^*]_0 \mapsto e_i \in \mathbb{Z}^{M_k^l}$ *extends to an isomorphism from* $(K_0(\mathscr{F}_\mathsf{X}), K_0^+(\mathscr{F}_\mathsf{X}), (\lambda_\mathsf{X})_*)$ *to* $(\Delta_\mathsf{X}, \Delta_\mathsf{X}^+, \delta_\mathsf{X})$.

For every $l \in \mathbb{N}$ denote by $B^l$ the linear map from $\mathbb{Z}^{M_1^l}$ to $\mathbb{Z}^{m(l+1)}$ given by

$$e_j \mapsto \sum_{i=1}^{m(l+1)} \left( I_l(i,j) - \sum_{a \in \mathfrak{a}} A_l(i,j,a) \right) e_i.$$

One can easily check that the following diagram commutes for every $l \in \mathbb{N}$:

$$\begin{tikzcd} \mathbb{Z}^{M_1^l} \arrow[r, "B^l"] \arrow[d, "I_1^l"'] & \mathbb{Z}^{m(l+1)} \arrow[d, "I_0^{l+1}"] \\ \mathbb{Z}^{M_1^{l+1}} \arrow[r, "B^{l+1}"] & \mathbb{Z}^{m(l+2)}. \end{tikzcd}$$

Hence the family $\{B^l\}_{l \in \mathbb{N}}$ induces a linear map $B$ from $\mathbb{Z}_{\mathsf{X}_1}$ to $\mathbb{Z}_{\mathsf{X}_0}$.

**Theorem 26.** *Let* $\mathsf{X}$ *be a one-sided shift space. Then*

$$K_0(\mathscr{D}_\mathsf{X} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}) \cong \mathbb{Z}_{\mathsf{X}_0} / B\mathbb{Z}_{\mathsf{X}_1},$$

*and*

$$K_1(\mathscr{D}_\mathsf{X} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}) \cong \ker(B).$$

*More precisely, the map*

$$[1_{\mathscr{E}_i^l}]_0 \mapsto e_i \in \mathbb{Z}^{m(l)}$$

*induces an isomorphism from* $K_0(\mathscr{D}_\mathsf{X} \rtimes_{\alpha,\mathscr{L}} \mathbb{N})$ *to* $\mathbb{Z}_{\mathsf{X}_0} / B\mathbb{Z}_{\mathsf{X}_1}$.

## 13. The ideal structure of $\mathscr{D}_\mathsf{X} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$

In this section, we will briefly describe the structure of the gauge invariant ideals of $\mathscr{D}_\mathsf{X} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$. By an ideal, we will always mean a closed two-sided ideal, and by a gauge invariant ideal, we mean an ideal $I$ such that $\gamma_z(I) \subseteq I$ for every $z \in \mathbb{T}$.

The lattice of the gauge invariant ideals of $\mathscr{D}_{\mathsf{X}} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$ has been described by Matsumoto in [28] in the case where $\mathsf{X}$ is of the form $\mathsf{X}_\Lambda$ for some two-sided shift space $\Lambda$ and this has been generalized to arbitrary one-sided shift spaces $\mathsf{X}$ by the first author in [5]. We will slightly reformulate the description here.

**Theorem 27.** *Let $\mathsf{X}$ be a one-sided shift space. Then there exist order-preserving bijections between each pair of the following lattices*:

1. *the lattice of gauge invariant ideals of $\mathscr{D}_{\mathsf{X}} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$*,
2. *the lattice of ideals $J$ of $\mathscr{F}_{\mathsf{X}}$, such that $s_u x s_u^*$, $s_u^* x s_u \in J$ for every $u \in \mathfrak{a}^*$ and every $x \in J$*,
3. *the lattice of ideals $I$ of $\mathscr{A}_{\mathsf{X}}$, such that $s_u^* x s_u \in I$ for every $u \in \mathfrak{a}^*$ and every $x \in I$*,
4. *the lattice of order ideals of $\Delta_{\mathsf{X}}$ invariant under $\delta_{\mathsf{X}}$*,
5. *the lattice of subsets $A$ of $\mathsf{X}$, such that $\sigma(A) \subseteq A$ and $\forall x \in A\ \exists l \in \mathbb{N} : \mathscr{P}_l(x) \subseteq A$.*

## 14. Examples

What we have described in this paper is the general structure of $\mathscr{D}_{\mathsf{X}} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$. It is of course also interesting to look at $\mathscr{D}_{\mathsf{X}} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$ for particular examples of $\mathsf{X}$. We will now briefly mention some papers where this has been done. The focus of these papers is mainly to compute the $K$-groups $K_0(\mathscr{D}_{\mathsf{X}} \rtimes_{\alpha,\mathscr{L}} \mathbb{N})$, $K_1(\mathscr{D}_{\mathsf{X}} \rtimes_{\alpha,\mathscr{L}} \mathbb{N})$ and $K_0(\mathscr{F}_{\mathsf{X}})$ and other invariants of $\mathscr{D}_{\mathsf{X}} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$ of $K$-theoretical nature for classes of non-sofic shift spaces (if $\mathsf{X}$ is sofic, then these invariants are easily computed by using the results of [6]). This has shed some light on the class of non-sofic shift spaces, a class of dynamical system which is far less understood than the class of sofic shift spaces.

In [37], Matsumoto has taken a closer look at $\mathscr{D}_{\mathsf{X}} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$ in the case where $\mathsf{X}$ is the Motzkin shift, and in [30] he examines $\mathscr{D}_{\mathsf{X}} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$ for the context-free shift. In [24], $\mathscr{D}_{\mathsf{X}} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$ is examined for the Dyck shift, and in [20] (see also [2]) $\mathscr{D}_{\mathsf{X}} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$ is examined for a class of shift spaces called $\beta$-shifts.

If $\Lambda$ is a two-sided shift space, then, as explained before, we can associate to it the $C^*$-algebra $\mathscr{D}_{\mathsf{X}_\Lambda} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$. But we can of course also look at the $C^*$-crossed product $C(\Lambda) \rtimes_\phi \mathbb{Z}$, defined in Section 2, where $\phi : C(\Lambda) \to C(\Lambda)$ is the map

$$f \mapsto f \circ \sigma.$$

Consider the condition:

$$(*) \text{ For each } u \in \mathscr{L}(\Lambda), \text{ there exists an } x \in \mathsf{X}_\Lambda \text{ such that } \mathscr{P}_{|u|}(x) = \{u\}.$$

It is proved in [8] that if $\Lambda$ satisfies the condition $(*)$, then $C(\Lambda) \rtimes_\phi \mathbb{Z}$ is a quotient of $\mathscr{D}_{\mathsf{X}_\Lambda} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$. This is used in [12,13] to relate the $K$-theory of $\mathscr{D}_{\mathsf{X}_\Lambda} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$ to the $K$-theory of $C(\Lambda) \rtimes_\phi \mathbb{Z}$ for these shift spaces, and in [11] to present $K_0(\mathscr{D}_{\mathsf{X}_\Lambda} \rtimes_{\alpha,\mathscr{L}} \mathbb{N})$, for the two-sided shift space $\Lambda$ of an aperiodic primitive substitution, as a stationary inductive limit of a system associated to an integer matrix defined from combinatorial data which can be computed in an algorithmic way (cf. [10]).

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