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Matrix results on the Khatri–Rao and Tracy–Singh products

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Abstract

We establish a connection between the Khatri–Rao and Tracy–Singh products introduced by Khatri and Rao (C.G. Khatri, C.R. Rao, Sankhya 30 (1968) 167–180) and Tracy and Singh (D.S. Tracy, R.P. Singh, Statistica Neerlandica 26 (1972) 143–157), respectively, and present further results including matrix equalities and inequalities involving the two products. Also, we give two statistical applications.

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1. Introduction

The Hadamard and Kronecker products play an important role in matrix methods for statistics and econometrics, see e.g. [1–3]. Relevant to these two matrix products the Khatri–Rao product for partitioned matrices, see [4,5], is claimed to be a generalized Hadamard product. Rao and Kleffe [6] has compiled several matrix equalities involving the Khatri–Rao product, which seem to be most existing results. Liu [3] has recently given a further result. The

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Tracy–Singh product, introduced also for partitioned matrices and used in an application to econometrics by Tracy and Singh [7], is a generalized Kronecker product.

The purpose of this present paper is to study matrix results on the Khatri–Rao and Tracy–Singh products. In Section 2 we introduce the definitions of the four above-mentioned products, and several elementary results. In Section 3 we establish a connection between the Khatri–Rao and Tracy–Singh products and present matrix inequalities involving these two products. In Section 4 we discuss statistical applications of some of the obtained results of Section 3. We complete the paper with a final section of concluding remarks.

2. Elementary results

2.1. Definitions and equalities

We introduce the definitions of four matrix products, namely the Kronecker, Hadamard, Tracy–Singh and Khatri–Rao products, and then give several equalities involving the Tracy–Singh and Khatri–Rao products.

Consider matrices $\mathbf{A} = (a_{ij})$ and $\mathbf{C} = (c_{ij})$ of order $m \times n$ and $\mathbf{B} = (b_{kl})$ of order $p \times q$. Let $\mathbf{A} = (A_{ij})$ be partitioned with A_{ij} of order $m_i \times n_j$ as the (i, j)th block submatrix and let $\mathbf{B} = (B_{kl})$ be partitioned with B_{kl} of order $p_k \times q_l$ as the (k, l)th block submatrix ($\sum m_i = m$, $\sum n_j = n$, $\sum p_k = p$ and $\sum q_l = q$). The four matrix products of $\mathbf{A}$ and $\mathbf{B}$ are defined as follows.

1. Kronecker product

$$\mathbf{A} \otimes \mathbf{B} = (a_{ij}\mathbf{B})_{ij},$$

where a_{ij} $\mathbf{B}$ is of order $p \times q$ and $\mathbf{A} \otimes \mathbf{B}$ of order $mp \times nq$.

2. Hadamard product

$$\mathbf{A} \odot \mathbf{C} = (a_{ij}c_{ij})_{ij},$$

where $a_{ij}c_{ij}$ is a scalar and $\mathbf{A} \odot \mathbf{C}$ is of order $m \times n$.

3. Tracy–Singh product

$$\mathbf{A} \circ \mathbf{B} = (A_{ij} \circ \mathbf{B})_{ij} = ((A_{ij} \otimes B_{kl})_{kl})_{ij},$$

where $A_{ij} \otimes B_{kl}$ is of order $m_ip_k \times n_jq_l$, $A_{ij} \circ B$ of order $m_ip \times n_jq$ and $A \circ B$ of order $mp \times nq$.

4. Khatri–Rao product

$$\mathbf{A} * \mathbf{B} = (A_{ij} \otimes B_{ij})_{ij},$$

where $A_{ij} \otimes B_{ij}$ is of order $m_ip_i \times n_jq_j$ and $A * B$ of order $(\sum m_ip_i) \times (\sum n_jq_j)$.

Note that Tracy and Singh's [7] definition of the Tracy–Singh product is to place $\mathbf{A} \circ B_{kl} = (A_{ij} \circ B_{kl})$ as the (k, l)th block submatrix of $\mathbf{A} \circ \mathbf{B}$. For a non-partitioned matrix $\mathbf{B}$, their $\mathbf{A} \circ \mathbf{B}$ is $\mathbf{A} \otimes \mathbf{B}$. The new definition advocated above of the Tracy–Singh product is different and is given in such a way (similar to that the right Kronecker product is defined): $A_{ij} \circ \mathbf{B} = (A_{ij} \otimes B_{kl})$ is located as the (i,j)th block submatrix. In fact, our theorems to be presented in the sequel remain the same for the two definitions, which link each other, of the Tracy–Singh product. However, we prefer to use the new definition. We see that for non-partitioned matrices $\mathbf{A}$ and $\mathbf{B}$ both $\mathbf{A} \circ \mathbf{B}$ and $\mathbf{A} * \mathbf{B}$ yield the Kronecker product. For $\mathbf{C} = (c_{ij})$ where c_{ij} is a scalar, we have

$$\mathbf{C} \circ \mathbf{B} = (c_{ij} \circ B)_{ij} = ((c_{ij} \otimes B_{kl})_{kl})_{ij} = ((c_{ij} B_{kl})_{kl})_{ij} = (c_{ij} B)_{ij} = \mathbf{C} \otimes \mathbf{B}$$

and

$$\mathbf{C} * \mathbf{B} = (c_{ij} B_{ij})$$

which can be viewed as a generalized Hadamard product. We have the following results on the Tracy–Singh and Khatri–Rao products.

Theorem 1. *Let* $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$, $\mathbf{E}$ *and* $\mathbf{F}$ *be compatibly partitioned matrices, then*

(a) $(\mathbf{A} \circ \mathbf{B})(\mathbf{D} \circ \mathbf{E}) = (\mathbf{AD}) \circ (\mathbf{BE})$

(b) $(\mathbf{A} \circ \mathbf{B})^+ = \mathbf{A}^+ \circ \mathbf{B}^+$ *for the Moore–Penrose inverse*

(c) $\mathbf{A} * \mathbf{B} \neq \mathbf{B} * \mathbf{A}$ *in general*

(d) $\mathbf{C} * \mathbf{B} = \mathbf{B} * \mathbf{C}$ *where* $\mathbf{C} = (c_{ij})$ *and* c_{ij} *is a scalar*

(e) $(\mathbf{A} * \mathbf{B})' = \mathbf{A}' * \mathbf{B}'$

(f) $(\mathbf{A} + \mathbf{D}) * (\mathbf{B} + \mathbf{E}) = \mathbf{A} * \mathbf{B} + \mathbf{A} * \mathbf{E} + \mathbf{D} * \mathbf{B} + \mathbf{D} * \mathbf{E}$

(g) $(\mathbf{A} * \mathbf{B}) * \mathbf{F} = \mathbf{A} * (\mathbf{B} * \mathbf{F})$

(h) $(\mathbf{A} * \mathbf{B}) \odot (\mathbf{D} * \mathbf{E}) = (\mathbf{A} \odot \mathbf{D}) * (\mathbf{B} \odot \mathbf{E})$

Proof. Straightforward. □

2.2. Miscellaneous results

We introduce several known results including Albert's theorem (to be employed to derive most of new results) and the others (to be generalized in Section 3). Without loss of generality, we consider

$$\mathbf{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}, \tag{1}$$

where $\mathbf{A}, A_{11}, A_{12}, A_{21}, A_{22}, \mathbf{B}, B_{11}, B_{12}, B_{21}$ and B_{22} are $m \times n$, $m_1 \times n_1$, $m_1 \times n_2$, $m_2 \times n_1$, $m_2 \times n_2$, $p \times q$, $p_1 \times q_1$, $p_1 \times q_2$, $p_2 \times q_1$ and $p_2 \times q_2$ ($m_1 + m_2 = m$, $n_1 + n_2 = n$, $p_1 + p_2 = p$ and $q_1 + q_2 = q$) matrices, respectively.

Furthermore we denote special matrices $\mathbf{M}$ and $\mathbf{N}$ as follows

$$\mathbf{M} = \begin{pmatrix} M_{11} & M_{12} \\ M'_{12} & M_{22} \end{pmatrix}, \quad \mathbf{N} = \begin{pmatrix} N_{11} & N_{12} \\ N'_{12} & N_{22} \end{pmatrix}, \tag{2}$$

where $\mathbf{M}$, M_{11}, M_{22}, $\mathbf{N}$, N_{11} and N_{22} are $m \times m$, $m_1 \times m_1$, $m_2 \times m_2$, $p \times p$, $p_1 \times p_1$ and $p_2 \times p_2$ symmetric matrices respectively, and M_{12} and N_{12} are $m_1 \times m_2$ and $p_1 \times p_2$ matrices, respectively.

We write $\mathbf{M} \geqslant \mathbf{P}$ in the *Löwner* ordering sense, which means $\mathbf{M} - \mathbf{P} \geqslant 0$ is positive semidefinite, for symmetric matrices $\mathbf{M}$ and $\mathbf{P}$ of the same order. Albert's [8] theorem, see also Refs. [9] or [10], asserts that: $\mathbf{M} \geqslant 0$ if and only if $M_{11} \geqslant 0, M_{12} = M_{11}M_{11}^{+}M_{12}$ and $M_{22} - M'_{12}M_{11}^{+}M_{12} \geqslant 0$. In particular, $\mathbf{M} > 0$ if and only if $M_{11} > 0$ and $M_{22} - M'_{12}M_{11}^{-1}M_{12} > 0$. Trenkler [11] established an elegant result on the Kronecker product, see also [12]

$$\mathbf{M} \geqslant \mathbf{P} \text{ is equivalent to } \mathbf{M} \otimes \mathbf{M} \geqslant \mathbf{P} \otimes \mathbf{P}, \tag{3}$$

where $\mathbf{M} \geqslant 0$ and $\mathbf{P} \geqslant 0$.

Faliva [13] seems to be the first one to observe the following connection between the Kronecker and Hadamard products, see also e.g. [3]

$$\mathbf{J}'(\mathbf{A} \otimes \mathbf{C})\mathbf{K} = \mathbf{A} \odot \mathbf{C}, \tag{4}$$

where $\mathbf{A}$ and $\mathbf{C}$ are of the same order $m \times n$, in general; $\mathbf{J}$ is the $m^2 \times m$ selection matrix and $\mathbf{K}$ is the $n^2 \times n$ selection matrix. Browne [14] early used the equality when $\mathbf{J} = \mathbf{K}$ with $m = n$.

Amemiya's [15] inequality on the Hadamard product is

$$\mathbf{A}'\mathbf{A} \odot \mathbf{C}'\mathbf{C} \geqslant (\mathbf{A}' \odot \mathbf{C}')(\mathbf{A} \odot \mathbf{C}), \tag{5}$$

where $\mathbf{A}$ and $\mathbf{C}$ are of order $m \times n$.

Schott's [16] Theorem 7.22 implies that: if $\mathbf{M} \geqslant 0$ is positive semidefinite with positive diagonal elements, then

$$\mathbf{N} > 0 \text{ implies } \mathbf{M} \odot \mathbf{N} > 0. \tag{6}$$

Liu [3] presented

$$\mathbf{M} \geqslant 0 \text{ implies } \mathbf{M} * \mathbf{M} \geqslant 0. \tag{7}$$

A well-known result on the Kronecker product, which can be viewed as a special case of Theorem 1(a), is as follows

$$(\mathbf{A} \otimes \mathbf{B})(\mathbf{D} \otimes \mathbf{E}) = (\mathbf{AD}) \otimes (\mathbf{BE}). \tag{8}$$

3. Main results

3.1. On the Tracy–Singh product

Theorem 2. *Let* $\mathbf{M} \geqslant \mathbf{P} \geqslant 0, \mathbf{N} \geqslant \mathbf{Q} \geqslant 0$, *and* $\mathbf{M}, \mathbf{P}, \mathbf{N}$ *and* $\mathbf{Q}$ *be compatibly partitioned matrices, then*

$$\mathbf{M} \circ \mathbf{N} \geqslant \mathbf{P} \circ \mathbf{Q} \geqslant 0, \tag{9}$$

$$\mathbf{M} \geqslant \mathbf{P} \textit{ is equivalent to } \mathbf{M} \circ \mathbf{M} \geqslant \mathbf{P} \circ \mathbf{P}. \tag{10}$$

Proof. To establish Eq. (9) we first consider $\mathbf{M} \circ \mathbf{N}$ with $M_{11} \circ \mathbf{N}$:

$$\mathbf{M} \circ \mathbf{N} = \begin{pmatrix} M_{11} \circ \mathbf{N} & M_{12} \circ \mathbf{N} \\ M_{12}' \circ \mathbf{N} & M_{22} \circ \mathbf{N} \end{pmatrix},$$
$$M_{11} \circ \mathbf{N} = \begin{pmatrix} M_{11} \otimes N_{11} & M_{11} \otimes N_{12} \\ M_{11} \otimes N_{12}' & M_{11} \otimes N_{22} \end{pmatrix}.$$

We prove the following statements (i), (ii) and (iii).

(i) $M_{11} \circ \mathbf{N} \geqslant 0$: Because $\mathbf{N} \geqslant 0$, Albert's theorem ensures that $N_{11} \geqslant 0, N_{22} - N_{12}' N_{11}^{+} N_{12} \geqslant 0$ and $N_{12} = N_{11} N_{11}^{+} N_{12}$. Given $M_{11} \geqslant 0$, we get $M_{11} \circ \mathbf{N} \geqslant 0$ by using Eq. (8) and Albert's theorem.

(ii) $M_{22} \circ \mathbf{N} - (M_{12}' \circ \mathbf{N})(M_{11} \circ \mathbf{N})^{+}(M_{12} \circ \mathbf{N}) \geqslant 0$: It follows from using Theorem 1 (b) and (a), $M_{22} - M_{12}' M_{11}^{+} M_{12} \geqslant 0$, $\mathbf{N} \geqslant 0$ and (i) above.

(iii) $M_{12} \circ \mathbf{N} = (M_{11} \circ \mathbf{N})(M_{11} \circ \mathbf{N})^{+}(M_{12} \circ \mathbf{N})$: By virtue of Theorem 1 (a) and (b), we establish

$$M_{12} \circ \mathbf{N} = M_{11} M_{11}^{+} M_{12} \circ \mathbf{N}\mathbf{N}^{+}\mathbf{N} = (M_{11} \circ \mathbf{N})(M_{11} \circ \mathbf{N})^{+}(M_{12} \circ \mathbf{N}).$$

Then, by Albert's theorem based on (i), (ii) and (iii) we obtain

$$\mathbf{M} \circ \mathbf{N} \geqslant 0 \quad \text{for } \mathbf{M} \geqslant 0 \text{ and } \mathbf{N} \geqslant 0.$$

Furthermore, this result implies that

$$\begin{aligned} \mathbf{P} \circ \mathbf{Q} \geqslant 0, \mathbf{M} \circ \mathbf{N} - \mathbf{P} \circ \mathbf{Q} &= \mathbf{M} \circ \mathbf{N} - \mathbf{M} \circ \mathbf{Q} + \mathbf{M} \circ \mathbf{Q} - \mathbf{P} \circ \mathbf{Q} \\ &= \mathbf{M} \circ (\mathbf{N} - \mathbf{Q}) + (\mathbf{M} - \mathbf{P}) \circ \mathbf{Q} \geqslant 0. \end{aligned}$$

To prove Eq. (10) we notice that $\mathbf{M} \circ \mathbf{M} \geqslant \mathbf{P} \circ \mathbf{P}$ for $\mathbf{M} \geqslant \mathbf{P}$. Also, $\mathbf{M} \circ \mathbf{M} \geqslant \mathbf{P} \circ \mathbf{P}$ implies $(\mathbf{x} \circ \mathbf{x})'(\mathbf{M} \circ \mathbf{M})(\mathbf{x} \circ \mathbf{x}) \geqslant (\mathbf{x} \circ \mathbf{x})'(\mathbf{P} \circ \mathbf{P})(\mathbf{x} \circ \mathbf{x})$, i.e. $(\mathbf{x}'\mathbf{M}\mathbf{x})^2 \geqslant (\mathbf{x}'\mathbf{P}\mathbf{x})^2$, and therefore $\mathbf{x}'\mathbf{M}\mathbf{x} \geqslant \mathbf{x}'\mathbf{P}\mathbf{x}$. for $\mathbf{M} \geqslant 0, \mathbf{P} \geqslant 0$ and any compatible vector $\mathbf{x}$. Hence, $\mathbf{M} \geqslant \mathbf{P}$. □

If we replace the Tracy–Singh product by the Kronecker product, Eq. (10) turns to be Eq. (3), which is given by Trenkler [11].

3.2. Connection between the Khatri–Rao and Tracy–Singh products

To establish Theorem 3 we define the following selection matrices $\mathbf{Z}_1$ of order $k \times r$ and $\mathbf{Z}_2$ of order $nq \times t$:

$$\mathbf{Z}_1 = \begin{pmatrix} I_{11} & 0_{11} & 0_{21} & 0 \\ 0 & 0 & 0 & I_{21} \end{pmatrix}', \quad \mathbf{Z}_2 = \begin{pmatrix} I_{12} & 0_{12} & 0_{22} & 0 \\ 0 & 0 & 0 & I_{22} \end{pmatrix}', \tag{11}$$

where $\mathbf{Z}_1'\mathbf{Z}_1 = \mathbf{I}_1$ and $\mathbf{Z}_2'\mathbf{Z}_2 = \mathbf{I}_2$ with $I_{11}, I_{21}, I_{12}, I_{22}, \mathbf{I}_1$ and $\mathbf{I}_2$ being $m_1p_1 \times m_1p_1$, $m_2p_2 \times m_2p_2$, $n_1q_1 \times n_1q_1$, $n_2q_2 \times n_2q_2$, $r \times r$ and $t \times t$ identity matrices ($k = mp$, $m = m_1 + m_2$, $n = n_1 + n_2$, $p = p_1 + p_2$, $q = q_1 + q_2$, $r = m_1p_1 + m_2p_2$ and $t = n_1q_1 + n_2q_2$), and $0_{11}, 0_{21}, 0_{12}$ and 0_{22} being $m_1p_1 \times m_1p_2$, $m_1p_1 \times m_2p_1$, $n_1q_1 \times n_1q_2$ and $n_1q_1 \times n_2q_1$ matrices of zeros.

Theorem 3 *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be partitioned as in Eq.* (1) *and* $\mathbf{M}$ *and* $\mathbf{N}$ *be partitioned as in Eq.* (2), *we have*

$$\mathbf{A} * \mathbf{B} = \mathbf{Z}_1'(\mathbf{A} \circ \mathbf{B})\mathbf{Z}_2, \tag{12}$$

and

$$\mathbf{M} * \mathbf{N} = \mathbf{Z}'(\mathbf{M} \circ \mathbf{N})\mathbf{Z}, \tag{13}$$

where $\mathbf{Z}_1$ *and* $\mathbf{Z}_2$ *in Eq.* (11) *both become the* $k \times r$ *selection matrix* $\mathbf{Z}$ *with* $\mathbf{Z}'\mathbf{Z} = \mathbf{I}_r$, *as* $I_{11} = I_{12}$ *and* $I_{21} = I_{22}$ *with* $m_i = n_i$ *and* $p_i = q_i$, $i = 1, 2$.

Proof. Using the definitions of the Khatri–Rao and Tracy–Singh products, we establish Theorem 3. □

Obviously, Eq. (12) generalizes Eq. (4), a result in Ref. [13], and Eq. (13) generalizes the result of Browne [14].

3.3. On the Khatri–Rao product

Theorem 4. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be partitioned as in Eq.* (1), *then*

$$\mathbf{A}'\mathbf{A} * \mathbf{B}'\mathbf{B} \geqslant (\mathbf{A}' * \mathbf{B}')(\mathbf{A} * \mathbf{B}). \tag{14}$$

Proof. Using Eq. (12) and Theorem 1(a), and noting that $\mathbf{I}_k \geqslant \mathbf{Z}_1\mathbf{Z}_1'$ we derive Eq. (14). □

For the Hadamard product, Eq. (14) reduces to Eq. (5), which is due to Amemiya [15].

Theorem 5. *Let* $\mathbf{M} \geqslant \mathbf{P} \geqslant 0, \mathbf{N} \geqslant \mathbf{Q} \geqslant 0$, *and* $\mathbf{M}, \mathbf{P}, \mathbf{N}$ *and* $\mathbf{Q}$ *be compatibly partitioned matrices, then*

$$\mathbf{M} * \mathbf{N} \geqslant \mathbf{P} * \mathbf{Q} \geqslant 0. \tag{15}$$

Let $\mathbf{M}$ *be partitioned as in Eq.* (2) *with* $M_{11} \nleqslant 0$ *and* $M_{22} \nleqslant 0$, *then*

$$\mathbf{M} \geqslant 0 \text{ is equivalent to } \mathbf{M} * \mathbf{M} \geqslant 0. \tag{16}$$

Proof. Using Eqs. (9) and (13) above we prove Eq. (15). Clearly, $\mathbf{M} \geqslant 0$ implies $\mathbf{M} * \mathbf{M} \geqslant 0$. We prove that $\mathbf{M} * \mathbf{M} \geqslant 0$ implies $\mathbf{M} \geqslant 0$ below. As

$$\mathbf{M} * \mathbf{M} = \begin{pmatrix} M_{11} \otimes M_{11} & M_{12} \otimes M_{12} \\ M_{12}' \otimes M_{12}' & M_{22} \otimes M_{22} \end{pmatrix} \geqslant 0,$$

Albert's theorem guarantees

$$\begin{aligned} &M_{11} \otimes M_{11} \geqslant 0, \\ &M_{22} \otimes M_{22} - (M_{12}' \otimes M_{12}')(M_{11} \otimes M_{11})^{+}(M_{12} \otimes M_{12}) \geqslant 0, \\ &M_{12} \otimes M_{12} = (M_{11} \otimes M_{11})(M_{11} \otimes M_{11})^{+}(M_{12} \otimes M_{12}). \end{aligned}$$

Then using Eq. (3) leads to

$$\begin{aligned} &M_{11} \geqslant 0, \\ &M_{22} - M_{12}' M_{11}^{+} M_{12} \geqslant 0, \\ &M_{12} = M_{11} M_{11}^{+} M_{12}, \end{aligned}$$

i.e. $\mathbf{M} \geqslant 0$. □

Here Eq. (16) is stronger than Eq. (7) given by Liu [3].

Theorem 6. *Let* $\mathbf{M} \geqslant 0$ *such that* $M_{11} > 0$ *and* $M_{22} > 0$, *then*

$$\mathbf{N} > 0 \text{ implies } \mathbf{M} * \mathbf{N} > 0.$$

Proof. Use Albert's theorem. □

We mention that Theorem 6 is an extension of (6) by Schott [16].

Theorem 7. *If* $\mathbf{M} \geqslant 0$ *and* $M_{11} = M_{12} = M_{22}$, *then*

$$\mathbf{M} * \mathbf{N} > 0 \text{ is equivalent to } M_{11} > 0 \text{ and } \mathbf{N} > 0. \tag{17}$$

Proof. Use Albert's theorem. □

Theorem 8. *Let* $\mathbf{M} > 0$ *and* $\mathbf{N} > 0$ *be* $m \times m$ *and* $p \times p$ *positive definite matrices partitioned as in Eq. (2),* $\mathbf{I}$ *be a* $r \times r$ *identity matrix,* $r = m_1p_1 + m_2p_2$, $m = m_1 + m_2$, $p = p_1 + p_2$ *and* $k = mp$, *then*

$$(\mathbf{M} * \mathbf{N})^{-1} \leqslant \mathbf{M}^{-1} * \mathbf{N}^{-1}; \tag{18}$$

$$\mathbf{M}^{-1} * \mathbf{N}^{-1} \leqslant \frac{(\lambda_1 + \lambda_k)^2}{4\lambda_1\lambda_k}(\mathbf{M} * \mathbf{N})^{-1}; \tag{19}$$

$$\mathbf{M} * \mathbf{N} - (\mathbf{M}^{-1} * \mathbf{N}^{-1})^{-1} \leqslant (\sqrt{\lambda_1} - \sqrt{\lambda_k})^2\mathbf{I}; \tag{20}$$

$$(\mathbf{M} * \mathbf{N})^2 \leqslant \mathbf{M}^2 * \mathbf{N}^2; \tag{21}$$

$$\mathbf{M}^2 * \mathbf{N}^2 \leqslant \frac{(\lambda_1 + \lambda_k)^2}{4\lambda_1\lambda_k}.(\mathbf{M} * \mathbf{N})^2; \tag{22}$$

$$(\mathbf{M} * \mathbf{N})^2 - \mathbf{M}^2 * \mathbf{N}^2 \leqslant \tfrac{1}{4}(\lambda_1 - \lambda_k)^2\mathbf{I}; \tag{23}$$

$$\mathbf{M} * \mathbf{N} \leqslant (\mathbf{M}^2 * \mathbf{N}^2)^{1/2}; \tag{24}$$

$$(\mathbf{M}^2 * \mathbf{N}^2)^{1/2} \leqslant \frac{\lambda_1 + \lambda_k}{2\sqrt{\lambda_1\lambda_k}}\mathbf{M} * \mathbf{N}; \tag{25}$$

$$(\mathbf{M}^2 * \mathbf{N}^2)^{1/2} - \mathbf{M} * \mathbf{N} \leqslant \frac{(\lambda_1 - \lambda_k)^2}{4(\lambda_1 + \lambda_k)}\mathbf{I}, \tag{26}$$

where $\lambda_1 \geqslant \cdots \geqslant \lambda_k$ *are the eigenvalues of* $\mathbf{M} \circ \mathbf{N}$ *of order* $k \times k$.

Proof. We obtain Theorem 8 by using Eq. (13), and replacing $\mathbf{H}$ by $\mathbf{M} \circ \mathbf{N} > 0$ and $\mathbf{X}$ by $\mathbf{Z}$ of Eq. (13) in the following inequalities, see e.g. [3]:

$$(\mathbf{X}'\mathbf{H}\mathbf{X})^{-1} \leqslant \mathbf{X}'\mathbf{H}^{-1}\mathbf{X};$$
$$\mathbf{X}'\mathbf{H}^{-1}\mathbf{X} \leqslant \frac{(\lambda_1 + \lambda_k)^2}{4\lambda_1\lambda_k}(\mathbf{X}'\mathbf{H}\mathbf{X})^{-1};$$
$$\mathbf{X}'\mathbf{H}\mathbf{X} - (\mathbf{X}'\mathbf{H}^{-1}\mathbf{X})^{-1} \leqslant (\sqrt{\lambda_1} - \sqrt{\lambda_k})^2\mathbf{I};$$
$$(\mathbf{X}'\mathbf{H}\mathbf{X})^2 \leqslant \mathbf{X}'\mathbf{H}^2\mathbf{X};$$
$$\mathbf{X}'\mathbf{H}^2\mathbf{X} \leqslant \frac{(\lambda_1 + \lambda_k)^2}{4\lambda_1\lambda_k}(\mathbf{X}'\mathbf{H}\mathbf{X})^2;$$

$$\mathbf{X}'\mathbf{H}^2\mathbf{X} - (\mathbf{X}'\mathbf{H}\mathbf{X})^2 \leqslant \frac{1}{4}(\lambda_1 - \lambda_k)^2\mathbf{I};$$

$$\mathbf{X}'\mathbf{H}\mathbf{X} \leqslant (\mathbf{X}'\mathbf{H}^2\mathbf{X})^{1/2};$$

$$(\mathbf{X}'\mathbf{H}^2\mathbf{X})^{1/2} \leqslant \frac{\lambda_1 + \lambda_k}{2\sqrt{\lambda_1\lambda_k}}\mathbf{X}'\mathbf{H}\mathbf{X};$$

$$(\mathbf{X}'\mathbf{H}^2\mathbf{X})^{1/2} - \mathbf{X}'\mathbf{H}\mathbf{X} \leqslant \frac{(\lambda_1 - \lambda_k)^2}{4(\lambda_1 + \lambda_k)}\mathbf{I},$$

where $\mathbf{H} > 0$ is a $k \times k$ matrix with eigenvalues $\lambda_1 \geqslant \cdots \geqslant \lambda_k > 0$, and $\mathbf{X}$ is a $k \times r$ matrix such that $\mathbf{X}'\mathbf{X} = \mathbf{I}$. □

4. Two applications

Sims et al. [17] has considered estimation and hypothesis testing in linear time series regressions with unit roots. Chambers et al. [18] has discussed limited information maximum likelihood estimation for analysis of survey data. They derived respectively two variance matrices which contain a Khatri–Rao product. We will find sufficient (and necessary) conditions for the two variance matrices to be strictly positive definite, which they did not study.

The first variance matrix can be written as

$$\Psi = \Omega * \mathbf{W},$$

where:

$$\Omega = \begin{pmatrix} \Sigma & \Sigma \\ \Sigma & \Sigma \end{pmatrix} \geqslant 0, \quad \Sigma \geqslant 0,$$

$$\mathbf{W} = \begin{pmatrix} \Gamma_1\Gamma_1' & \Gamma_1\Gamma_2' \\ \Gamma_2\Gamma_1' & \Gamma_2\Gamma_2' \end{pmatrix} \geqslant 0, \quad \Gamma_1\Gamma_1' > 0.$$

For the definitions of the relevant submatrices above and detail background, see Refs. [17,19]. Obviously, Theorem 5 ensures in an algebraic approach that $\Psi \geqslant 0$, as $\Omega \geqslant 0$ and $\mathbf{W} \geqslant 0$ (both due to Albert's theorem). In practice, an important question is to examine when $\Psi > 0$ is positive definite. By using Eq. (17) in Theorem 7 we get the answer: $\Psi > 0$ is equivalent to $\Sigma > 0$ with $\mathbf{W} > 0$, i.e. $\Sigma > 0$ with $\Gamma_2\Gamma_2' - \Gamma_2\Gamma_1'(\Gamma_1\Gamma_1')^{-1}\Gamma_1\Gamma_2' > 0$, as $\Gamma_1\Gamma_1' > 0$ is assumed.

The second variance matrix is

$$\Lambda = \mathbf{P} * \mathbf{Q},$$

where

$$\mathbf{P} = \begin{pmatrix} R & R \\ R & I_n \end{pmatrix} \geqslant 0, \quad \mathbf{Q} = \begin{pmatrix} h & \mathbf{c}' \\ \mathbf{c} & \Sigma \end{pmatrix},$$

$R = I_n - 1/N\, \mathbf{1}_n \mathbf{1}_n' > 0$, $N > n+1$, $h > 0$ is a scalar, $\Sigma > 0$ is a $(q+1) \times (q+1)$ variance matrix, $\mathbf{c}$ is a $(q+1) \times 1$ vector.

Also, Chambers et al. [18] uses $t_1 = h - \mathbf{c}'\Sigma^{-1}\mathbf{c}$ (which is a scalar), and contains necessary details. We are interested in when $\Lambda > 0$. Based on Theorem 6, we specify $\Lambda > 0$ if $t_1 > 0$ which is equivalent to $\mathbf{Q} > 0$ as $\Sigma > 0$ ($\Lambda \geqslant 0, \Lambda$ 0, if $t_1 = 0$). Such a (sufficient) condition is useful and efficient because it is quite easy to check.

5. Concluding remarks

1. The results in this paper are related to those on the Kronecker and Hadamard products or more general than some of them. We have only two applications of the results on the Khatri–Rao product and expect to have more.

2. Theorem 3 is an extension of the connection between the Hadamard and Kronecker products, which is effective to be used in deriving results of the Hadamard product based on those of the Kronecker product, see e.g. [3]. Similarly, applying Theorem 3 we can derive more inequalities involving the Khatri–Rao product from those involving the Tracy–Singh product based on matrix results like the inequalities satisfying such as $\mathbf{X}'\mathbf{X} = \mathbf{I}$ used in e.g. Theorem 8. See also Theorem 4.

3. Albert's theorem plays an essential role in deriving our results, especially Eqs. (9) and (16).

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