Compactly supported solutions of two-scale difference equations

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Abstract

We consider Lebesgue-integrable, compactly supported solutions of two-scale difference equations and investigate the relations between translates of these solutions. A detailed study of corresponding invariant subspaces leads to new observations concerning the factorization of the refinement mask and certain spectral properties of corresponding coefficient matrices. In particular, new necessary conditions for the existence of integrable, compactly supported solutions are derived. © 1998 Elsevier Science Inc. All rights reserved.

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1. Introduction

A two-scale difference equation is a functional equation of the form

\[ \varphi(t/2) = \sum_{v=0}^{n} c_v \varphi(t-v), \]  

(1.1)

where \( c_v \) are given real or complex constants with \( c_0 c_n \neq 0 \) and \( n \geq 1 \). A function \( \varphi \) satisfying (1.1) for all real \( t \), is called refinable.
Functional equations of type (1.1) arise in many contexts, in the construction of wavelets as well as in interpolating subdivision schemes. There are a lot of papers studying these equations extensively (see e.g. [1–10]). In particular, several special cases of solutions are investigated. For subdivision schemes for instance, compactly supported solutions \( \varphi \in C(\mathbb{R}) \) (or more generally, \( \varphi \in L^p(\mathbb{R}) \)) with \( L^2 \)-stable (or \( L^p \)-stable) integer translates are considered ([10,6]).

In what follows, we are interested in (1.1) as a functional equation, and consider solutions with the following property.

**Definition 1.1.** A refinable function \( \varphi \) is called E-solution (essential solution) of (1.1), if it is a not identically vanishing, Lebesgue-integrable and compactly supported function. Two E-solutions \( \varphi_1 \) and \( \varphi_2 \) are not considered as different, if there exists a constant \( c \), such that \( \varphi_1 = c \varphi_2 \) almost everywhere.

As shown in [2], the assumptions in the definition yield that \( \text{supp} \, \varphi \subseteq [0,n] \).

By Fourier transform of (1.1), we obtain

\[
\hat{\varphi}(2u) = P(e^{-iu})\hat{\varphi}(u) \tag{1.2}
\]

with \( \hat{\varphi}(u) := \int_{-\infty}^{\infty} \varphi(t)e^{-itu} \, dt \), and with the refinement mask (or the two-scale symbol)

\[
P(z) := \frac{1}{2} \sum_{v=0}^{n} c_v z^v. \tag{1.3}
\]

Assuming that (1.1) is given with real coefficients \( c_v \), in [2] it is proved that:

(i) if \( |P(1)| < 1 \) or \( P(1) = -1 \), then (1.1) has no E-solution;
(ii) if \( P(1) = 1 \), then (1.1) has at most one E-solution;
(iii) if \( |P(1)| > 1 \), and if an E-solution \( \varphi \) exists, then \( P(1) = 2^m \) for some non-negative integer \( m \). If, in the last case, the coefficients \( c_v \) \((v = 0, \ldots, n)\) are replaced by \( 2^{-m}c_v \) in (1.1) then the new two-scale difference equation possesses a continuous solution \( g \), and \( \varphi \) is the \( m \)th derivative of \( g \),

\[
\varphi(x) = \frac{d^m}{dx^m} g(x),
\]

almost everywhere.

Hence, we make the following assumption.

**Assumption 1.1.** Throughout the paper, we assume that \( P(1) = 1 \).

Then, for the Fourier transform \( \hat{\varphi}(u) \) of a refinable function \( \varphi \), we obtain by repeated application of (1.2),

\[
\hat{\varphi}(u) = \prod_{j=1}^{\infty} P(e^{-iu/2^j}),
\]
where we have assumed that $\dot{\phi}(0) = \int_0^t \phi(t) \, dt = 1$ (see [2,11]).

For $t = 0$ and $t = n$, (1.1) simplifies to $\phi(0) = c_0 \phi(0)$ and $\phi(n) = c_n \phi(n)$, respectively.

**Assumption 1.2.** Throughout the paper (disregarding the exceptional case of step functions), we assume that $c_0 \neq 1, c_n \neq 1$.

This implies together with the foregoing equations that the E-solution $\phi$ satisfies $\phi(0) = \phi(n) = 0$.

For a refinable function $\phi$, we introduce the vector
$$
\psi(t) := (\phi(t), \phi(t + 1), \ldots, \phi(t + n - 1))^T.
$$

Since $\text{supp } \phi \subseteq [0,n]$, it suffices to consider $\psi(t)$ for $0 \leq t \leq 1$. Further, in view of $\phi(0) = \phi(n) = 0$, there is an exact equivalence between $\phi$ and the vector $\psi$ (cf. [5], Proposition 1).

As in [5,2,3,10,6,1], we introduce the $1 \times 2$ block Toeplitz matrices $A_0 := (c_{2j-k})$, $j,k = 0,\ldots,n-1$ and $A_1 := (c_{2j-k+1})$, $j,k = 0,\ldots,n-1$, where $c_j = 0$ for $j < 0$ and $j > n$, respectively, i.e.,

$$
A_0 := \begin{pmatrix}
0 & 0 & \cdots & 0 \\
0 & c_1 & c_0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & c_{n-2} & c_{n-3} & 0 \\
0 & \cdots & 0 & c_n & c_{n-1}
\end{pmatrix},
A_1 := \begin{pmatrix}
c_1 & c_0 & 0 & \cdots & 0 \\
c_3 & c_2 & c_1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & c_n & c_{n-1} & c_{n-2} \\
0 & \cdots & 0 & 0 & c_n
\end{pmatrix}.
$$

Observe that the both matrices $A_0$ and $A_1$ contain $M := (c_{2j-k-1})_{j,k=0}^{n-2}$ as a submatrix with the following peculiarity: If $M$ possesses the spectrum $\text{spec}(M)$, then $\text{spec}(A_0) = \text{spec}(M) \cup \{c_0\}$ and $\text{spec}(A_1) = \text{spec}(M) \cup \{c_n\}$.

With the notations above, (1.1) can be written in the vector form

$$
\psi \left( \frac{t}{2} \right) = A_0 \psi(t), \quad \psi \left( \frac{t+1}{2} \right) = A_1 \psi(t)
$$

for $0 \leq t \leq 1$.

The equations in (1.5) imply that

$$
\psi \left( \frac{t}{2} \right) = A_0^{t} \psi(0), \quad \psi \left( \frac{t+1}{2} \right) = A_1^{t} \psi(t);
$$

and for $t = 0$ and $t = 1$, respectively,

$$
\psi(0) = A_0 \psi(0), \quad \psi(1) = A_1 \psi(1).
$$

Introducing the vector $\psi(t) := (\phi(t), \ldots, \phi(t + n - 2))^T$, relations (1.7) reduce to the single relation $\psi(1) = M \psi(1)$. In the case $\psi(1) \neq 0$, which is valid for a nontrivial continuous solution (cf. [3], Proposition 2.1), $\psi(1)$ is necessarily
a right eigenvector of \( M \) corresponding to the eigenvalue 1, and, of course, 1 is also an eigenvalue of the both matrices \( A_0 \) and \( A_1 \).

Starting with an eigenvector \( \psi(1) \) of \( M \) corresponding to 1, we can recursively compute values of \( \varphi \) at dyadic rationals by means of (1.5). This dyadic interpolation method is extensively explained in [3,1]. It also applies if \( \varphi \) has linearly dependent integer translates, while the subdivision algorithm usually does not work in this case (cf. e.g. [4]).

The matrix \( M \) may have the eigenvalue 1 with a multiplicity greater than 1. In this case, only one particular linear combination of corresponding eigenvectors can lead to an E-solution \( \varphi \) (see Example 2.1).

In the following, it is also convenient to introduce the infinite matrix 
\[
A := (c_{j+k})_{j,k \geq 0}
\]
and the infinite column vector \( \psi(t) := (\varphi(t+j))_{j \geq 0} \) of a refinable function \( \varphi \), so that (1.1) can be written in the form
\[
\psi \left( \frac{t}{2} \right) = A \psi(t)
\]
for \(-\infty < t \leq 1\).

Micchelli and Prautzsch [1] and Colella and Heil [5] succeeded in establishing necessary and sufficient conditions for the existence of continuous solutions of (1.1). They extensively studied the linear space \( W \subset \mathbb{C}^n \),

\[
W := \text{space } \{ \psi(t) - \psi(0) : t \in [0,1] \}.
\]

Conditions could be expressed in terms of the joint spectral radius \( \rho(A_0|_W, A_1|_W) \) of the two matrices \( A_0 \) and \( A_1 \) restricted to the subspaces \( W \) (cf. [5]). As shown in [5], Proposition 3, \( W \) is the smallest subspace of \( \mathbb{C}^n \) invariant under both \( A_0 \) and \( A_1 \), which contains the vector \( \psi(1) - \psi(0) \). So, if \( \psi(0), \psi(1) \) are determined by (1.7), then \( W \) can be constructed without knowing \( \psi \) explicitly.

The space \( W \) is uniquely determined by its orthogonal complement \( L_0 \subset \mathbb{C}^n \),

\[
L_0 := \text{span}\{w \in \mathbb{C}^n : w^T \psi(t) = w^T \psi(0), t \in [0,1]\}.
\]

In [3], the special case \( L_0 = \text{span}\{1, \ldots, 1\}^T \) has been considered. Further, let

\[
\mathcal{L}_0 := \text{span}\{w = (w_j)_{j \geq 0} : w^T \psi(t) = w^T \psi(0), t \in (-\infty,1]\}
\]

with \( \psi(t) := (\varphi(t+j))_{j \geq 0} \) satisfying (1.8). In particular, for \( w \in \mathcal{L}_0 \) we have by definition

\[
\sum_{j=0}^{\infty} w_j \varphi(t+j) = c
\]

for all \( t \leq 1 \) with a fixed constant \( c \). Since \( \psi(t) \) has only finitely many components different from zero, there arise no convergence problems.
Concerning E-solutions, we extend the spaces $L_0$ and $L_0$, respectively, to

$$L := \text{span}\{w \in \mathbb{C}^n : w^T \psi(t) - c, t \in [0, 1], \text{a.e.}\},$$  

(1.12)

$$L := \text{span}\{w = (w_j)_{j \geq 0} : w^T \psi(t) = c, t \in (-\infty, 1], \text{a.e.}\}$$  

(1.13)

with certain constants $c$ which depend on $w$ (or $w$), but not on $t$. For simplicity, we restrict ourselves to these extended spaces, though several of the following results are even valid with respect to $L_0$, $L_0$, and we usually drop the restriction “almost everywhere” tacitly. Let us emphazise that (1.1) shall be satisfied in any case for all real $t$.

In Section 2, the properties of $L$ and $L$ are extensively studied. A general characterization of elements of $L$ will be presented. Similar results can be found in [4].

Knowing the structure of vectors contained in $L$, we are able to derive new consequences on eigenvectors of our matrices $A_0, A_1$ and $A$, and on zeros of the refinement mask $P(z)$ from (1.3) in Section 3. Usually, papers dealing with refinement equations are restricted to the case $P(-1) = 0$, i.e., that $P(z)$ possesses the factor $z + 1$. If the subdivision scheme associated with $\{c_t\}_{t=0}^n$ converges uniformly (or in $L'(\mathbb{R})$), then $P(-1) = 0$ is necessarily satisfied (see [10,6]). Here, we drop this assumption and consider also refinement masks with $P(-1) \neq 0$; however, we show that the refinement mask $P(z)$ of an E-solution of (1.1) always possesses a factor $p(z)$, which is a certain modification of $(z + 1)$. This factor $p(z)$ can be considered as the refinement mask of a piecewise step function. Moreover, it follows that each E-solution can be represented as a finite linear combination of integer translates of a simpler E-solution of (1.1) with a refinement mask containing the factor $(z + 1)$. The arguments can be pushed a little further, also allowing assertions on multiple zeros of $P(z)$.

Finally, in Section 4, the structure of $L$ implies consequences on eigenvectors of the coefficient matrices $A_0, A_1$ and $A$. In particular, it will be shown that the matrices $A_0, A_1$ have the eigenvalue $1$ also in the general case; however, in case of continuous E-solutions, they cannot possess proper root vectors belonging to the eigenvalue $1$. As corollaries, we obtain new statements on the nonexistence of E-solutions of (1.1). The results will be explained by examples.

2. Invariant spaces

We consider the functional equation (1.1) satisfying Assumptions (1.1) and (1.2) and possessing an E-solution $\varphi$. In this section, we want to derive some properties of the corresponding spaces $L$ and $L$ in (1.12) and (1.13), respectively. We start with the following basic theorem.
Theorem 2.1. Let \( \varphi \) be an E-solution of (1.1), and let Assumptions 1.1 and 1.2 be satisfied. Then we have
\[
\sum_{v \in \mathbb{Z}} \varphi(t + v) = \int_{0}^{a} \varphi(s) \, ds = \varphi(0)
\]
(2.1)
almost everywhere for \( t \in \mathbb{R} \).

Proof. Let \( \varphi(t) := \sum_{v=0}^{n-1} \varphi(t + v) \). Then by (1.1) it follows for \( t \in [0, 1] \) that
\[
\varphi\left(\frac{t}{2}\right) + \varphi\left(\frac{t+1}{2}\right) = 2\varphi(t),
\]
(2.2)
since the left hand-side is equal to
\[
\varphi\left(\frac{t}{2}\right) + \varphi\left(\frac{t+1}{2}\right) = \sum_{v=0}^{n-1} \left[ \varphi\left(\frac{t}{2} + v\right) + \varphi\left(\frac{t+1}{2} + v\right) \right]
\]
\[
= \sum_{v=0}^{n-1} \sum_{k=0}^{n} c_k \varphi(t + 2v - k) + \varphi(t + 1 + 2v - k)
\]
\[
= \sum_{k=0}^{n} c_k \sum_{v=0}^{2n-1} \varphi(t + v - k) = 2\varphi(t).
\]

With regard to Klemmt [12], it follows that the only Lebesque-integrable solution \( \varphi(t) \) of (2.2) is
\[
\varphi(t) = \int_{0}^{1} \varphi(s) \, ds
\]
almost everywhere for \( t \in [0, 1] \). According to the definition of \( \varphi(t) \), this is our assertion. \( \square \)

Remark 2.1. (1) In papers dealing with the construction of wavelets, (2.1) is usually assumed to be true (see e.g. [13,2]). If \( \varphi(0) \neq 0 \), then (2.1) implies that \( \varphi \) satisfies the moment condition of order 1, i.e., constants can be reproduced by integer translates of \( \varphi \).

(2) For continuous solutions, it was already shown by Fichtenholz, cf. [14], pp. 789–790, that (2.2) has constant solutions only. The proof of Fichtenholz also works for Riemann-integrable functions, where (2.1) is valid for all \( t \in \mathbb{R} \).

(3) Let us mention that according to Gupta, cf. [15], p. 420, there exist non-integrable solutions \( \varphi(t) \) satisfying (2.2), for instance \( \varphi(t) = \cot(\pi t) \) \((0 < t < 1)\). In fact, there exist infinitely many nonintegrable solutions, namely, given an arbitrary function \( \varphi(t) \) for \( \frac{1}{2} < t \leq 1 \), it can be continued such that (2.2) holds.
For $n = 1$, we have $\varphi(t) = \phi(t) = c$, and (1.1) implies either $c = 0$ or $c_0 = c_1 = 1$, which contradicts Assumption 1.2. Hence, in (1.1) we have in fact $n \geq 2$.

For the next considerations, we repeat the following notation from linear algebra. A vector $v$ is called a left root vector of height $k \geq 2$ belonging to the eigenvalue $\lambda$ of a quadratic matrix $M$ if

$$v^T(M - \lambda I)^k = 0^T, \quad v^T(M - \lambda I)^{-1} \neq 0^T.$$ 

The left eigenvector $v$ of $M$ belonging to $\lambda$ is the (improper) left root vector of height 1. In the following, if we speak about root vectors, then we mean a proper root vector (of height $k \geq 2$).

**Theorem 2.2.** Let $\varphi$ be an E-solution of (1.1) with Assumptions 1.1 and 1.2. Then for the space $L$ in (1.12) the following assertions are satisfied.

(i) The vector $e := (1, \ldots, 1)^T$ is contained in $L$.

(ii) If $w \in L$, then $(w^T A_0)^T \in L$ and $(w^T A_1)^T \in L$.

(iii) Let the solution $\varphi$ be bounded in neighbourhoods of the points $k$ ($k = 0, \ldots, n$). If $\lambda \in \mathbb{C}$ with $|\lambda| > 1$ is an eigenvalue of the coefficient matrix $A_0$ (or $A_1$), then the left eigenvectors and left root vectors of $A_0$ (or $A_1$) corresponding to $\lambda$ are contained in $L$.

(iv) Let the solution $\varphi$ be continuous in the points $k$ ($k = 0, \ldots, n$). Then the left eigenvectors and left root vectors of $A_0$ (or $A_1$) corresponding to an eigenvalue $\lambda$ with $|\lambda| = 1$ are contained in $L$.

(v) If $w \in \mathbb{C}^n$ is a left eigenvector of both $A_0$ and $A_1$, corresponding to the eigenvalue $0$, then $w \in L$.

(vi) The dimension of $L$ is at most $n - 1$.

**Proof.** Assertions (i) and (ii) immediately follow from (1.12), (1.5) and from Theorem 2.1.

(iii) Let $\lambda$ with $|\lambda| > 1$ be an eigenvalue of $A_0$, and $w^T A_0 = \lambda w^T$. Then we find iteratively by (1.5) and (1.6)

$$w^T \psi(t) = \frac{1}{\lambda} w^T A_0 \psi(t) = \frac{1}{\lambda^2} w^T A_0 \psi\left(\frac{t}{2}\right) = \cdots = \frac{1}{\lambda^k} w^T \psi\left(\frac{t}{2^k}\right).$$

If $k$ goes to infinity, it follows, by boundedness of $\psi$ in the neighbourhood of 0, that $w^T \psi(t) = 0$, i.e., $w \in L$ with the constant $c = 0$ in (1.12). Assuming that $A_0$ possesses a root vector of height 2, $w^T A_0 = \hat{\lambda}(w^T + w^T)$, we can use the same argument, observing that

$$w^T \psi(t) = \left(\frac{1}{\lambda} w^T A_0 - w^T\right) \psi(t) = \frac{1}{\lambda} \hat{w}^T \psi\left(\frac{t}{2}\right).$$
So, it follows that $\tilde{w} \in L$. Analogously, we can derive the assertion for all root vectors. The same arguments apply for eigenvectors and root vectors of $A_1$, using (1.6).

(iv) For $|\lambda| = 1$, we obtain the assertion in a similar manner, since the assumed continuity of $\varphi$ implies that $\psi$ is continuous in 0. In case of $\lambda \neq 1$, we again find $c = 0$ in (1.12).

(v) This assertion easily follows from (1.5).

(vi) If there were $n$ linearly independent vectors $w^{(k)}$ $(k = 0, \ldots, n - 1)$ in $L$, then $w^{(k)T} \psi(t) = c^{(k)}$ for $k = 0, \ldots, n - 1$ would yield a solution $\psi(t)$ with constant components, i.e., $\varphi(t + j) = C_j$ for $j = 0, \ldots, n - 1$ and $0 \leq t \leq 1$ a.e. But (1.1) yields for $0 \leq t < 1$ that $\varphi(t/2) = c_0 \varphi(t)$; hence, by Assumption 1.2, it follows that $C_0 = 0$, i.e., $\varphi(t)$ vanishes identically on $[0, 1)$. Applying (1.1) recursively, we find that $\varphi$ vanishes identically on $[0, n]$, in contrast to our assumption. 

\textbf{Remark 2.2.} (1) The boundedness of $\varphi$ in (iii) for $|\lambda| > 1$ can be weakened by boundedness in right neighbourhoods of $k$ for eigenvalues of $A_0$, and boundedness in left neighbourhoods of $k$ for eigenvalues of $A_1$.

(2) Analogously, in (iv), for $A_0$ we need continuity of $\psi(0)$ from the right; and for $A_1$, continuity of $\psi(1)$ from the left is sufficient.

(3) Eigenvectors of $A_0$ (or $A_1$) corresponding to eigenvalues $\lambda$ with $0 < |\lambda| < 1$ can also be contained in $L$. For example, consider (1.1) with $c_0 = c_1 = c_3 = c_4 = 1/2$, $c_2 = 0$. Then $A_0$ possesses the eigenvalues $1, 1/2, 1/2, -1/2$ with corresponding left eigenvectors $w_1 = (1, 1, 1, 1)^T$, $w_2 = (1, 0, 0, -1)^T$, $w_3 = (1, -1, 0, 1)^T$ and $w_4 = (1, 1, -2, 1)^T$. Observing that $(0, 1, 1, 1)^T$ is a right eigenvector of $A_0$ to the eigenvalue 1, we find $\psi(1) = \psi(0) = (1, 0, 0, -1)^T$. For the space $W$ introduced after (1.8), we easily check that $W = \text{span} \{ (1, 0, 0, -1)^T \}$, since it is already invariant under both $A_0$ and $A_1$. Hence, $L$ is spanned by $w_1$, $w_3$ and $w_4$.

(4) If the first five statements of Theorem 2.2 yield $n$ linearly independent vectors of a formally constructed space $L$, then (1.1) has no $E$-solution.

Analogous considerations yield the following Corollary.

\textbf{Corollary 2.1.} Let $\varphi$ be an $E$-solution of (1.1) with Assumptions 1.1 and 1.2. Then for the space $L$ in (1.13) we have:

(i) The vector $e := (1, 1, \ldots)^T$ is contained in $L$.
(ii) If $w \in L$, then $(w^T A)^T \in L$.
(iii) Let the solution $\varphi$ be bounded in neighbourhoods of the points $k$ $(k = 0, \ldots, n)$. If $\lambda \in \mathbb{C}$ with $|\lambda| > 1$ is an eigenvalue of $A$, then the left eigenvectors and left root vectors of $A$ corresponding to $\lambda$ are contained in $L$.
(iv) Let the solution $\varphi$ be continuous in the points $k$ $(k = 0, \ldots, n)$. Then the left eigenvectors and left root vectors of $A$ corresponding to an eigenvalue $\lambda$. 

with $|\lambda| = 1$ are contained in $\mathcal{L}$.

(v) If $w$ is a left eigenvector of both $A$ and $A' := (c_{j-k+1})_{j,k=0}^{\infty}$ corresponding to the eigenvalue 0, then $w \in \mathcal{L}$.

(vi) If $w \in \mathcal{L}$ then $w \in L$ for the corresponding restriction.

Moreover, we have the following theorem.

**Theorem 2.3.** Let $\varphi$ be an E-solution of (1.1) with Assumptions 1.1 and 1.2. Then for the spaces $L$, $\mathcal{L}$ in (1.12) and (1.13), respectively, the following assertions are satisfied.

(i) If $w = (w_j)_{j=0}^{n-1}$ is a left eigenvector or left root vector of $A_0$ to the eigenvalue $\lambda$ with $\lambda \neq c_n$, then $w$ is uniquely extendable to a left eigenvector or left root vector $w = (w_j)_{j \geq 0}$ of $A$.

(ii) If $w = (w_j)_{j=0}^{n-1}$ is a left eigenvector or left root vector of $A_1$ to the eigenvalue $\lambda$, then $w$ is uniquely extendable to a left eigenvector or left root vector $w = (w_j)_{j \geq 0}$ of $A'$.

(iii) If $(w_j)_{j \geq 0} \in \mathcal{L}$, then we have $(w_{j+k})_{j \geq 0} \subset \mathcal{L}$ for every $k \in \mathbb{N}$.

Proof. (i) If $w$ is a left eigenvector of $A_0$ corresponding to $\lambda \neq 0$ and $\lambda \neq c_n$, then we have

$$\sum_{j=0}^{n-1} w_j c_{j-k} = \lambda w_k$$

for $k = 0, \ldots, n - 1$. From this equation, $w$ can successively be extended to a vector $w = (w_j)_{j \geq 0}$ such that $\sum_{j=0}^{\infty} w_j c_{j-k} = \lambda w_k$ is also satisfied for $k \geq n,$ since $j$ runs in fact up to $\lfloor (n + k)/2 \rfloor \leq k$. This means, that $w$ is an eigenvector of $A$ to $\lambda$. For $\lambda = 0$, $w$ can also be extended to $w = (w_j)_{j \geq 0}$, since the components $w_k$ $(k = n, n + 1, \ldots)$ can be found successively from $\sum_{j=0}^{k} w_j c_{j-k+n} = 0$ in view of $c_n \neq 0$. Actually, there are two equations determining $w_k$ by $w_l$ $(0 \leq l \leq k - 1)$, namely

$$k \sum_{j=0}^{l} w_j c_{j-2k+n} = 0, \quad k \sum_{j=0}^{l} w_j c_{j-2k+n-1} = 0.$$

But the second equation is a consequence of the former ones, since both equations are contained in the system $(w_j)_{j=k-n+1}^{k} A_0 = 0$ with $\det A_0 = 0$. Same ideas apply for root vectors.

(i') For left eigenvectors of $A_1$ we can prove the assertion analogously as
(i) considering $A' := (c_{2i-j+1})_{i,j=0}^{\infty}$ instead of $A$.

(ii) Replacing $t$ by $t-k$ in (1.11) with an arbitrary $k \in \mathbb{N}$, we obtain
\[ \sum_{j=-k}^{\infty} w_{j+k} \varphi(t+j) = c \]
for $t \leq k+1$ and, in view of $\varphi(t) = 0$ for $t < 0$,
\[ \sum_{j=0}^{\infty} w_{j+k} \varphi(t+j) = c \quad (2.3) \]
for $t \leq 1$. Hence, (ii) holds.

(iii) First, it can easily be seen that an $E$-solution $\varphi(t)$ can neither vanish identically in $[0, 1]$ nor in $[n-1, n]$, otherwise it would vanish for all $t \in [0, n]$. By (2.3), for a certain $t_0 \in [0, 1]$ with $\varphi(t_0 + n - 1) \neq 0$, we find
\[ \sum_{j=0}^{n-1} w_{j+k} \varphi(t_0 + j) = c. \]
This recursion formula shows that $w_v$ for $v \geq n$ is uniquely determined by the initial values $w_1, \ldots, w_{n-1}$, hence there can be at most one extension from $w$ to $w$.

Assertion (iv) follows from (ii) and from Corollary 2.1 (vi). Finally, (v) is a consequence of (iii) and the definition of $L$. □

Remark 2.3. (1) Using Theorem 2.3 (i) and Corollary 2.1 (iii), (iv) we find: if $\varphi$ is bounded in neighbourhoods of the points $k$ ($k = 0, \ldots, n$), and if $w \in L$ is a left eigenvector or left root vector of $A_0$ (or $A_1$) to the eigenvalue $\lambda$ ($|\lambda| > 1$), then $w$ is uniquely extendable to $w \in L$. If we replace the boundedness condition by continuity in all points $k$ ($k = 0, \ldots, n$), then this assertion also holds for eigenvalues $\lambda$ with $|\lambda| = 1$.

(2) If $(w_j)_{j \geq 0}^T A = \lambda (w_j)_{j \geq 0}^T$, then we also have $(w_{j+1})_{j \geq 0}^T A' = \lambda (w_{j+1})_{j \geq 0}^T$.

(3) For $\lambda = c_n$, it can happen that an eigenvector of $A_0$ corresponding to $\lambda$ cannot be extended to an eigenvector of $A$, but then it can always be extended to a root vector of $A$ corresponding to $\lambda$. For example, consider (1.1) with $c_0 = c_2 = 1/2$, $c_1 = 3/4$ and $c_3 = 1/4$. Then $(-2, 1, -2)^T$ is a left eigenvector of $A_0$ to $1/4$. This eigenvector cannot be extended to an eigenvector of $A$, since there is no $x$, such that $(1, -2, x)A_1 = \frac{1}{4}(1, -2, x)$. But, since $(1, -2, x)^T$ is a root vector of $A_1$, it can be extended to a root vector of $A'$; hence $(-2, 1, -2)^T$ can be extended to a root vector of $A$.

(4) If a formally constructed space $\mathcal{L}$ has dimension $n$, then (1.1) has no $E$-solution.

Considering solutions of (1.1) with linearly independent integer translates, we have dim $W = n - 1$. Hence, dim $L = 1$, and by Theorem 2.1 it follows that
Let $L = \text{span}\{(1, \ldots, 1)^T\}$ and $\mathcal{L} = \text{span}\{e\} = \text{span}\{(1, 1, \ldots)^T\}$. By Theorem 2.2 (ii),

$$(1, \ldots, 1)^T A_0 = \left(\sum_v c_{2v}, \sum_v c_{2v+1}, \ldots\right) = \lambda (1, \ldots, 1).$$

Taking into account that $2P(1) = \sum_v c_v = 2$ according to Assumption 1.1, it follows that the first sum rule, cf. [3],

$$\sum_v c_{2v} = \sum_v c_{2v+1} \quad (2.4)$$

is satisfied, and $\lambda$ must be 1.

In the following, we are interested in the structure of elements of $L$ and $\mathcal{L}$, and in consequences for the refinement mask.

\textbf{Theorem 2.4.} Let $\varphi$ be an $E$-solution of (1.1) with Assumptions 1.1 and 1.2, let $\mathcal{L}$ be defined as in (1.13), and let $w = (w_j)_{j \geq 0} \in \mathcal{L}$. Then there are $\zeta_k \in C \setminus \{0\}$ $(k = 1, \ldots, l)$ such that the $j$th element of $w$ has the form

$$w_j = \sum_{k=1}^l d_k(j) \zeta_k^j \quad (j \geq 0) \quad (2.5)$$

with $l < n$, and where $d_k(j)$ are polynomials of degree $v_k$ in $j$. In case of $v_k > 0$, the vectors $(j^r \zeta_k^j)_{j \geq 0}$ $(0 \leq v \leq v_k)$ belong to $\mathcal{L}$.

\textbf{Proof.} (1) According to Theorem 2.3 (ii), by $w \in \mathcal{L}$, the vectors $(w_{j+m})_{m \geq 0}$ $(j \in \mathbb{N}_0)$ are also contained in $\mathcal{L}$, i.e.,

$$\sum_{m=0}^{\infty} w_{j+m} \varphi(t+m) = c \quad (t \leq 1). \quad (2.6)$$

Observe that $\varphi(t)$ is not a constant function for $t \in [0, 1]$, since $c_0 \neq 1$ by Assumption 1.2. Hence, we can choose $t_0, t_1 \in [0, 1]$, such that (2.6) is satisfied and $\varphi(t_0) \neq \varphi(t_1)$. Considering (2.6) for these $t_0$ and $t_1$, and putting $b_m := \varphi(t_0 + m) - \varphi(t_1 + m)$, we obtain

$$\sum_{m=0}^{n-1} b_m w_{j+m} = 0 \quad (2.7)$$

for all $j \in \mathbb{N}_0$. This is a difference equation with constant coefficients, where in particular, $b_0 \neq 0$. Hence, the solution is of the form (2.5) for $j \in \mathbb{N}_0$, and the numbers $\zeta_k$ $(k = 1, \ldots, l)$ are the pairwise different zeros of the characteristic polynomial of the difference equation, i.e.,

$$\sum_{m=0}^{n-1} b_m x^m = b_p \prod_{k=1}^l (z - \zeta_k)^{v_k+1},$$

where $b_{n-1} \neq 0$. This shows that $w_j$ is a polynomial of degree $v_k$ in $j$.
where the index $p$ of the coefficient $b_p$ is determined by $p = \max\{j: b_j \neq 0\}$. In view of Corollary 2.1 (i), both Eqs. (2.6) and (2.7) are satisfied at least for $w_j = 1$ ($j \in \mathbb{N}_0$), and hence, $p \geq 1$. Further, $b_0 \neq 0$ implies that $\zeta_k \neq 0$ for all $k$. The polynomials $d_k$ in (2.5) are of degree $v_k \leq \mu_k$, where $\mu_k + 1$ is the multiplicity of the zero $\zeta_k$ in the characteristic polynomial. (The zero polynomial $d_k(j) = 0$ is included with degree $v_k = -1$.)

(2) If we assume first that the zeros $\zeta_k$ are simple, then the coefficients $d_k$ in (2.5) are independent from $j$. Replacing $j$ successively by $j+1, j+2, \ldots, j+l-1$ in (2.5), the arising equations can be written in form of the system

$$
\begin{pmatrix}
1 & \zeta_1 & \ldots & \zeta_{l-1} \\
\vdots & \vdots & \ddots & \vdots \\
1 & \zeta_l & \ldots & \zeta_{l-1}
\end{pmatrix}
\begin{pmatrix}
d_1, \\
d_2, \\
\vdots \\
d_l
\end{pmatrix}
= (w_j, \ldots, w_{j+l-1}),
$$

where the determinant $\det(\zeta_k^{l-1})_{k=1}^{l}$ is the Vandermondiand of $\zeta_1, \ldots, \zeta_l$ and therefore different from zero. Hence, all $(d_k(j))_{j \geq 0}$ $(k = 1, \ldots, l)$ are linear combinations of $(w_{j+k})_{j \geq 0}$ $(k = 0, \ldots, l - 1)$, i.e., in case of $d_k \neq 0$, also $(\zeta_k(j))_{j \geq 0}$ is contained in $\mathcal{L}$.

(3) In the same manner, we can conclude for variable $d_k(j)$, where instead of the Vandermondiand the confluent Vandermondiand appears. In this case, for $v_k > 0$, we can enlarge $j$ such that all coefficients of $d_k(j)$ are different from zero.

Remark 2.4. (1) For $(\zeta^j)_{j \geq 0} \in \mathcal{L}$, we have by Theorem 2(ii)

$$
\sum_{j=0}^{\infty} \zeta^{j+k} \varphi(t+j) = c
$$

for all $k \in \mathbb{N}_0$. Hence, for $k = 0$ and $k = 1$, respectively,

$$
0 = \sum_{j=0}^{\infty} \zeta^j \varphi(t+j) - \sum_{j=0}^{\infty} \zeta^{j+1} \varphi(t+j) = (1-\zeta)c,
$$

so that $c = 0$ for $\zeta \neq 1$. Only for $\zeta = 1$, there can be $c \neq 0$. In fact, from $e^t \psi(t) = 1$ for $\int_0^t \varphi(t) \, dt = 1$ (see Theorem 2.1) it follows that

$$
\mathcal{L} = \{ \mathbf{w}: \mathbf{w}^T \psi(t) = 0, t \in (-\infty, 1], \text{a.e.} \} \cup \{ e \}.
$$

(2) Instead of $\mathcal{L}$, in [4], Theorem 6.4, the structure of

$$
N_{\psi} := \left\{ \lambda: \sum_{x \in \mathbb{Z}} \lambda_x \varphi(x-x) = 0, \ x \in \mathbb{R}^s \right\}
$$

was considered, with a result similar to Theorem 2.4. Observe that for $s = 1$, $N_{\varphi} \subset \mathcal{L}_0$ in view of the substitution $x = -j$. More precisely, we have
\( \mathcal{L}_0 = N_\varphi \cup \{ e \} \). The investigation of \( N_\varphi \) was especially addressed in detail for the cube spline by Dahmen and Micchelli [16].

**Example 2.1.** We consider (1.1) with the coefficients \( c_0 = c_6 = \frac{1}{2}, \) \( c_3 = 1 \) and \( c_1 = c_2 = c_4 = c_5 = 0 \), i.e., \( P(z) = \frac{1}{4}(1+z^2)^2 \).

The corresponding matrices \( A_0 \) and \( A_1 \) read

\[
A_0 = \begin{pmatrix}
\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}, \quad A_1 = \begin{pmatrix}
0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & \frac{1}{2} \\
0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & \frac{1}{2} & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{2}
\end{pmatrix}.
\]

The matrix \( A_0 \) possesses the eigenvalue \( 1 \) with multiplicity \( 2 \) with corresponding right eigenvectors \( v^{(0)} = (0, 1, 2, 0, 2, 1)^T \) and \( v^{(1)} = (0, -1, -2, 6, -2, -1)^T \). Analogously, \( (1, 2, 0, 2, 1, 0)^T \) and \( (-1, -2, 6, -2, -1, 0)^T \) are right eigenvectors of \( A_1 \) corresponding to \( 1 \). But only the linear combination \( \psi(0) = (0, 1, 2, 3, 2, 1)^T = \frac{3}{2} v^{(0)} + \frac{1}{2} v^{(1)} \) leads to an E-solution of (1.1), namely

\[
\phi(t) = \begin{cases}
t, & 0 \leq t < 3, \\
(6-t), & 3 \leq t \leq 6, \\
0, & \text{otherwise.}
\end{cases}
\]

We observe that \( \psi(1) - \psi(0) = (1, 1, 1, -1, -1, -1)^T \) is a right eigenvector of both \( A_0 \) and \( A_1 \) to be the eigenvalue \( \frac{1}{2} \), such that \( W := \text{span} \{ (1, 1, 1, -1, -1, -1)^T \} \). Hence, \( L \) is given by

\[
L = \text{span} \left\{ (1, 0, 0, 1, 0, 0)^T, (0, 1, 0, 0, 1, 0, 0)^T, (0, 0, 1, 0, 0, 1, 0)^T, (0, 0, 0, 0, 1, -1, 0)^T, (0, 0, 0, 0, 1, -1)^T \right\}.
\]

In particular, the left eigenvectors of \( A_0 \) and \( A_1 \) corresponding to \( 1 \) and \(-1\), are contained in \( L \). The vectors in \( L \) can be extended to vectors in \( \mathcal{L} \), namely,

\[
\mathcal{L} = \text{span} \left\{ (1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 0, \ldots)^T, (0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0, \ldots)^T, (0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0, \ldots)^T, (0, 0, 0, 1, -1, 0, 2, -2, 0, 3, -3, 0, 4, -4, \ldots)^T, (0, 0, 0, 0, 1, -1, 0, 2, -2, 0, 3, -3, 0, 4, -4, \ldots)^T \right\}.
\]

Considering the proof of Theorem 2.4, the characteristic polynomial of the difference equation (2.7) reads
\[
(1 + z + z^2 - z^3 - z^4 - z^5) = (z - 1)(z - e^{2\pi i/3})(z - e^{4\pi i/3})^2.
\]

Hence, \( \mathcal{L} \) is also spanned by the vectors \( e := (1, 1, \ldots)^T, \ (e^{2\pi i j/3})_{j \geq 0}, \ (je^{2\pi i j/3})_{j \geq 0} \), \( (e^{4\pi i j/3})_{j \geq 0} \) and \( (je^{4\pi i j/3})_{j \geq 0} \).

3. Factorization of the mask

As before, let \( \varphi \) be an E-solution of (1.1) under the Assumptions 1.1 and 1.2, and let \( \mathcal{L} \) be defined as in (1.13). Knowing the structure of vectors contained in \( \mathcal{L} \), we shall derive consequences on eigenvectors of the infinite matrix \( A := (c_{ij})_{i,j \geq 0} \) and on zeros of the refinement mask \( P(z) \) defined in (1.3). We shall show that for each E-solution \( \varphi \), the refinement mask necessarily contains a polynomial factor, which is a certain modification of \( (1 + z) \).

Let \( \sqrt{\zeta} \) denote an arbitrarily chosen, but then fixed value of the two square roots of \( \zeta \) (\( \zeta \neq 0 \)).

Theorem 3.1. Let \( (\xi_j)_{j \geq 0} \in \mathcal{L} (\zeta \neq 0) \) and
\[
\alpha = \alpha(\zeta) := \sum_{j=0}^{\infty} \xi_j c_{2j}, \quad \beta = \beta(\zeta) := \sum_{j=0}^{\infty} \xi_j c_{2j-1}.
\]

Then, one of the following 4 cases arises:

(i) \( \alpha = \beta = 0 \); then \( P(\pm \sqrt{\zeta}) = 0 \), and \( (\xi_j)_{j \geq 0} \) is a left eigenvector of \( A \) to the eigengvalue 0.

(ii) \( \beta = \alpha \sqrt{\zeta} (\alpha \neq 0) \); then \( P(-\sqrt{\zeta}) = 0 \), and \( ((\sqrt{\zeta})_j)_{j \geq 0} \in \mathcal{L} \).

(iii) \( \beta = -\alpha \sqrt{\zeta} (\alpha \neq 0) \); then \( P(\sqrt{\zeta}) = 0 \), and \( ((-\sqrt{\zeta})_j)_{j \geq 0} \in \mathcal{L} \).

(iv) \( \beta^2 \neq x^2 \zeta \); then both \( ((\sqrt{\zeta})_j)_{j \geq 0} \) and \( ((-\sqrt{\zeta})_j)_{j \geq 0} \) belong to \( \mathcal{L} \).

Proof. If \( (\xi_j)_{j \geq 0} \in \mathcal{L} \), then
\[
(\xi_j^T)_{j \geq 0} A = \left( \sum_{j=0}^{\infty} \xi_j^T c_{2j-1} \right)^T = (\alpha, \beta, \zeta \alpha, \zeta \beta, \zeta^2 \alpha, \zeta^2 \beta, \ldots)
\]

\[
= \frac{1}{2} \left( \alpha + \frac{\beta}{\sqrt{\zeta}} \right) (\sqrt{\zeta})_j + \frac{1}{2} \left( \alpha - \frac{\beta}{\sqrt{\zeta}} \right) (-\sqrt{\zeta})_j
\]

\[
= P(\sqrt{\zeta}) (\sqrt{\zeta})_j + P(-\sqrt{\zeta}) (-\sqrt{\zeta})_j
\]

also belongs to \( \mathcal{L} \), according to Corollary 2.1.

Case (i): From \( \alpha = \beta = 0 \) it obviously follows that \( (\xi_j^T)_{j \geq 0} \) is a left zero vector of \( A \). Moreover, we find \( \alpha \pm \beta/\sqrt{\zeta} = P(\pm \sqrt{\zeta}) = 0 \).

Case (ii): For \( \beta = \alpha \sqrt{\zeta} (\alpha \neq 0) \), the vector \( (\xi_j^T)_{j \geq 0} A \) is equal to \( (\alpha(\sqrt{\zeta})_j)_{j \geq 0} \) and \( P(\sqrt{\zeta}) \neq 0 \). Hence, \( (\sqrt{\zeta})_j \in \mathcal{L} \), and by \( \alpha - \beta/\sqrt{\zeta} = 0 \) we obtain that \( P(\sqrt{\zeta}) = 0 \). Analogously, we can conclude in the case (iii).
Case (iv): By Theorem 2.3 (ii), the shifted vector
\[(\zeta^{j+1})^T_{j \geq 0} A - (\beta, \zeta \beta, \zeta^2 \beta, \ldots)\]
also belongs to \(\mathcal{L}\). Multiplying the matrix consisting of the two row vectors \((\zeta^j)^T_{j \geq 0} A\) and \((\zeta^{j+1})^T_{j \geq 0} A\) by
\[
\begin{pmatrix}
\alpha & \beta \\
\beta & \zeta \alpha
\end{pmatrix}^{-1} = \frac{1}{\zeta \alpha^2 - \beta^2} \begin{pmatrix}
\zeta \alpha & -\beta \\
-\beta & \alpha
\end{pmatrix},
\]
it follows that the vectors
\[
(1, 0, \zeta, 0, \zeta^2, 0, \ldots)^T, \quad (0, 1, 0, \zeta, 0, \zeta^2, \ldots)^T,
\]
and therefore also both the vectors \(((\pm \sqrt{\zeta})^j)^T_{j \geq 0}\) belong to \(\mathcal{L}\). □

Considering the more general case if \((j^j \zeta^j)^T_{j \geq 0} \in \mathcal{L}\) with \(v > 0\), analogous assertions as in Theorem 3.1 can be derived.

For \(v \in \mathbb{N}_0\), we define
\[
N^{(v)} := \{z \in \mathbb{C}: D^v P(z) = 0, \mu = 0, \ldots, v - 1\} \tag{3.3}
\]
the set of \(v\)-fold zeros of \(P(z)\), where \(P(z)\) is the refinement mask in (1.3), and where \(D\) denotes the usual differential operator \(D := \frac{d}{dz}\).

We state the following generalization of Theorem 3.1.

**Theorem 3.2.** Assume that \((j^j \zeta^j)^T_{j \geq 0} \in \mathcal{L}\) \((v \in \mathbb{N}_0, \zeta \neq 0)\). Then there exist integers \(r, \tilde{r} \) with \(0 \leq r, \tilde{r} \leq v + 1\), such that

1. \((j^\mu (\sqrt{\zeta})^j)^T_{j \geq 0} \in \mathcal{L}\) for \(\mu = 0, \ldots, r - 1\) and \(\sqrt{\zeta} \in N^{(v - r + 1)}\).
2. \((j^\mu (-\sqrt{\zeta})^j)^T_{j \geq 0} \in \mathcal{L}\) for \(\mu = 0, \ldots, \tilde{r} - 1\) and \(-\sqrt{\zeta} \in N^{(v - \tilde{r} + 1)}\).

**Example 3.1.** Let us apply Theorem 3.1 to Example 2.1. For \(\zeta_1 = 1\), we obtain \(\alpha(1) = \beta(1) = 1\). Hence, putting \(\sqrt{\zeta_1} = 1\), it follows by case (ii) that \(P(-1) = 0\).

For \(\zeta_2 = e^{2\pi i/3}\) and \(\sqrt{\zeta_2} = e^{\pi i/3}\), we find \(\alpha(e^{2\pi i/3}) = 1, \beta(e^{2\pi i/3}) = e^{4\pi i/3}\). Thus, we have to apply case (iii) yielding that \(P(e^{2\pi i/3}) = 0\) and that \((-e^{2\pi i/3})^T_{j \geq 0} = (e^{4\pi i/3})^T_{j \geq 0} \in \mathcal{L}\). Finally, from \(\zeta_3 = e^{4\pi i/3}\) with \(\sqrt{\zeta_3} = e^{2\pi i/3}\) it follows that \(\alpha(e^{4\pi i/3}) = 1, \beta(e^{4\pi i/3}) = e^{2\pi i/3}\). Hence, by case (ii), we obtain \(P(e^{5\pi i/3}) = 0\) and \((e^{2\pi i/3})^T_{j \geq 0} \in \mathcal{L}\).

In some papers considering compactly supported solutions of refinement equations (see e.g. [3]), the first sum rule (2.4) is assumed, yielding that \(e : = (1, \ldots, 1)^T\) is a left eigenvector of both \(A_0\) and \(A_1\). In particular, it follows that the corresponding refinement mask \(P(z)\) possesses the factor \((z + 1)\). For integrable solutions of (1.1) with compact support in \([0, n]\) and with linearly independent integer translates, the first sum rule (2.4) is satisfied everytime (see Section 2). Now, we show that a refinement mask \(P(z)\) yielding an
E-solution \( \varphi \), necessarily possesses a factor \( p(z) \) which is a certain modification of \( (z+1) \).

**Theorem 3.3.** Assume that (1.1) with Assumptions 1.1 and 1.2 possesses an E-solution \( \varphi \). Then there is a number \( k \in \mathbb{N}_0 \), such that \( p(z) := p_k(z) \) is a factor of the refinement mask \( P(z) \), where \( p_k(z) \) is obtained iteratively in the following manner:

1. \( p_0(z) := (z + 1)/2; \)
2. \( p_l(z) \) is obtained by replacing \( z \) by \( z^2 \) in \( p_{l-1}(z) \) or in a polynomial factor of \( p_{l-1}(z) \) \( (l = 1, 2, \ldots, k) \).

The resulting factor \( p_k(z) \) of \( P(z) \) is a polynomial satisfying \( p_k(1) = 1 \), and all its zeros are roots of \(-1\) of order \( 2^r \) with \( r \in \{0, \ldots, k\} \).

**Proof.** With the notation \( e := (1, 1, 1, \ldots)^T \), we have that \( e \in \mathcal{L} \) in view of Corollary 2.1 (i). Hence, we can apply Theorem 3.1 to \( \zeta = 1 \). By \( 2P(1) = \alpha + \beta = 2 \) (with \( \alpha, \beta \) defined in (3.1)) we obtain

\[
 e^T A = (\alpha, \beta, \alpha, \beta, \ldots) = e^T + (\alpha - 1)((-1)^j)_{j \geq 0}^T.
\]

Let us first consider the case that \( \alpha = \beta = 1 \); i.e., the case (ii) of Theorem 3.1. Then, \( P(-1) = 0 \), i.e., \( p_0(z) = (z + 1)/2 \) is a factor of the mask \( P(z) \), and the last statement of (ii) (in Theorem 3.1) gives no new relation in view of \( \zeta = 1 \).

For \( \alpha \neq 1 \), we have \( \beta^2 = (2 - \alpha)^2 \neq \alpha^2 \), and case (iv) of Theorem 3.1 is applicable. Hence, \( ((-1)^j)_{j \geq 0} \in \mathcal{L} \). Now the procedure can be repeated with \( \zeta = -1 \).

We compute \( \alpha(-1), \beta(-1) \) and check which of the four cases of Theorem 3.1 can be applied. For \( \alpha(-1) = \beta(-1) = 0 \), it follows that \( P(\pm i) = 0 \), i.e., \( p_1(z) = (1 + z^2)/2 \). If one of the last three cases occurs, then, again (at least one) new element of \( \mathcal{L} \) is found, namely \( (i')_{j \geq 0} \) or \( ((-1)^j)_{j \geq 0} \), or both of them. The procedure must then be applied to the remaining roots \( \zeta = i \) and \( \zeta = -i \), respectively, or to both of them, and so on. Observe, that this algorithm only produces numbers \( \zeta \), which are roots of unity, namely roots of \(-1\) of order \( 2^r \) in the \( r \)th step. Moreover, all elements \( e, ((-1)^j)_{j \geq 0}, \ldots \) of \( \mathcal{L} \), found iteratively by this procedure, are linearly independent. Since the dimension of \( \mathcal{L} \) is finite, we must arrive at case (i) (of Theorem 3.1) after finitely many steps, and the procedure stops. The algorithm is completely described by determining what to do when arriving at one of the cases (i)-(iv), after replacing the factor \( z - \zeta \) of \( p_{l-1}(z) \) by \( z^2 - \zeta = (z - \sqrt{\zeta})(z + \sqrt{\zeta}) \).

**Case (i) means:** \( (z^2 - \zeta) \) is a factor of \( P(z) \) and the procedure stops for this \( \zeta \).

**Case (ii) means:** \( (z + \sqrt{\zeta}) \) is a factor of \( P(z) \) and we continue with the factor \( z - \sqrt{\zeta} \) applying Theorem 3.1 to \( ((\sqrt{\zeta})^j)_{j \geq 0} \).

**Case (iii) means:** \( (z - \sqrt{\zeta}) \) is a factor of \( P(z) \) and we continue with the factor \( z + \sqrt{\zeta} \) applying Theorem 3.1 to \( ((-\sqrt{\zeta})^j)_{j \geq 0} \).
Case (iv) means: We continue with both factors \((z + \sqrt{\zeta})(z - \sqrt{\zeta})\) applying Theorem 3.1 to both \((\sqrt{\zeta})_{j \geq 0}\) and \(((\sqrt{\zeta})_{j \geq 0}\).

Since some factors \(z - \zeta\) can be gathered up, we have exactly the procedure as described in the theorem leading to a factor \(p(z) = p_k(z)\) with the mentioned properties. \(\square\)

Remarks 3.1. (1) If the refinement mask of an E-solution \(\varphi\) does not possess the factor \(p_0(z) = (1 + z)/2\), then in view of the foregoing case (i), it has symmetric zeros on the unit circle. Hence, the condition that \(P(z)\) has no symmetric zeros on the unit circle, is necessary for linear independence as well as for Riesz stability of integer translates of \(\varphi\) (see e.g. [7], Theorem 3.3).

(2) Characterising the roots \(e^{\pi p/q}\) of \(-1\) by \(p/q\), we can interpret the result of Theorem 3.3 by means of the tree graph in Fig. 1, in order to get the zeros of \(p(z)\). Geometrically, the endpoints of a certain finite subtree of the graph with invariant root 1 form the set \(R\) of zeros of a possible factor \(p(z) = p_k(z)\) in the theorem.

(3) Using the just mentioned characterization of zeros, the simplest zero set corresponding to \(p_0(z)\) is \(R = \{1\}\). Other examples for zero sets are \(R = \{1/2, 3/2\}\) corresponding to \(p_1(z) = (z^2 + 1)/2\) and \(R = \{1/8, 9/8, 5/4, 3/2\}\) corresponding to \(p_3(z) = C(z^2 + e^{5\pi/4})(z - e^{5\pi/4})(z + i)\) with a certain constant \(C\).

The factor \(p(z)(= p_k(z))\) found in Theorem 3.3 can also be characterized as follows:

Theorem 3.4. The polynomial \(p(z)\) can be found by the iteration process in Theorem 3.3 if and only if it is of the form

\[
\begin{align*}
&1 \\
&\downarrow \\
&\frac{1}{2} \\
&\downarrow \\
&\frac{1}{4} \quad \frac{5}{4} \\
&\downarrow \quad \downarrow \\
&\frac{1}{8} \quad \frac{9}{8} \quad \frac{5}{8} \quad \frac{13}{8} \\
&\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
&\frac{3}{8} \quad \frac{11}{8} \quad \frac{7}{8} \quad \frac{15}{8}
\end{align*}
\]

Fig. 1. The directed graph of successive square roots, beginning from \(-1 = e^{\pi i}\) where \(p/q\) stands for \(e^{\pi p/q}\).
\[ p(z) = \frac{q(z^2)}{2q(z)}, \quad (3.4) \]

where \( q(z) \) is a polynomial of the same degree and possessing a set \( Q \) of zeros with the following property: \( Q \) contains roots of unity with powers of 2 as root exponent, and it is closed regarding to the operation \( z \to z^2 \) (i.e., for \( z \in Q \) it follows that \( z^2 \in Q \)). Moreover, denoting the zero set of \( p(z) \) by \( R \) and introducing the set \( \hat{Q} \) of all square roots of elements of \( Q \),

\[ Q := \{ z : z^2 \in Q \}, \]

we have the relations:

\[ R = \hat{Q} \setminus Q, \quad Q = \{ z^2 : j \in \mathbb{N}, z \in R \}. \]

**Proof.** For \( m = 1 \), the representation is valid with \( q(z) = z - 1 \). For \( m > 1 \), the construction of \( p(z) \) can be described in the following way:

\[
\begin{align*}
2^k + 1 &= q_1(z)r_1(z) \to q_1(z)r_1(z^2) = q_2(z)r_2(z) \to \ldots \\
&= q_k(z)r_k(z) \to q_k(z)r_k(z^2) = 2p(z)
\end{align*}
\]

with polynomials \( q_i(z), r_i(z) \) \((i = 1, \ldots, k)\) and \( q_i(z)r_i(z^2) = q_{i+1}(z)r_{i+1}(z) \) \((i = 1, \ldots, k - 1)\). We find recursively

\[
2p(z) = (z^2 + 1)\prod_{i=1}^{k} r_i(z),
\]

so that (3.4) is obtained with \( q(z) = (z^2 - 1)\prod_{i=1}^{k} r_i(z) \). Denoting the set of zeros of \( p(z) \) by \( R := \{ \xi_1, \ldots, \xi_m \} \), and the zero set of \( q(z) \) by \( Q := \{ \sigma_1, \ldots, \sigma_m \} \), Eq. (3.4) shows that

\[
\{ \xi_v, \sigma_v : v = 1, \ldots, m \} = \{ \sqrt{\sigma_v}, -\sqrt{\sigma_v} : v = 1, \ldots, m \},
\]

i.e., \( R \cup Q = \hat{Q} \) with \( \hat{Q} := \{ \sqrt{\sigma_v}, -\sqrt{\sigma_v} : v = 1, \ldots, m \} \). Thus, for every \( v \), there exist numbers \( a_v, b_v \in \{ 1, \ldots, m \} \) with \( \xi_v = \sigma_{a_v}, \sigma_v = \sigma_{b_v} \). Hence, the set \( \{ \sigma_1, \ldots, \sigma_m \} \) is the closure of the set \( \{ \xi_1^2, \ldots, \xi_m^2 \} \) by the operation \( z \to z^2 \), i.e., \( Q = \{ z^2 : j \in \mathbb{N}, z \in R \} \). Since all \( \xi_v \) are roots of unity with a power of 2 as root exponent, the \( \sigma_v \) have the same property. Finally, since the \( 2m \) elements of \( \hat{Q} \) are different, it follows that \( R = \hat{Q} \setminus Q \).

The second direction of the proof follows conversely. \( \square \)

**Remark 3.2.** (1) In [4], Lemma 6.6, it was already shown that a finite set \( R \) of complex numbers with the property \( R \subseteq R^2 := \{ z^2 : z \in R \} \) and cardinality \( n \) can only consist of elements which are roots of unity or zero. Further, for any element \( z \in R \) there are integers \( k, l \) with \( 1 \leq l < k \leq n \), such that \( z^{2k} = z^{2l} \).

(2) In the special case \( 2p(z) = (z^{2^n} + 1) \), (3.4) is satisfied with \( q(z) = z^{2^n} - 1 \). Of course, \( P(z) \) can have several factors of the type \( p(z) \), such that the zeros \( \xi_v \) can also appear with a higher multiplicity.
(3) If we assume that \( p(z) \) has real coefficients only, then \( p(z) \) can be decomposed into factors of the form \( z^2 + a_k z + 1 \). The coefficients \( a_k \) can be found iteratively from \( a_0 := 0, a_1 := \sqrt{2 - a_{1-1}}, \) since

\[
(z^2 + \sqrt{2 - a_k z} + 1)(z^2 - \sqrt{2 - a_k z} + 1) = z^4 + a_k z^3 + 1.
\]

The factor \( p_k(z) \) of \( P(z) \), found iteratively from \((z + 1)/2\) as described in Theorem 3.3, can be considered as the refinement mask of a special compactly supported step function. This is the exceptional case, where Assumption 1.2 is not satisfied.

Let

\[
q(t) = \sum_{i=0}^{k} b_i \chi(t - v) \quad (t \in \mathbb{R})
\]

with \( b_0 b_k \neq 0 \) \((k \geq 1)\), and where \( \chi(t) \) is the characteristic function of the interval \([0, 1)\), i.e.,

\[
\chi(t) = \begin{cases} 
1, & t \in [0, 1), \\
0, & t \notin [0, 1).
\end{cases}
\]

Then according to Lawton et al. [17] we have the following proposition.

**Proposition 3.1.** The function \( \phi \) of the form (3.5) is refinable, if and only if the corresponding refinement mask \( P(z) \) is of the form \( 2P(z) = Q(z^2)/Q(z) \), where \( Q(z) := (z - 1)(\sum_{i=0}^{k} b_i z^i) \) with the coefficients \( b_v \) in (3.5), and where the zero set \( \{z \in \mathbb{C} : Q(z) = 0\} \) is closed regarding the operation \( z \rightarrow z^2 \).

Observe that the zeros of \( Q(z) \) in Proposition 3.1 can be arbitrary roots of unity. The factorization \( P(z) = p(z)Q(z) \) with \( p(z) \) in (3.4), given by Theorem 3.4, allows to simplify the E-solution of (1.1). In order to show this, we need the following theorem.

**Theorem 3.5.** Let \( P(z) \) and \( \hat{P}(z) \) be polynomials of the form

\[
P(z) = P_1(z)P_2(z^2), \quad \hat{P}(z) = P_1(z)P_2(z),
\]

where \( P_2(z) = \sum_{i=0}^{k} r_i z^i \) with \( r_0 \neq 0 \) and with \( P(1) = \hat{P}(1) = 1 \). Then we have: Eq. (1.1) with the refinement mask \( P(z) \) possesses an E-solution \( \phi \) if and only if (1.1) with \( \hat{P}(z) \) possesses an E-solution \( \tilde{\phi} \), and we have, for \( t \in \mathbb{R} \),

\[
\phi(t) = \sum_{i=0}^{k} r_i \tilde{\phi}(t - v). \quad (3.6)
\]
Proof. Since $P(1) = \tilde{P}(1) = 1$, there exist unique compactly supported distributions $\varphi, \tilde{\varphi}$ satisfying (1.1) with the refinement masks $P(z), \tilde{P}(z)$, respectively (see e.g. [11], Theorem 1.1, [2,18]). The Fourier transforms of $\varphi, \tilde{\varphi}$ are given by

$$
\hat{\varphi}(u) = \prod_{j=1}^{\infty} P_1(e^{-iu/2^j}) P_2(e^{-2iu/2^j}) = P_2(e^{-iu}) \prod_{j=1}^{\infty} P_1(e^{-iu/2^j}) P_2(e^{-iu/2^j}),
$$

$$
\hat{\tilde{\varphi}}(u) = \prod_{j=1}^{\infty} P_1(e^{-iu/2^j}) P_2(e^{-iu/2^j}),
$$

where the infinite products converge uniformly on every compact subset of $\mathbb{C}$ (cf. [11]). Hence,

$$
\hat{\varphi}(u) = P_2(e^{-iu}) \hat{\tilde{\varphi}}(u).
$$

Inverse Fourier transform yields that

$$
\varphi(t) = \sum_{v=0}^{k} r_v \tilde{\varphi}(t - v).
$$

This equation shows that if $\tilde{\varphi}$ is an E-solution, then so is $\varphi$; vice versa, if $\varphi$ is an E-solution, then, by $r_0 \neq 0$, the function $\tilde{\varphi}$ can recursively be constructed by means of (3.6) as a nonvanishing locally Lebesgue-integrable function. Since $\tilde{\varphi}$ is compactly supported, $\varphi$ is an E-solution. □

We easily conclude that, if $\tilde{\varphi}(t)$ is continuous, then also $\varphi(t)$ is continuous. Observe, that $P_1(z)$ in Theorem 3.5 is not necessarily a polynomial as in the example $P(z) = \frac{1}{2} (1 + z^4), \tilde{P}(z) = \frac{1}{2} (1 + z), P_1(z) = (1 + z^2)^{-1}.$

By Theorem 3.4, each E-solution $\varphi$ of (1.1) has a refinement mask of the form $P(z) = p(z)Q(z)$ with $2p(z) = q(z^2)/q(z)$. The polynomial $q(z)$ possesses the simple zero 1, i.e., $q(z) = (z - 1)r(z)$ with $r(1) \neq 0$ (see the proof of Theorem 3.4). We can apply Theorem 3.5 as in the following proposition.

**Proposition 3.2.** Let $\varphi$ be an E-solution of (1.1) and let Assumptions 1.1 and 1.2 be satisfied. Assume that the corresponding refinement mask $P(z)$ has the representation $P(z) = p(z)Q(z)$. Further, let $p(z)$ be a polynomial factor of the form

$$
p(z) = \frac{(1 + z)r'(z^2)}{2r(z)}
$$

with $p(1) = 1$, and with $r(z) := \sum_{v=0}^{k} r_v z^v$ and $r_0 \neq 0$. Then (1.1) corresponding to the refinement mask $P(z) := ((z + 1)/2)Q(z)$ possesses an E-solution $\tilde{\varphi}$ and we have
Proof. We apply Theorem 3.5 as follows: Putting
\[ P_1(z) = \frac{(z + 1)Q(z)}{2r(z)}, \quad P_2(z) = r(z), \]
we have \( P(z) = P_1(z)P_2(z^2) \). Hence, the refinement mask \( \tilde{P}(z) = P_1(z)P_2(z) \) also provides an E-solution \( \tilde{\phi} \) of (1.1), and the assertion follows. \( \Box \)

Remark 3.3. (1) By Proposition 3.2, each E-solution \( \phi \) of (1.1) can be represented as a finite linear combination of integer translates of an E-solution \( \bar{\phi} \) with a refinement mask \( \bar{P}(z) \) containing the factor \( p_0(z) = (z + 1)/2 \). This argument can even be pushed a little further, showing that each E-solution can be given as a finite linear combination of integer translates of a refinable function \( \bar{\phi} \) with linearly independent integer shifts (see [6], Theorem 5.3). (2) If \( P(z) = P_1(z)P_2(z^2) \) then it can also be represented as
\[ P(z) = \frac{P_1(z)}{P_2(z^2)}P_2(z^2)P_2(z^4) \]
and we can apply Theorem 3.5 with \( P_2(z)P_2(z^2) \) instead of \( P_2(z) \). Analogously, \( P(z) = P_1(z)P_2(z^2) \) can be reduced to the original case replacing \( P_2(z) \) by \( P_2(z)P_2(z^2) \ldots P_2(z^{2^{n-1}}) \).

Example 3.2 (to Theorem 3.5). Let \( P(z) = \frac{1}{2}(z^2 + \sqrt{2}z + 1)(z^4 - \sqrt{2}z^2 + 1) \). A corresponding solution \( \phi \) of (1.1) is
\[
\phi(t) = \begin{cases} 
1, & t \in [0, 1) \cup [5, 6); \\
1 - \sqrt{2}, & t \in [1, 2) \cup [4, 5); \\
2 - \sqrt{2}, & t \in [2, 4); \\
0, & \text{otherwise}.
\end{cases}
\]
Considering the simplified mask
\[ \tilde{P}(z) = \frac{1}{2}(z^2 + \sqrt{2}z + 1)(z^2 - \sqrt{2}z + 1) = \frac{1}{2}(z^4 + 1) \]
providing the solution \( \tilde{\phi} = \chi_{[0,4]} \) (where \( \chi \) denotes the characteristic function), it follows that
\[ \phi(t) = \tilde{\phi}(t) - \sqrt{2}\tilde{\phi}(t - 1) + \tilde{\phi}(t - 2). \]
Analogously as in the foregoing theorem, we have the following theorem.
Theorem 3.6. Let (1.1) possess an E-solution \( \varphi(t) \) and assume that the corresponding refinement mask \( P(z) \) factorizes
\[
P(z) = (z^l + 1)Q(z) \quad (l \in \mathbb{N}).
\]
Then (1.1) with the refinement mask \( (z^{ml} + 1)Q(z) \) \((m \in \mathbb{N})\) also provides an E-solution \( \Phi(t) \), namely
\[
\Phi(t) = \sum_{\nu=0}^{m-1} \varphi(t - \nu t).
\]

The case \( m = 2^k, \nu = 1 \) can be treated with both Theorems 3.5 and 3.6 in view of
\[
(1 + z)(1 + z^2) \cdots (1 + z^{2^{k-1}}) = \sum_{\nu=0}^{2^k-1} z^\nu.
\]

4. Eigenvectors of the coefficient matrix

In the considerations above, the structure of the spaces \( L \) and \( \mathcal{L} \) has been used for deriving assertions on zeros of the refinement mask corresponding to E-solutions. As seen in Section 2, eigenvectors and root vectors of \( A_0, A_1 \) and \( A \) can be special elements of \( L \) and \( \mathcal{L} \), respectively. Now, we want to investigate conversely, if given zeros of the refinement mask imply consequences for the eigenvalues and eigenvectors of the matrices \( A_0, A_1 \) and \( A \).

First, we show that there are no (proper) root vectors of \( A_0 \) and \( A_1 \) corresponding to the eigenvalue 1 in case of continuous E-solutions.

Theorem 4.1. Let \( \varphi \) be a continuous E-solution of (1.1). Then the Jordan block in the Jordan decomposition of \( A_0 \) (or \( A_1 \)) belonging to the eigenvalue 1 is the identity matrix, i.e., there are no root vectors of \( A_0 \) (or \( A_1 \)) belonging to the eigenvalue 1.

Proof. We only show the assertion for \( A_0 \).

(1) Observe that by (1.7), \( A_0 \) possesses the eigenvalue 1. Let us assume that there are vectors \( w := (w_j)_{j=0}^{n-1} \) and \( \tilde{w} := (\tilde{w}_j)_{j=0}^{n-1} \) with
\[
w^T A_0 = w^T, \quad w^T A_0 = \tilde{w}^T + w^T,
\]
so that \( w \) is an eigenvector and \( \tilde{w} \) is a root vector of height 2 to the eigenvalue 1. We show that this assumption leads to a contradiction. By Theorem 2.2 (iv), \( w \) and \( \tilde{w} \) are contained in \( L \). Further, both the vectors can be extended to \( w = (w_j)_{j=0}^\infty \in \mathcal{L}, \quad \tilde{w} = (\tilde{w}_j)_{j=0}^\infty \in \mathcal{L} \) by Theorem 2.3 (i) and Corollary 2.1 (iv). As we have shown in Theorem 2.4, \( w \) and \( \tilde{w} \) can be represented in the form...
and also all \((j^i\xi_k^i)_{j \geq 0}\), appearing in (4.2) are contained in \(\mathcal{P}\). We can suppose that, for each \(k\), \(d_k(j)\) and \(\tilde{d}_k(j)\) do not both vanish identically.

(2) First, let us assume that all \(d_k, \tilde{d}_k\) in (4.2) are independent of \(j\). According to (3.2), we have

\[
\left(\varepsilon_k^j\right)_{j \geq 0}^T A = \sqrt{\epsilon_k^j} \left(\sqrt{\xi_k^j}\right)_{j \geq 0}^T + \delta_k \left(-\sqrt{\xi_k^j}\right)_{j \geq 0}^T
\]

with

\[
\gamma_k := P(\sqrt{\epsilon_k^j}) = \frac{1}{2} \sum_{j=0}^{n} c_j(\sqrt{\xi_k^j})^j, \quad \delta_k := P(-\sqrt{\epsilon_k^j}) = \frac{1}{2} \sum_{j=0}^{n} c_j(-\sqrt{\xi_k^j})^j.
\]

Now, (4.1)–(4.3) yield for \(j \in \mathbb{N}_0\)

\[
\left(\sum_{k=1}^{l} d_k(\gamma_k^j(\sqrt{\xi_k^j})^j + \delta_k(-\sqrt{\xi_k^j})^j)\right) = \sum_{k=1}^{l} d_k \varepsilon_k^j, \quad (4.5)
\]

\[
\left(\sum_{k=1}^{l} \tilde{d}_k(\gamma_k^j(\sqrt{\xi_k^j})^j + \delta_k(-\sqrt{\xi_k^j})^j)\right) = \sum_{k=1}^{l} (d_k + \tilde{d}_k) \varepsilon_k^j. \quad (4.6)
\]

Since \((\sqrt{\epsilon_k^j})_{j \geq 0}\) and \((-\sqrt{\epsilon_k^j})_{j \geq 0}\) \((k = 1, \ldots, l)\) on the left-hand side and \((\xi_k^j)_{j \geq 0}\) \((k = 1, \ldots, l)\) on the right-hand side are linearly independent vectors, it follows that some elements of the set \(\{\gamma_k, \delta_k : k = 1, \ldots, l\}\) must vanish. On the other side, for each \(k \in \{1, \ldots, l\}\), there exist numbers \(p_k \in \{0, 1\}\), \(q_k \in \{1, \ldots, l\}\) such that

\[
\zeta_k = (-1)^{p_k} \sqrt{\epsilon_{q_k}}, \quad (4.7)
\]

where the corresponding coefficients

\[
\epsilon_k := \begin{cases} 
\gamma_{q_k} & \text{for } p_k = 0, \\
\delta_{q_k} & \text{for } p_k = 1
\end{cases}
\]

do not vanish. Comparing the coefficients in (4.5) and (4.6), we find \(d_k = \epsilon_k d_{q_k}\) and \(d_k + \tilde{d}_k = \epsilon_k d_{q_k}\). Iterating the first equation, we must arrive at a cycle, say with \(m\) steps \((1 \leq m < l)\). Then, after changing the notation of the indices, we find (disregarding the preperiod)

\[
d_1 = \epsilon_1 d_2, \quad d_2 = \epsilon_2 d_3, \quad \ldots, d_m = \epsilon_m d_1.
\]
which implies either $d_1 = \cdots = d_m = 0$ or $\epsilon_1 \epsilon_2 \cdots \epsilon_m = 1$. In the last case, from

$$d_1 + \tilde{d}_1 = \epsilon_1 \tilde{d}_2, \quad d_2 + \tilde{d}_2 = \epsilon_2 \tilde{d}_3, \quad \ldots, d_m + \tilde{d}_m = \epsilon_m \tilde{d}_1.$$ 

we obtain by elimination

$$\tilde{d}_1 = \epsilon_1 (d_2 - d_3), \quad \tilde{d}_2 = \epsilon_2 (d_3 - 2d_3), \quad \ldots,$$

$$\tilde{d}_n - (m - 1) d_m = \epsilon_m (d_1 - md_1)$$

and so

$$\tilde{d}_1 = \epsilon_1 \epsilon_2 \cdots \epsilon_m (d_1 - md_1) = d_1 - md_1.$$ 

It follows as before that $d_1 = 0$ and hence, by $\epsilon_i \neq 0$, that $d_2 = d_3 = \cdots = d_m = 0$. If $m < l$, we can conclude in any case that also the other $d_n$ must vanish. But this contradicts the fact that the eigenvector $w$ is non-vanishing.

(3) Next, we consider the case that $d_k(j)$ and $\hat{d}_k(j)$ are linear in $j$, i.e.,

$$d_k(j) = a_k + jb_k, \quad \hat{d}_k(j) = \tilde{a}_k + j\tilde{b}_k$$

with $a_k, b_k, \tilde{a}_k, \tilde{b}_k$ independent of $j$. Eq. (4.3) implies by differentiation with respect to $\zeta_k$ (which, in this connection, can be considered as a variable) and multiplication with $j_0$,

$$(j_0 \zeta_k j)^T A = \left( (j_0 \zeta_k j + \frac{j}{2} \gamma_k \sqrt{j_0 \zeta_k} j + \frac{j}{2} \delta_k \sqrt{j_0 \zeta_k} j) \right)^T$$

So (4.1)–(4.3) yield for $j \in \mathbb{N}_0$,

$$\sum_{k=1}^l a_k \left( j_0 \sqrt{j_0 \zeta_k} j + \delta_k \sqrt{-j_0 \zeta_k} j \right)$$

$$+ b_k \left( (j_0 \sqrt{j_0 \zeta_k} j + \frac{j}{2} \gamma_k \sqrt{j_0 \zeta_k} j + \frac{j}{2} \delta_k \sqrt{j_0 \zeta_k} j \right) = \sum_{k=1}^l (a_k + jb_k) j_0 \zeta_k$$

and an analogous equation with $\tilde{a}_k, \tilde{b}_k$ instead of $a_k, b_k$ on the left-hand side and

$$\sum_{k=1}^l (a_k + jb_k + \tilde{a}_k + j\tilde{b}_k) j_0 \zeta_k$$

on the right-hand side. With the same arguments as before, we can derive that $\zeta_k$ is of the form (4.7) and a comparison of the coefficients of $j_0 \zeta_k$ yields

$$b_k = \frac{1}{2} \epsilon_k b_{\eta_k}, \quad b_k + \tilde{b}_k = \frac{1}{2} \epsilon_k \tilde{b}_{\eta_k}$$

with $\epsilon_k$ as in (4.8). Now, we can conclude as before (with $(\epsilon_k/2)$ instead of $\epsilon_k$) that all $b_k$ vanish. Moreover, we find $\tilde{b}_k = \epsilon_k \tilde{b}_k$ and $a_k = \epsilon a_k$ with $\epsilon = \epsilon_1 \cdots \epsilon_m$, that either $a_k = 0$ or $\tilde{b}_k = 0$. In the last case, $a_k + \tilde{a}_k = \epsilon_k \tilde{a}_k$ implies as before $a_k = 0$. Hence, in any case, we have a contradiction to $w \neq 0$.

(4) For polynomials $d_k(j), \hat{d}_k(j)$ of higher degree the contradiction follows in the same manner. □
With the help of the foregoing theorem we simply observe the following corollary.

**Corollary 4.1.** If the matrix $A_0$ (or the matrix $A_1$) has nonsimple Jordan blocks corresponding to the eigenvalue 1, then (1.1) does not possess a continuous $E$-solution $\varphi$.

This case really can happen as is shown by Example 4.1.

**Example 4.1.** We consider (1.1) with the coefficients $c_0 = -c_4 = a$, $c_1 = c_3 = 1$, $c_2 = 0$, with $a \in \mathbb{R}$, $|a| < 1$, $a \neq 0$. We simply observe that the Assumptions 1.1 and 1.2 are satisfied. However, (1.1) does not possess a continuous $E$-solution, since the corresponding matrix $A_0$ contains a nonsimple Jordan block corresponding to 1,

\[
A_0 = \begin{pmatrix}
0 & 1 & 0 & 1 \\
-a & 0 & 1 & 0 \\
a & 0 & -a & 1 \\
0 & -a & 0 & 0
\end{pmatrix}
\]

Next, assuming that the refinement mask $P(z)$ possesses a factor $p_k(z)$ (as described in Theorem 3.3), we are interested in consequences for eigenvectors and root vectors of the coefficient matrix $A$.

**Theorem 4.2.** Let (1.1) have an $E$-solution with Assumptions 1.1 and 1.2. Then there exists an integer $k > 0$ such that $(e^T A^k)^T$ is a left eigenvector of $A$ corresponding to the eigenvalue 1, and for $k \geq 1$, the vectors $(e^T A^{v+1} - e^T A^v)^T$ ($v = 0, \ldots, k - 1$) are left root vectors of height $k - v$ of $A$ corresponding to the eigenvalue 0. Here again, $e := (1, 1, \ldots)^T$.

**Proof.** By Corollary 2.1, we have $(e^T A - e^T)^T \in \mathcal{L}$. Let $P(z)$ possess the factor $p_k(z)$ found after $k$ iterations as described in Theorem 3.3. Recalling the proof of Theorem 3.3, we observe that, by iterative application of (4.3), $(e^T A - e^T)A^k = 0^T$, for $k \geq 0$, where $0 := (0, 0, \ldots)^T$, whereas $(e^T A - e^T)A^{k-1} \neq 0^T$ for $k \geq 1$. Hence, $(e^T A^k)^T$ is a left eigenvector of $A$ corresponding to the eigenvalue 1, and by $(e^T A^{v+1} - e^T A^v)A^{k-v} = 0^T$, we obtain the assertion concerning the root vectors of $A$ belonging to 0. □

**Remark 4.1.** (1) Obviously, we have the identity
between the eigenvector $e^T A^k$ of $A$ corresponding to $1$, $e$, and the root vectors of $A$ corresponding to $0$. It can easily be seen (by repeated application of (4.3)) that $e^T A^k$ is a $2^k$-periodic vector. Writing $e^{(v)}^T = e^T A^{v+1} - e^T A^v$ and $e^{(v)} = (e^{(v)}_j)_{j=0}^{n-1}$ for $v = 0, \ldots, k-1$, it follows that $(e^{(v)}_j)_{j=0}^{n-1}, (e^{(v)}_{j+2^k - 1})_{j=0}^{n-1}$ also belong to $L$. Moreover, they must be root vectors of height $k - v$ of $A_0$ corresponding to $0$.

(2) In the case $2P(z) = z^{2^k} + 1$, we observe that

$$e^{(v)} = (2^{v_0}, 0, \ldots, 0, 2^{v_1}, 0, \ldots, 0, \ldots, 0, 2^{v_{k-1}}, 0, 2^{v_k}, 0, \ldots)$$

In particular, the $e^{(v)}$ ($v = 0, \ldots, k-1$) are linearly independent.

Now, let $w \in \mathcal{L}$, i.e., the elements of $w$ possess the representation (2.5). Then we have:

**Proposition 4.1.** If $w \in \mathcal{L}$ with the representation (2.5) is an eigenvector (or root vector) of $A$ corresponding to an eigenvalue $\lambda \neq 0$. Then the $\zeta_k$ occurring in the representation (2.5) of $w$ are roots of unity and the set $\{\zeta_k: k = 1, \ldots, l\}$ is closed regarding to the operation $z \mapsto z^2$.

**Proof.** Let $w^T A = \lambda w^T$ with $\lambda \neq 0$. Assume first that the $d_k$ in (2.5) are independent of $j$. Then we obtain, analogously as in the proof of Theorem 4.1,

$$\sum_{k=1}^{l} d_k (\sqrt{\zeta})^j + \delta_k (-\sqrt{\zeta})^j = \lambda \sum_{k=1}^{l} d_k \zeta_k^j \quad (j \geq 0)$$

with $\gamma_k$ and $\delta_k$ as in (4.4). Hence, with the same argument as in the proof of Theorem 4.1, the assertion follows. Similar ideas apply for polynomials $d_k(j)$ and for root vectors. □

**Remark 4.2.** (1) In the case $\lambda = 0$, not only roots of unity but arbitrary $\zeta_k$ can appear.

(2) Assertions on eigenvectors and root vectors of $A$ to the eigenvalue $0$ can be concluded by proving the converse of Theorems 3.1 and 3.2. If both, $\sqrt{\zeta}$ and $-\sqrt{\zeta}$, are zeros of the refinement mask $P(z)$, then $(\zeta_j^j)_{j=0}^{n-1}$ is a left eigenvector of $A$ corresponding to $0$, since $\sum_{j=0}^{n} c_j (\sqrt{\zeta})^j = 0$ and $\sum_{j=0}^{n} c_j (-\sqrt{\zeta})^j = 0$ imply that by (4.3),

$$(\zeta_j^j)_{j=0}^{n} A = P(\sqrt{\zeta}) \left((\sqrt{\zeta})^j\right)_{j=0}^{n} + P(-\sqrt{\zeta}) \left((-\sqrt{\zeta})^j\right)_{j=0}^{n} = 0^T.$$
Analogously, if \( \pm \sqrt{\xi} \in N^{(v)} \) (with \( N^{(v)} \) defined in (3.3)), then \( (j_n^{\mu})_{\mu \geq 0} \) \((\mu = 0, \ldots, v - 1)\) are eigenvectors of \( A \) corresponding to the eigenvector 0.

References