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Two segment classes with Hamiltonian visibility graphs

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Abstract

We prove that the endpoint visibility graph of a set of disjoint segments that satisfy one of two restrictions, always contains a simple Hamiltonian circuit. The first restriction defines the class of *independent* segments: the line containing each segment misses all the other segments. The second restriction specifies *unit lattice* segments: unit length segments whose endpoints have integer coordinates.

1. Introduction

It has been conjectured [2] that the visibility graph for a set of non-collinear disjoint line segments always contains a simple Hamiltonian circuit.[2] Mirzaian first proved this for what we call *hulled* segments: segments each of which touches the convex hull of the segments [2]. Later we found an alternative proof of this result [4].

In this paper we prove the conjecture for two more classes of segments, which we call "independent segments" and "unit lattice segments."[3] A set of segments is called *independent* if for each segment s in the set, the line containing s does not meet any other segment in the set. The proof for this class is not difficult. A set of *unit lattice* segments are disjoint segments with endpoints on the integer lattice, and each of unit

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[2]This conjecture has been formulated by several researchers independently of Mirzaian [2]: Toussaint [7], and (later) in [4]. Circuits through line segments were first studied in 1985; see [6].
[3]These results were first described in [5].

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length (so all segments are vertical or horizontal). Our proof for this case is more involved, but still elementary in the tools employed.

We now define the visibility graph more precisely. The *endpoint visibility graph* (or just *visibility graph*) G of a set S of closed, disjoint line segments has a node for each segment endpoint, and an arc between two nodes x and y if $[x, y] \cap S = \{x, y\}$ or $[x, y]$: the intersection is either just the two endpoints, or the entire closed segment. We say that the two endpoints x and y are *visible* to each other, or that they *see* each other. Note that visibility is blocked by even grazing contact with a segment, but that G contains an arc corresponding to each segment in S.

A simple Hamiltonian cycle is a Hamiltonian cycle embedded in the plane that does not touch itself: it corresponds to a simple polygon. Under our definition of visibility, the graph for a set of collinear segments does not contain a Hamiltonian cycle, so we will exclude this case when appropriate.[4]

2. Independent Segments

We first prove Hamiltonicity for sets of independent segments. An example set is shown in Fig. 1.

Theorem 2.1. *For any set S of $n > 1$ independent segments, there exists a circumscribing Hamiltonian cycle C in the visibility graph of S such that every segment on the convex hull of S is included in C.*

Proof. First, we show the base case, where $n = 2$. Since the two segments are independent, both segments must lie in their convex hull. Therefore there exists a Hamiltonian cycle that follows the convex hull, includes both segments, and is circumscribing.

We assume that our theorem is true for up to $n-1$ segments. Now suppose that S contains n independent segments. For $s \in S$, let $L_s = \{s\} \cup \{s' \in S \mid s' \text{ is left of } s\}$ and $R_s = \{s\} \cup \{s' \in S \mid s' \text{ is right of } s\}$. Choose an $s \in S$ such that neither $L_s = \{s\}$ nor $R_s = \{s\}$. If no such $s \in S$ exists, then for all $s \in S$, s is on the convex hull of S. In this case we can find a Hamiltonian cycle C such that C follows the convex hull of S, and therefore includes every segment in S and is circumscribing.

Otherwise, suppose $L = L_s$ contains k segments. Then $1 < k < n$. $R = R_s$ contains $n-k+1$ segments, and $1 < n-k+1 < n$. So there exists a circumscribing Hamiltonian cycle C_L of the set L such that every segment on the convex hull of L is included in C_L. Similarly there exists a circumscribing Hamiltonian cycle C_R of the set R such that every segment on the convex hull of R is included in C_R. See Fig. 1. Now remove s from both C_L and C_R and then glue C_L and C_R together. Since s had at least one

[4] Collinear segments are hulled; unit lattice segments might be collinear.

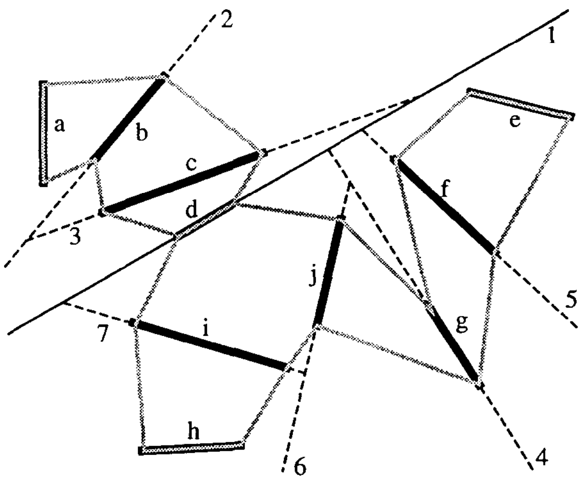

Fig. 1. Independent segments, with paths shown before the final merge. The first partition is the line containing d, the second through b, etc. After the first partition with $s=d$, $L_s=\{a,b,c,d\}$, and $R_s=\{d,e,f,g,h,i,j\}$.

segment to its right and at least one segment to its left, s is not on the convex hull of S. So we have a circumscribing Hamiltonian cycle C of S such that every segment on the convex hull of S is included in C. □

Fig. 1 shows a set of independent segments, along with the Hamiltonian cycle implied by viewing the above induction proof as a recursive algorithm. It should be clear from the proof that we do not need the line containing *every* segment of S to partition the set: we only need at each stage at least one segment with this property. One could base an alternative definition of independence on this observation and still prove our result, at the cost of defining a somewhat unnatural class of segments.

2.1. Algorithm

There are two algorithmic issues: determining if a set of segments is independent, and running the recursive algorithm implied by the proof. The recursion leads to an $O(n^2)$ algorithm if applied naively, as there would be no guarantee that the dividing segment chosen splits the sets into balanced halves. This algorithm can be improved by using half-plane range searching algorithms to split more intelligently. For example, we can achieve $O(n^{3/2}\log n)$ by using a result of Matoušek and Welzel [3]. They show how to preprocess points in $O(n^{3/2}\log n)$ time so that queries asking for the number of points above a line can be answered in $O(\sqrt{n}\log n)$ time. With n such queries, we could find a segment whose line bisects the set of segments. So we obtain the recurrence $T(n)=2T(n/2)+O(n^{3/2}\log n)$, whose solution is $T(n)=O(n^{3/2}\log n)$.

Perhaps more interesting is checking for independence. We can easily perform this check in $O(n^2)$ time, by checking for each segment whether its containing line meets any other segment. This algorithm can be improved by using techniques for processing the segments for ray shooting. For example, a result of Agarwal [1, Theorem 6.11, p. 224] results in an $O(n^{3/2} \text{ polylog } n)$ independence testing algorithm: preprocess the segments in $O(n^{3/2} \log^{\omega} n)$ time ($\omega < 4.33$), using $O(n^{3/2})$ storage, and then shoot a ray along each segment forward and backwards, at a total query cost of $O(n^{3/2} \log^2 n)$ time.

We leave it open whether these algorithms may be improved to $o(n^{3/2})$.

3. Unit lattice segments

Let S be a set of n unit lattice segments. We say that a column of the integer lattice is nonempty if at least one endpoint of a segment $s \in S$ lies in the column, and we can number the nonempty columns $1, 2, \ldots, m$ from left to right, where m is the number of nonempty columns. Additionally, we say that a_i is the ith segment endpoint from the bottom in column a.

Unit lattice segments are "almost" independent, and it is likely that a recursive algorithm is possible, similar in spirit to that just presented for independent segments. However, there are a number of complications not present with independent segments, and we have chosen a more direct construction.

Our proof proceeds in three stages, each removing assumptions from the previous stage. First we assume that each column contains at least two endpoints, and there are an even number of columns. This permits a simple monotone oscillating path, discernable in the first 10 columns of Fig. 2b. Second, we remove the assumption of an even number of columns. The last odd column is integrated into the path by zigzagging horizontally; see column 11 of Fig. 2b. Finally, we remove the assumption of at least two endpoints per column, and consider sections of one endpoint per column.

We begin by proving two lemmas necessary for the basic oscillating path.

3.1. Top and bottom edges

The top and bottom edges between two columns will be used to connect column $2j-1$, to $2j$, $1 \leqslant j \leqslant m/2$. In Fig. 2, columns 1–2, 3–4, 5–6, 7–8, and 9–10 are so connected.

Lemma 3.1. *Let S be a set of $n \geqslant 1$ unit segments. For all adjacent columns a and $b = a+1$, with $1 \leqslant a < m$, the top endpoints in columns a and b are visible to each other, and the bottom endpoints in columns a and b are visible to each other.*

Proof. Let a_i, b_j be the top endpoints in columns a and b, respectively. Suppose a_i and b_j do not see each other. Then there exists some segment s that blocks visibility

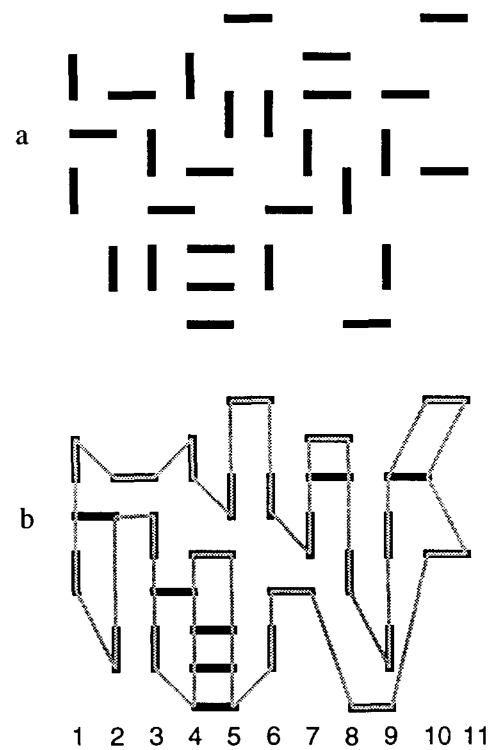

Fig. 2. (a) A set of segments with at least two endpoints per column. (b) Hamiltonian path constructed by algorithm.

between a_i and b_j. So some part of s must lie between columns a and b. Thus, s must be horizontal, with one endpoint a_x in column a and one endpoint b_y in column b. Since s must block visibility between a_i and b_j, either a_x lies below a_i in column a and b_y lies above b_j in column b, or a_x lies above a_i in column a and b_y lies below b_j in column b. Both of these are contradictions, since a_i and b_j are the top endpoints of columns a and b, respectively. Therefore, a_i and b_j see each other.

Similarly, the bottom endpoints of columns a and b are visible to each other. □

3.2. Isthmuses

We define an *isthmus* to be a pair of disjoint visibility edges, (a_i, b_j) and (a_{i+1}, b_{j+1}), between two adjacent columns a and $b=a+1$, where a_i and a_{i+1}, and b_j and b_{j+1}, are adjacent in their respective columns. Isthmuses will be used to connect column $2j$ to $2j+1$, $1 \leqslant j < m/2$. In Fig. 2b, isthmuses connect columns 2–3, 4–5, 6–7 and 8–9.

Lemma 3.2. *An isthmus exists between any two adjacent columns, if each column contains at least two endpoints.*

Proof. If there is no horizontal segment between adjacent columns a and b, then any pair of adjacent endpoints in column a form an isthmus with any pair of adjacent endpoints in column b, because no visibility edge between the columns is blocked.

So assume now that $s=(a_i, b_j)$ is a horizontal segment. We consider two further cases. First, suppose that there are endpoints in both column a and column b to one side of s. Without loss of generality, let a_{i+1} and b_{j+1} be the endpoints to this side and adjacent to a_i and b_j, respectively. Then a_{i+1} and b_{j+1} can see one another, because if there was an intervening blocking segment, then one of its endpoints would either be between a_i and a_{i+1} or between b_j and b_{j+1}.

The second case occurs when all of the endpoints in column a are to one side of s and all of the endpoints in column b are to the other side of s. Without loss of generality, let a_{i+1} and b_{j-1} be the endpoints adjacent to a_i and b_j, respectively. Then (a_i, b_{j-1}) and (a_{i+1}, b_j) must be visibility edges for the same reason as above.

Because there are at least two endpoints in each column, this exhausts all cases, and completes the proof. □

3.3. Two endpoints per column, m even

We can now show Hamiltonicity for sets of unit lattice segments with an even number m of columns and with at least two endpoints in each column.

Theorem 3.3. *Let S be a set of unit lattice segments such that m is even, and for all a, $1 \leqslant a \leqslant m$, column a contains at least two endpoints. Then the visibility graph of S has a simple Hamiltonian cycle.*

Proof. By Lemma 3.2, we can find an isthmus between all adjacent columns a and $a+1$, where a is even and $2 \leqslant a \leqslant m-2$. Call this set of isthmuses X. In each column b, let b_j, b_{j+1} be the isthmus endpoints, and let b_t be the top endpoint. Clearly, we have vertical paths along column b from b_j to b_1, and from b_{j+1} to b_t. These paths together include all of the endpoints in column b. Also, there is a path along column 1 from the top endpoint to the bottom endpoint using all the endpoints in column 1, and there is a similar path along column m. Let Y be the set of all of these paths along the columns. Finally, by Lemma 3.1, the visibility edges (a_1, b_1) and (a_s, b_t) exist, where a and b are adjacent columns with a odd and $1 \leqslant a \leqslant m$, and a_s and b_t are the top endpoints in their respective columns. Call this set of top and bottom edges Z.

If we join the sets X, Y, and Z, we obtain a simple Hamiltonian cycle for S. See the first 10 columns of Fig. 2b. □

3.4. Last two columns: m odd

Theorem 3.3 assumes that m, the number of columns in S, is even. Therefore the cycle uses the top and bottom visibility edges between columns $m-1$ and m, and continues through the endpoints along column m. If m is odd, however, the path described in the proof of Theorem 3.3 would use the isthmus between columns $m-1$ and m, and would continue to the top and bottom of column m. This would prevent the path from closing to form a cycle. We now give an algorithm to deal with this final column.

For m odd, let $a=m-1$ and $b=m$, and let a_1, b_1 and a_s, b_t be the bottom and top endpoints, respectively, of columns a and b. Sort the endpoints in columns a and b, with the exception of a_1 and a_s, into a list L, lowest to highest, choosing arbitrarily between a_i and b_i if they are at the same height. Let $\pi=a_1, L, a_s$. We claim that π joins with the rest of the path to form a Hamiltonian cycle. For example, in Fig. 2b, $a=10$ and $b=11$, with $a_s=a_3$ and $b_t=b_2$; here $L=(b_1, a_2, b_2)$ and $\pi=(a_1, b_1, a_2, b_2, a_3)$.

The path enters column a at a_1 and a_s. If the first element of L is in column a, then this first element must be a_2, and a_1 can see a_2. If the first element of L is in column b, then it must be b_1, and a_1 can see b_1 by Lemma 3.1.

Now suppose we are at some endpoint a_i in L, where a_i is not the last element of L. If the next endpoint in L is a_{i+1}, we can continue along π since a_i sees a_{i+1}. Otherwise, if the next endpoint in L is b_j, suppose a_i and b_j do not see each other. Then there must be a horizontal segment between columns a and b, with endpoints vertically and strictly between a_i and b_j. But then b_j would not follow a_i in L. Therefore, a_i and b_j see each other, and we can continue along π. If we are at some endpoint b_k in L, where b_k is not the last element of L, we can also continue along π by a similar argument as above.

Finally, if the last element of L is a_{s-1}, a_{s-1} and a_s see each other. If the last element is b_t, then b_t sees a_s by Lemma 3.1. Therefore, π can join the rest of the path to form a Hamiltonian cycle.

We note that it is possible to fall into the above situation when m is even. This can occur when there is a column that contains only one segment endpoint, which we consider in the following section. However, the above algorithm works for this case as well.

3.5. One endpoint per column

We now consider the last aspect of the proof, columns containing just one endpoint. Here it is more difficult to keep upper and lower paths separated. Let columns a and z each contain more than two endpoints, and suppose all the columns between a and z, columns $b,\ldots, y$, contain exactly one endpoint each; we will call this set of columns C_1.

In both columns a and z, a pair of *terminal* endpoints will be distinguished: a_i and a_j $(i<j)$, and z_k and z_ℓ $(k<\ell)$. The path to the left will terminate at and determine a_i and a_j, but we have a choice for the terminals in column z that resume the path to the right. These will be chosen to be adjacent, as if at the right end of isthmus, or at the top and bottom of the column if z is the last column.

The task is then to find two disjoint paths, π_A from a_i to z_k, and π_B from a_j to z_ℓ, which together touch every endpoint in C_1. Because there is just one endpoint in each of these columns, we will dispense with subscripts, calling them b etc.

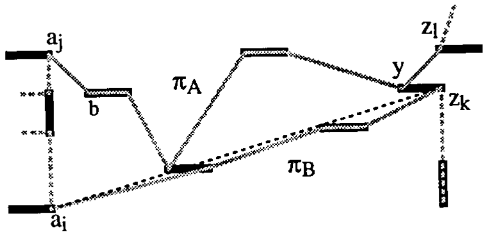

Fig. 3. One endpoint per column.

Let A be the set of endpoints of C_1 that lie strictly above the line through a_i and z_k, and let B be the set of endpoints on or below this line. (Either A or B might be empty.) Then simply let π_A start at a_j, include all endpoints in A left to right, and terminate at z_ℓ, and similarly let π_B start at a_i, include all endpoints in B left to right, and terminate at z_k. See Fig. 3. The reason this works is as follows. A and B are disjoint and partition the endpoints in C_1, so π_A and π_B are disjoint and together cover C_1. The only issue is whether each pair of consecutive vertices of the paths can see one another. Let p and q be two consecutive vertices of π_A. Suppose some segment s intersects the open segment (p, q), and so blocks visibility. Then one of s's endpoints r must be on or above the line through p and q, and therefore in A. Since the segments are unit length, r must lie between p and q horizontally. So π_A would include r between p and q, contradicting our assumption that p and q are consecutive vertices on π_A.

We assumed the existence of columns a and z surrounding C_1, but their absence causes no difficulties. If $b=1$, so there are no terminals to the left of b, simply start π_A and π_B at $a_i = a_j = b$. And similarly if $y = m$, the last column, end π_A and π_B at $z_k = z_\ell = y$.

3.6. *Summary*

The entire Hamiltonian cycle for an arbitrary set of unit lattice segments can be constructed left to right, as follows. For any contiguous group of columns from the left with two or more endpoints per column, connect via the path oscillating between extreme edges and isthmuses (Section 3.3). If this group includes all columns, either we are finished, or the last (odd) column needs to be adjusted as described in Section 3.4. If on the other hand the group is adjacent to a group C_1 of one-endpoint columns, connect across C_1 as just described. If C_1 does not include the m-th column, then to its right we have another group of columns with two or more endpoints. We proceed as before, and repeat until all columns are consumed.

An example is shown in Fig. 4. Columns 1–3 contain more than one end point, and are connected using a 2–3 isthmus (indicated by short dashes). Column 4 has just one endpoint. The dashed line shows the "$a_i z_k$" line partitioning the endpoints into A (one endpoint) and B (empty). We place the terminals adjacent in column 5, as at the right

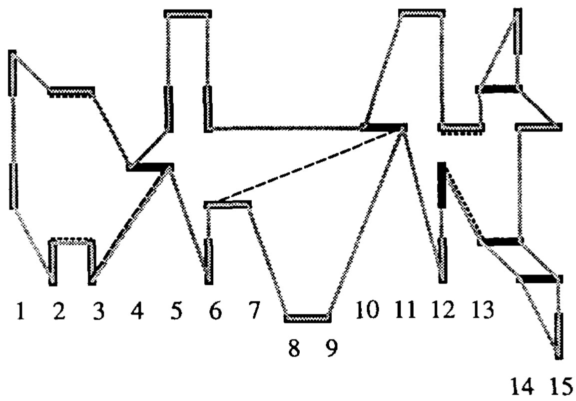

Fig. 4. A set of segments that exercise all aspects of the algorithm. Columns 4, and 7–10, contain just one endpoint.

end of an isthmus. Columns 5–6 again contain more than one endpoint. Columns 7–10 form a one-endpoint group, and again the dashed lines shows the A/B partitioning line. Here A contains one endpoint, and B the remainder. Column 11 contains only two endpoints, treated as the right end of an isthmus. Columns 11–15 each contain more than one endpoint. An isthmus is used for 12–13 (shown with short dashes), leaving column 15 "odd." This column is integrated by vertically sorting the endpoints in columns 14 and 15.

Theorem 3.4. *For any set S of $n > 1$ noncollinear unit lattice segments, there exists a simple Hamiltonian cycle in the visibility graph.*

3.7. *Algorithm*

The proof leads to a straightforward $O(n \log n)$ algorithm. The integer-coordinate endpoints are first sorted horizontally, and then vertically within each column. The coordinates can then be replaced by indices of sorted rank, effectively removing empty rows and columns, as clearly this transformation does not affect visibility between adjacent rows or columns. The remainder of the algorithm implied by the proof is linear-time:

1. Identifying top and bottom endpoints in a column (Section 3.1) is constant-time.
2. Finding an isthmus between adjacent columns (Section 3.2) requires checking the local vicinity of each straddling horizontal segment, and is thus linear-time.
3. Merging the sorted $(m-1)$-st and m-th columns (Section 3.4) is linear in the number of endpoints in those columns.
4. Partitioning C_1 by the $a_i z_k$ line for the one endpoint per column case (Section 3.5) is linear in the size of C_1.

Finally, we note two features of the Hamiltonian cycle produced by our algorithm: it is not necessarily monotone with respect to the horizontal (because of the possible zigzagging between the last two columns), and it is not usually circumscribing (some segments are exterior to the cycle, as in Fig. 2, but not in Fig. 4). We do not know if unit lattice segments always admit cycles that are monotone, or circumscribing.

4. Discussion

It must be admitted that the evidence for the simple Hamiltonicity of segment visibility graphs is weak. The conjecture has now been proved for three highly restricted classes: hulled, independent, and unit lattice segments. We have been unable to prove it for the following natural class. Define a set of disjoint segments *shellable* if they may be ordered $s_1, s_2, \ldots, s_n$ such that for each i, $1 < i \leqslant n$, s_i lies in the exterior of the convex hull of segments $s_1, \ldots, s_{i-1}$. We leave the status of this problem, as well as the general conjecture, open.

Acknowledgements

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References

[1] P.K. Agarwal, Intersection and Decomposition Algorithms for Planar Arrangements (Cambridge Univ. Press, New York, 1991).

[2] A. Mirzaian, Hamiltonian triangulations and circumscribing polygons of disjoint line segments, Comput. Geom. Theory Appl. (1993) 15–30.

[3] J. Matoušek and E. Welzl, Good splitters for counting points in triangles, J. Algorithms 13 (1992) 307–319.

[4] J.O'Rourke and J. Rippel, A class of segments whose visibility graphs are Hamiltonian, Technical Report 12, Smith College, 1991.

[5] J. O'Rourke and J. Rippel, Segments visibility graphs: Some results, in: Proceedings of the 4th Canadian Conference on Computational Geometry, Memorial University of Newfoundland (1992).

[6] D. Rappaport, H. Imai and G.T. Toussaint, Computing simple circuits from a set of line segments, Discrete Comput. Geom. 5 (1990) 289–304.

[7] G. Toussaint, Personal communication, 1992.