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The Helly bound for singular sums ☆

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Abstract

The singularity graph of a finite ring has the ring elements as vertices with edges joining pairs whose difference is not invertible. In this paper we will establish a bound for the number of sums which can be generated by a clique in the singularity graph of Z_n, the ring of integers modulo n. When n has at least three prime factors, there are always cliques based on Helly families of sets which realize $n-\phi(n)$ sums, where ϕ denotes the Euler totient function. When n has exactly three prime factors, this bound is best possible.© 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

For any integer n, let Z_n be the finite ring of integers mod n. Generally, we will take $\{0,1,2,3,\ldots,n-1\}$ as the set of elements of Z_n. The *singularity graph* SING(n) has these elements as its vertices with an edge between two vertices x and y iff $x-y$ is singular (i.e., not invertible) modulo n. A subset S of Z_n is *singular* iff the difference of any two of its members is singular—that is, iff S is a clique (complete subgraph) of the singularity graph SING(n). The goal of this paper is to investigate the number of sums that can arise from a singular set in Z_n.

This problem is motivated by the study of *sum covers*, sets whose pairwise sums yield all elements of Z_n. Sum covers play an important role in certain geometric and combinatorial problems [5,8], and are related to the much studied Sidon sets and additive bases of additive number theory [1,2,6]. Any set which contains a pair of elements whose difference is relatively prime to the modulus n can be standardized by an affine bijection $x \to ax+b$ into a set containing 0 and 1 [8]. It is natural to wonder if sets that are as “rich” as sum covers must always contain a coprime pair. Indeed, the

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smallest integer n for which a singular sum cover exists in Z_n is $n=2310$, the product of the first five primes [8]. The techniques in [8] are very nonconstructive and give no insight into how many sums can be generated by a singular set when a singular sum cover fails to exist. That question will be explored here. Along the way we will derive a characterization of cliques in the singularity graph when n has at most three prime factors, and study a general family of cliques that arise from Helly families (also called intersecting families) [9,11,14].

Two tools will be basic to this study. The first of these is the notion of a transitive edge-coloring [7]. Let $G=(V,E)$ be a finite simple graph (no loops or multiple edges) and C a finite set of "colors". A *C-edge-multicoloring* of G is an assignment of a *nonempty* subset of C to each edge of G. Since all colorings considered here are of this type, for brevity, we will speak just of a *C-coloring* of G. When only the cardinality $d=|C|$ of C and not the specific color set C is important, we will refer to a *d-coloring*. A C-coloring of G is *transitive* provided that for each color c, if edges xy and yz have c as an assigned color, then $x=z$ or xz is an edge of G and xz has c as an assigned color.

The condition that x and y are adjacent in SING(n) iff $x-y$ is not invertible mod n means that there is an edge between x and y iff $\gcd(x-y,n)>1$. Such an edge is then colored with all common prime divisors of n and $x-y$. This coloring is transitive because if xy and yz are edges colored p, then p is a prime dividing n which also divides $x-y$ and $y-z$. Hence p divides the sum $x-z=(x-y)+(y-z)$, so the edge xz exists and is colored p. Thus SING(n) is a transitively edge-colored graph whose color set C is the set of all prime divisors of n.

The second tool is the notion of factored abelian group introduced in [8]. This is introduced in Section 5 and involves the representation of Z_n as a direct product via the Chinese Remainder Theorem. This representation is closely related to the cocategorical product of complete graphs [13], which has been studied by several authors [4,10,12], mostly from the complementary point of view. West [15, p. 376] gives a succinct treatment of the main properties of this product and its connection with other graphical representations.

2. The easy cases

For any subset S of Z_n, the *k-fold sum* of S is the set of sums $x_1+x_2+\cdots+x_k$ where the x_i are (not necessarily distinct) elements S. Let $S_k(n)$ denote the maximum cardinality of the k-fold sum of a singular set in Z_n. As it turns out, $S_k(n)$ is easily evaluated if $k\geqslant 3$ or if $d\leqslant 2$ (where d is the number of prime divisors of n). In the second case, a simple lemma on transitive 2-colorings is useful. A clique in a transitively C-colored graph G is *principal* iff there is some color which appears on all of its edges.

Lemma 2.1. *Every clique K in a transitively 2-colored graph is principal.*

Proof. Let the colors be r and s, and suppose there is an edge vw in K which is not assigned r as a color. Then vw is colored s. Let x be any third vertex of K. The edges vx and wx cannot both receive r as transitivity would then imply that vw receives r. Hence at least one of vx or wx must receive s. But then the other also receives s by transitivity since vw is colored s. This shows that all edges at v must be colored s. Hence if x and y are any two vertices in K, vx and vy are colored s, so by transitivity, xy must also receive s. Hence s occurs on every edge in K. □

This lemma was implicitly observed by Graham and Sloane in [5].

Theorem 2.2. (i) *If n has only one or two prime factors of which p is the smaller, then $S_k(n) = n/p$ for all $k \geqslant 2$.*

(ii) *If n has three or more prime factors and $k \geqslant 3$, then $S_k(n) = n$.*

Proof. (i) If S is a singular set, then S is a clique in SING(n). By the lemma, S is principal. That is, there is a prime divisor p of n such that $p \mid (x - y)$ for all x and y in S. Thus, S lies in a coset $\langle p \rangle + a$ of the cyclic subgroup of Z_n generated by p. It follows that the k-fold sum of S lies in the coset $\langle p \rangle + ka$ which has cardinality n/p. This is achieved if S is the subgroup $\langle p \rangle$ and the cardinality is maximized when p is the smaller prime divisor.

(ii) Suppose p, q, and r are distinct prime divisors of n. Let S consist of all integers $\bmod\, n$ which are divisible by at least two of these primes. Then any two share a common prime divisor with n, so S is a singular set. Since $\gcd(pq, qr, pr) = 1$, there are integral coefficients a, b, and c with $apq + bqr + cpr = 1$. Multiplying this by any t in Z_n, we see that any element of Z_n is a sum of three elements of S. Adding $k - 3$ zeroes yields t as a sum of k elements of S. Hence $S_k(n) = n$ as claimed. □

Thus it remains to determine $S_k(n)$ when $k = 2$ and $d \geqslant 3$, where d is the number of prime divisors of n. We will show in Theorem 6.5 below that the Helly bound $S_2(n) \geqslant n - \phi(n)$ holds when $d \geqslant 3$ and equality holds when $d = 3$. When $d = 4$, the results of [8] show that $S_2(n) < n$, and in the final section we will consider some explicit examples in this case. For $d \geqslant 5$, the Helly bound $S_2(n) \geqslant n - \phi(n)$ can always be improved, but the constructions are somewhat involved and will be deferred to a later paper. Constructions in [8] show that $S_2(n) = n$ when $n = 6rst$, where r, s, and t are distinct primes larger than 3. However, it seems unlikely that $S_2(n)$ will equal n for all n with $d \geqslant 5$ prime divisors.

In contrast to the difficulty of determining the largest number of sums from a clique, it is very easy to determine the largest cliques—they are principal. Recall that for any graph $\omega(G)$ denotes the maximum order of a clique in G and that $\chi(G)$ denotes the chromatic number of G. Clearly, for any graph $\omega(G) \leqslant \chi(G)$ (cf. [15]). For singularity graphs, equally is easily seen to hold. (This can also be viewed as determining the value of $S_1(n)$.)

Scholium 2.3. For any singularity graph SING(n), we have $\omega=\chi=n/p$ where p is the smallest prime divisor of n.

Proof. Let $\langle p\rangle$ denote the cyclic subgroup of Z_n generated by p. Then $\langle p\rangle$ is a clique of order n/p all of whose edges are colored p. Thus $\omega \geqslant n/p$. Now any set of the form $\{kp, kp+1, kp+2, \ldots, kp+(p-1)\}$ is an independent set since any two members differ in absolute value by an amount less than the smallest prime divisor p of n. As k ranges from 0 to $(n/p)-1$, these n/p sets form the color classes of a proper vertex coloring of SING(n). Hence $\chi \leqslant n/p$. □

3. Reduction to the square-free case

Two vertices in a transitively edge-colored graph G are *clones* iff they are joined by an edge which receives every color used in G. The *closed neighborhood* $N[v]$ consists of v and all vertices adjacent to v. In a general graph, vertices with $N[v]=N[w]$ are called (true) *twins*. As shown below, clones are twins. Moreover, transitivity implies that the clone relation, like the twin relation, is an equivalence relation on the vertices. The lemma says that whether there is an edge between two vertices—and if so, then the colors on it—are determined completely by the clone classes of those vertices. Hence we can unambiguously contract each clone class to a single vertex.

Lemma 3.1. *Let G be a transitively edge-colored graph.*

(i) *if v and w are clones, then they have the same closed neighborhood N and for every x in N, the edges vx and wx have the same colors.*
(ii) *if v and w are clones and y and z are clones, then there is an edge wz iff there is an edge vy and if these edges exist, they receive the same colors.*

Proof. (i) Suppose v and w are clones and x is a neighbor of v. It suffices to show that x is also a neighbor of w and every color c on vx is also on wx. Since vw receives all colors, it is colored c. Thus wx exists and is colored c by transitivity.

(ii) follows at once from (i). □

Theorem 3.2. *Suppose n is square-free and m is any integer all of whose prime divisors are among those of n. Then*

(i) SING(mn) *can be formed from* SING(n) *by cloning each element of* SING(n) *m times.*
(ii) $S_2(mn)=mS_2(n)$.

Proof. Any element v of Z_{mn} can be written uniquely as $v=a+sn$ where a is in Z_n and $0\leqslant s<m$. Thus v is a clone of a in SING(mn) because by construction, every prime divisor of mn is a divisor of n and hence of $sn=v-a$. This establishes (i).

It follows from the lemma that any maximal clique K in Z_{mn} must contain all clones in Z_{mn} of each of its elements. Thus $K' := K \cap Z_n$ is a clique in Z_n and K consists of the union of all cosets of the form $a + \langle n \rangle$ where $\langle n \rangle$ is the cyclic subgroup generated by n and a is in K'. Each pairwise sum $a + b$ in $K' + K'$ then yields a coset $(a + b) + \langle n \rangle$ of sums from $K + K$. Obviously, two sums yield different cosets iff the two sums are different mod n. As each coset contains m elements, it follows that K has m times as many pairwise sums in Z_{mn} as K' does in Z_n. Thus $S_2(mn) \leqslant mS_2(n)$. Equality holds because any maximal clique in SING(n) can be extended to a maximal clique in Z_{mn} by adjoining all clones of its elements. □

4. Classification of cliques for $d = 3$

A C-coloring of G is *supertransitive* provided that for each color c, if edges wx and yz have c as an assigned color, then $w = z$ or wz is an edge of G and wz has c as an assigned color. For each color c, look at the spanning subgraph $G[c]$ of G whose edges are those edges which receive c as a color (among possibly others). The transitive condition is equivalent to requiring that the components of $G[c]$ are complete. The nontrivial components—those with at least one edge—of $G[c]$ will be called the *c-parts* of G. The supertransitive condition is equivalent to saying there is at most one part per color. It is interesting to note that the minimum number of colors required to supertransitively color a graph G is just the minimum number of cliques required to cover the edges of G. And this number is the celebrated intersection number of G. (Cf. [15, p. 374] and [3]).

The transitively 3-colored clique shown in Fig. 1 will be called a *Fano clique* because it is the slope pattern in the sense of [7] of the affine plane of order 2. That is, parallel lines get the same color. This coloring is, of course, also the unique 1-factorization of K_4. Although both interpretations generalize, the interest in the Fano clique here is that it is essentially the only transitive clique on three colors that is neither supertransitive nor principal.

Theorem 4.1. *Any clique in a clone-free transitively 3-colored graph is either principal, supertransitive, or a Fano clique.*

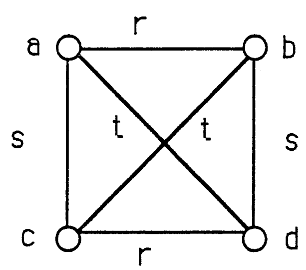

Fig. 1. The Fano clique.

Proof. Let K be a clique in a graph G with a clone-free transitive $\{r,s,t\}$-coloring. Suppose K is neither principal nor supertransitive. Then there are two edges ab and cd colored, say with r, that are not in the same r-part. Each of the four edges between $\{a,b\}$ and $\{c,d\}$ must be colored either s or t or possibly both. But none of them get color r. There are two cases to consider, the first leading to a contradiction and the second leading to the Fano clique.

Case 1: At least one of ab and cd has a second color.

Say ab is colored with both r and s. The edges ac and bc cannot both be colored with t, or by transitivity ab would have all three colors. Hence either ac or bc must be colored with s. Thus c is in the same s-part as a and b. The same argument using ad and bd implies d is also in the same s-part as a and b. Hence a, b, c, and d all lie in the same s-part. By the "non-principal" condition, there must be a vertex v in K not in this s-part. Hence no edge from v to a, b, c, or d is colored s. Thus all these edges are either r or t. If va and vb were both t, then ab would be colored t by transitivity, and a and b would be clones. Hence one of va or vb must be colored r. Similarly, one of vc or vd must be colored r. But then ab and cd are in the same r-part as v.

Case 2: Neither ab nor cd has another color besides r.

Then edges ac and ad must have different colors. Without loss of generality, let ac be s and ad be t. Likewise, ad and bd must get different colors, so bd is s. Also bd and bc must get different colors, so bc is t.

Suppose there were a fifth vertex v in K. Since ab and cd are in different r-parts, v is not in one of these two r-parts. Without loss of generality, suppose v is not in the r-part of ab. Then neither va nor vb can be colored r. Hence they must be either s or t. Both cannot be the same or ab would get a second color by transitivity. There is enough symmetry that we can assume without loosing generality that va is s and vb is t. Thus since va and ac are s and since vb and bc are t, it follows that vc is colored both s and t.

Now what color is vd? If it is r, then by transitivity vc is also r. This puts all three colors on vc, making v and c clones. If vd is either s or t, then by transitivity, cd must also share that color, contrary to the case assumption that cd is colored only r. Hence there is no color available for vd.

Thus there can be no fifth vertex and a, b, c, d are the only vertices in K. The opening paragraph of this case showed the six edges have the colors shown in Fig. 1. It only remains to show that there are no additional colors on these edges. But any additional color would force, by transitivity, all edges to share that color, making K principal. Hence K must be the Fano 1-factorization shown in Fig. 1. □

5. Helly cliques in factored groups

A family of sets such that any two have nonempty intersection is called a *Helly family* [14] (or an *intersecting family* [9,11]). Now suppose G is a transitively C-colored graph. For any two vertices v and w, let $C(v,w)$ denote the set of colors assigned to edge vw if v and w are adjacent. If v and w are not adjacent, take $C(v,w)=\emptyset$. Let $\mathscr{F}$ be a Helly family of colors, and let v be any "root" vertex in G. Define $K(v,\mathscr{F})$ to be the set of all vertices x of G such that the color set $C(v,x)$ belongs to $\mathscr{F}$. Since $\mathscr{F}$ has the Helly property, any two edges vx and vy for x and y in $K(v,\mathscr{F})$ share a common color. Hence the edge xy exists and has this color. Whence $K(v,\mathscr{F})$ is a clique. Such cliques will be called *Helly cliques*. Notice that (maximal) principal cliques are Helly cliques in which the Helly family consists of all color subsets which contain a fixed common color. (In set-theoretic language, this is sometimes called a *principal filter*.)

A Helly family of subsets of C is *maximal* iff it is not contained in a larger Helly family of subsets of C. The following facts are well-known and easy to verify:

(i) if a Helly family $\mathscr{F}$ is maximal, then every superset of a set in $\mathscr{F}$ is in $\mathscr{F}$;
(ii) a Helly family $\mathscr{F}$ is maximal iff every set or its complement is in $\mathscr{F}$;
(iii) a maximal Helly family $\mathscr{F}$ has cardinality $|\mathscr{F}|=2^{d-1}$ where $d=|C|$.

The next goal is to show that maximal Helly families yield maximal cliques and that any supertransitive clique lies inside a Helly clique. This is not true in general but is true for a large class of transitively colored graphs which include the singularity graphs.

Let $A_1,A_2,\dots,A_d$ be finite sets. The *meet graph* $MG(A_1,A_2,\dots,A_d)$ has as its vertices all vectors of length d whose ith coordinate is an element of A_i. Two vectors u and v are adjacent iff they agree in at least one coordinate, and in that case the edge between them is "colored" by the set of coordinate indices in which they agree. Obviously, if u and v agree in the ith coordinate and v and w agree in the ith coordinate, then so do u and w, so this is a transitive coloring. The sets A_i are called *stacks* and their cardinalities $s_i=|A_i|$ are the *stack sizes*. When only the size of each stack matters and not its specific structure, the meet graph can be denoted by $MG(s_1,s_2,s_3,\dots,s_d)$.

If the stacks are abelian groups (written additively), then the meet graph has the additional structure of an abelian group with coordinatewise addition. Notice that adjacency can now be defined by saying that u and v are adjacent iff $u-v$ is zero in some coordinate and if so, the edge uv is colored by the set of all coordinate indices in which $u-v$ is zero. The entire structure—set of vectors, graph, edge-coloring, and group structure—will be called simply a *factored* (*abelian*) *group* and denoted by the product notation $G=A_1\times A_2\times\cdots\times A_d$. If n is square free, say, $n=p_1p_2p_3\dots p_d$, then the Chinese Remainder Theorem says that SING(n) is the factored group where the ith stack A_i is the cyclic group of order p_i. In fact, a factored group is a singularity graph iff its stack sizes $s_i=|A_i|$ are distinct primes. Hence this will be referred to as the *numeric case*. To avoid degeneracy, we always assume the stacks sizes are all at least 2. Mimicing number-theoretic terminology, the coordinate indices for a meet graph or factored group will be called *places*. In the numeric case, the places correspond to the prime divisors of n. These are, of course, just the colors used on the edges, so the set of all places will be denoted by C.

In passing from singularity graphs to factored groups, we lose the possibility of clones, which are more of a technical nuisance than a real generality. But we gain a much richer class of structures. (See [8] for more details on this discussion.)

For a factored group $G = A_1 \times A_2 \times \cdots \times A_d$, denote the maximum number of k-fold sums of a clique in G by $S_k(G)$. Theorem 2.2 extends quite naturally with essentially the same proof:

$$\text{If } d = 2, \text{ then } S_k(G) = s_2 \text{ where } s_1 \leqslant s_2; \quad \text{and}$$

$$\text{If } k \geqslant 3 \text{ and } d \geqslant 3, \text{ then } S_k(G) = s_1 s_2 s_3 \ldots s_d.$$

Notice that the clique used to prove the second assertion is a Helly clique. For completeness, it is also worth mentioning that the analogue of Scholium 2.3 also holds for a general factored group G:

$$\text{For all } d \geqslant 1, \quad S_1(G) = \omega = \chi = |G|/s_1.$$

Here the independent sets can be taken as $\{t + (i, i, \ldots, i): i = 0, \ldots s_1 - 1\}$ as t runs over elements of the subgroup $\{0\} \times A_2 \times \cdots \times A_d$.

Our goal can now be extended to determining $S_2(G)$ for every factored group, and we will achieve this in Theorem 6.4 for $d = 3$ factors. The following easy lemma is helpful in simplifying notation. There are other automorphisms, but they will not be needed here.

Lemma 5.1. *In any factored group, translation is a color preserving automorphism of the group. Moreover, it preserves the number of pairwise sums of any set.*

Proof. Consider a translation $x \to x + t$. For any two vectors u and v, $u - v = (u + t) - (v + t)$, so u and v are joined by an edge iff their images are, and if so, the two edges receive the same colors. Now for any subset S of vectors, a sum $x + y$ of elements in S is mapped into the sum $(x + t) + (y + t) = (x + y) + 2t$ from the image $S + t$ of S. Hence the sums are also translated, by $2t$ rather than t. Since this is a bijection, it preserves cardinality. $\square$

Theorem 5.2. *Let $\mathscr{F}$ be a Helly family in C. Then $\mathscr{F}$ is a maximal Helly family iff $K(v, \mathscr{F})$ is a maximal clique in $MG(s_1, s_2, \ldots, s_d)$.*

Proof. ($\to$) Pick a vector w not in $K(v, \mathscr{F})$. Let A be the (possibly empty) set of places where w and v agree. Then A is not in $\mathscr{F}$ since w is not in $K(v, \mathscr{F})$. Hence $C \backslash A$ is in $\mathscr{F}$ by maximality property (ii). Let u agree with v on $C \backslash A$; but on A, let u take values different from v (and hence w). Then u is in $K(v, \mathscr{F})$ and is not adjacent to w.

($\leftarrow$) Suppose $\mathscr{F}$ is not a maximal Helly family. Let $\mathscr{H}$ be a Helly family that properly contains $\mathscr{F}$, and let A be a set in $\mathscr{H}$ that is not in $\mathscr{F}$. Define a vector w which agrees with v on A and disagrees with v off A. Then A is precisely the set of colors assigned to edge vw. Thus w is not in $K(v,\mathscr{F})$, but w is in the clique $K(v,\mathscr{F}\cup\{A\})$ which also contains $K(v,\mathscr{F})$. Hence $K(v,\mathscr{F})$ is not maximal. $\square$

One consequence of this is that "Helly" and "maximal" commute. That is, any Helly clique that is maximal in the class of Helly cliques is maximal in the class of all cliques.

Theorem 5.3. *Every supertransitive clique K in a meet graph lies in some Helly clique.*

Proof. For each place p in C, recall that $K[p]$ is the spanning subgraph of K formed by using all the edges of K that are colored p. Since K is supertransitive, $K[p]$ has at most one nontrivial component—the unique p-part of K. Let C_1 be the set of places for which $K[p]$ has a p-part, and let C_0 be the set of places for which $K[p]$ consists of isolated vertices. Define a vector v as follows. For p in C_1, all the vectors in the unique p-part of K agree in the place p. Assign this common value to the pth place of v. For p in C_0, assign the pth place of v arbitrarily.

Let $\mathscr{F}:=\{C(v,x)\colon x\in K\}$. We claim that $\mathscr{F}$ is a Helly family. Indeed, given x and y in K, they are joined by an edge xy. Let i be any "color" on this edge. The edge xy then lies in a component of the monochrome subgraph $K[i]$, so this component is nontrivial and hence an i-part of K. Since K is supertransitive, this is the unique i-part of K. Moreover, this i-part consists of all vectors in K which take the value v_i in the ith place. Hence x and y agree with v in the ith place as well as with each other. We have just shown that wherever x and y agree with each other, they also agree with v. That is, $C(x,y)\subseteq C(v,x)\cap C(v,y)$. Since $C(x,y)$ is nonempty, this establishes the Helly property.

It is clear from the definitions that K lies inside $K(v,\mathscr{F})$ as desired. $\square$

The property that "wherever x and y agree with each other, they also agree with v" is characteristic of supertransitivity and usually fails in general Helly cliques. This is the basis for the next result.

Theorem 5.4. *If $d>3$, no maximal clique is supertransitive.*

Proof. Since a supertransitive clique lies in a Helly clique, a supertransitive maximal clique must be a maximal Helly clique. Thus it suffices to show that no maximal Helly clique is supertransitive. Consider $K(v,\mathscr{F})$ where $\mathscr{F}$ is a maximal Helly family. $\mathscr{F}$ cannot contain both $\{1,2\}$ and $\{3,4\}$. Without loss of generality, $\mathscr{F}$ does not contain $\{1,2\}$. Then $\mathscr{F}$ does contain $C\backslash\{1,2\}$ by property (ii) of maximal Helly families. Let v' denote the vector obtained by deleting the first two coordinates of v. Then these four vectors lie in $K(v,\mathscr{F})$, since they all agree with v on a set in $\mathscr{F}$:

$$[0,0,v'],\quad [0,1,v'],\quad [1,0,v'],\quad \text{and}\quad [1,1,v'].$$

The first and last pairs are joined by an edge colored 1. But there is no edge colored 1 between them. Hence $K(v,\mathscr{F})$ is not supertransitive. $\square$

One consequence of this is that "supertransitive" and "maximal" do not commute. There are cliques that are maximal in the class of supertransitive cliques, but when $d > 3$, there are no supertransitive maximal cliques.

6. The Helly bound

We now return to the central question of determining the number of sums of a Helly clique. A place p is a *support place* for a Helly family $\mathscr{F}$ on C iff there are sets A and B in $\mathscr{F}$ with $A \cap B = \{p\}$.

Lemma 6.1. *Let $\mathscr{F}$ be a maximal Helly family on C. Any intersection of two sets in $\mathscr{F}$ contains a support place for $\mathscr{F}$.*

Proof. Let A' and B' be any two sets in $\mathscr{F}$. Select A and B in $\mathscr{F}$ with $A \subseteq A'$ and $B \subseteq B'$ so that $A \cap B$ is minimal. If $A \cap B$ contains two places, say, p and q, then by minimality of the intersection $A \cap B$, the set $A \backslash \{p\}$ must not be in $\mathscr{F}$. Thus by maximality property (ii), $C \backslash (A \backslash \{p\}) = (C \backslash A) \cup \{p\}$ is in $\mathscr{F}$, and this set intersects A in $\{p\}$ alone. Hence p is a support place contained in $A' \cap B'$. $\square$

Theorem 6.2. *Let $G = A_1 \times A_2 \times \cdots \times A_d$ with stack sizes $2 \leqslant s_1 \leqslant s_2 \leqslant s_3 \leqslant \cdots \leqslant s_d$. Let $K = K(v, \mathscr{F})$ where $\mathscr{F}$ is a maximal Helly family on $C = \{1, 2, 3, \ldots, d\}$. Then*

$$|K + K| = N - ST,$$

where N is the product of s_p over all places p; S is the product of $s_q - 1$ over all support places q; T is the product of s_r over all nonsupport places r.

Proof. By Lemma 5.1, we may apply a translation to assume without loss of generality that $v = 0$.

First, we claim that $K + K$ consists of all vectors which vanish on at least one support place p of $\mathscr{F}$. Indeed, suppose u and w are in K. Then u vanishes on a set A in $\mathscr{F}$, and w vanishes on a set B in $\mathscr{F}$. By Lemma 6.1, $A \cap B$ contains a support place p for $\mathscr{F}$. Clearly, $u + w$ vanishes at p.

Conversely, suppose z is a vector vanishing at some support place p of $\mathscr{F}$. Then, by definition of support, there are sets A and B in $\mathscr{F}$ with $A \cap B = \{p\}$. Thus $A \subseteq (C \backslash B) \cup \{p\}$. Since $\mathscr{F}$ is maximal, property (i) implies that $(C \backslash B) \cup \{p\}$ lies in $\mathscr{F}$. Without loss of generality, we may assume $A = (C \backslash B) \cup \{p\}$, so $A \cup B = C$. Define a vector u to be 0 on A and agree with z on B. Also, define a vector w to be 0 on B and agree with z on A. Note that these definitions are consistent since z is 0 on $A \cap B = \{p\}$ and are complete since $A \cup B = C$. Also, clearly $u + w = z$. Finally, note that $A \subseteq C(u, 0)$, so by property (i) of maximal Helly families, $C(u, 0)$ belongs to $\mathscr{F}$. Thus u is in $K = K(0, \mathscr{F})$. Likewise, w is in K. This establishes the claim.

Imagine now that the support places are listed first and that the nonsupport places are listed last. Each vector can be subdivided into a *head* vector containing the values on the support and a *tail* containing the nonsupport values. In this terminology, the claim shows that $K+K$ is the set of vectors whose heads contain at least one 0 and whose tails are arbitrary. Let M be the product of s_q over all support places q. There are then M possible head vectors. Exactly S of these do not vanish at all on the support, so $M-S$ vanish somewhere on the support. Also, there are exactly T tail vectors. A vector in $K+K$ is obtained by independently combining a head vanishing somewhere on the support with an arbitrary tail. The number of ways to do this is $(M-S)T = MT - ST = N - ST$. $\square$

As noted above, principal cliques are Helly cliques arising from the principal filter $\mathscr{P}$ consisting of all sets containing a fixed point p. This p is the unique support place of $\mathscr{P}$. When $d \geqslant 3$, a Helly family $\mathscr{F}$ with full support can be obtained from $\mathscr{P}$ just be replacing $\{p\}$ by its complement $C\backslash\{p\}$. Indeed, if q is any other place, then $\{p,q\}$ and $C\backslash\{p\}$ lie in $\mathscr{F}$ and intersect in q, so q is a support place. If $d \geqslant 3$, there are at least two other places, say q and r, besides p. Thus $\{p,q\}$ and $\{p,r\}$ are in $\mathscr{F}$ and intersect in $\{p\}$, making p a support place. Ironically, $\{p\}$ is the *only* set in $\mathscr{P}$ that can be exchanged with its complement to yield a new Helly family. These considerations prove the following:

Lemma 6.3. *For all $d \geqslant 3$, there is a Helly family on $C = \{1,2,3,\ldots,d\}$ whose support is all of C.*

A (finite) group is *Boolean* provided every element is its own inverse—or equivalently, the group is a direct product of copies of Z_2.

Theorem 6.4. *Let $G = A_1 \times A_2 \times \cdots \times A_d$ with stack sizes $2 \leqslant s_1 \leqslant s_2 \leqslant s_3 \leqslant \cdots \leqslant s_d$. If $d \geqslant 3$, then $S_2(G) \geqslant N - S$ where N is the product of s_p over all places p and S is the product of $s_p - 1$ over all places p. Moreover, equality holds if A_i is Boolean for all i or if $d = 3$.*

Proof. The required Helly bound $S_2(G) \geqslant N - S$ follows at once from Theorem 6.2 and Lemma 6.3.

Suppose all the stacks A_i are Boolean. Let U denote the set of all vectors in G which are never 0 in any coordinate. Then $|U| = S$. Let K be any clique in the Boolean factored group G. We will show that $K+K$ misses all points in U. Indeed, for $u \in U$, if $v + w = u$, then $w = -v + u = v + u$, since $v = -v$. Since u is not zero anywhere, $w = v + u$ disagrees with v in every place. Hence v and $w = v + u$ cannot lie in any clique together, so u cannot occur in any clique sum $K+K$.

Now suppose $d = 3$. Notice that $N - S$ counts the number of "singular" vectors in G—i.e., those which vanish in at least one place. Obviously, increasing a stack size will always allow more such vectors. Since the case $s_1 = s_2 = s_3 = 2$ was handled in the preceding paragraph, we may assume that $s_3 \geqslant 3$. Thus $N-S \geqslant 2\cdot2\cdot3-1\cdot1\cdot2=10$. Now the meet graph $MG(s_1,s_2,s_3)$ underlying G is a transitively 3-colored graph without clones. Hence any clique is principal, supertransitive, or the Fano clique. The Fano clique has only four points and hence can generate at most 10 sums—6 along edges and 4 by doubling vertices. Thus a Fano clique cannot beat the Helly bound. But principal cliques and supertransitive cliques lie inside Helly cliques by Theorem 5.3. Hence when $d=3$, the maximum value of $S_2(G)$ is attained by a Helly clique K, and by Theorem 6.2, $N-S$ is the maximum 2-sum set size for a Helly clique. □

Let us now apply these results to the numeric case.

Theorem 6.5. *For any integer n with at least three prime factors*, $S_2(n) \geqslant n-\phi(n)$. *Moreover*, *equality holds if n has exactly three prime factors.*

Proof. First consider the case that n is square-free. In this case, SING(n) is a factored abelian group by the Chinese Remainder Theorem. Examining the Helly bound in Theorem 6.4 in this case reveals that $N=n$ and $S=\phi(n)$, yielding the theorem in the square-free case. In general, write $n=mn'$ where n' is the square-free part of n. Since all prime divisors of m also divide n', we have $\phi(n)=m\phi(n')$. Thus using Theorem 3.2,

$$S_2(n)=S_2(mn')=mS_2(n') \geqslant m(n'-\phi(n'))=mn'-m\phi(n')=n-\phi(n)$$

as desired, and equality holds when n has exactly three prime factors. □

7. A characterization of maximal Helly cliques

In this section, it is convenient to think of a vector v in a meet graph as a set of ordered pairs $\{(i,v_i)\}$. This is what analysts would call the "graph" of the "function" $i \to v_i$. Thus the meet-graph literally becomes an intersection graph since v and w are adjacent iff they agree in some place p and that happens iff the ordered pairs (p,v_p), and (p,w_p) are the same—i.e., the graphs of v and w have nonempty intersection. We can even write $v\cap w$ meaningfully and say $v\cap w\neq\emptyset$ iff v and w are adjacent. Thus a clique K is just a Helly family of graphs. Unfortunately, we cannot say K is maximal iff the Helly family is maximal since supersets of function graphs are not function graphs. Nonetheless, a useful analogue of Lemma 6.1 does hold.

For any clique K, define $J(K):=\{v\cap w\colon v \text{ and } w\in K\}$. This is a family of (nonempty) sets of ordered pairs. Two vectors v and w are *tangent* iff $v\cap w$ is a singleton. The *support* of a clique K is the set of places at which pairs of vectors in K are tangent. That is, a place p is in the support of K iff there are two vectors v and w in K which agree at p and disagree elsewhere.

Lemma 7.1. *If K is a maximal clique in a meet graph*, *then every minimal member of* $J(K)$ *is singleton.*

Proof. Suppose v and w are vectors in K for which $v \cap w$ is minimal. By reordering coordinates, we may assume that v and w agree on $C(v,w) = \{1,2,3,\ldots,k\}$. By symmetry, we may assume for concreteness that v is all 0's and w is 0 on places 1 through k and 1 on places $k+1$ through d. Thus $v \cap w = \{(1,0),(2,0),\ldots,(k,0)\}$. We will show that $k > 1$ leads to a contradiction.

Define u to be 0 at place 1 and 1 elsewhere. Since $v \cap u = \{(1,0)\}$, the minimality of $v \cap w$ implies that u is not in the clique K. Since K is maximal, there is a vertex z in K not adjacent to u. This z is thus not 0 at the first place, and differs from 1 on all other places. Now w and z must agree somewhere since they are both in the clique K. But w is 0 at the first place, so 1 is not in $C(z,w)$. Furthermore, w is 1 from $k+1$ to d, so the places of agreement must be between 2 and k. Thus $z \cap w \subseteq \{(2,0),(3,0),\ldots,(k,0)\}$, contrary to minimality of $v \cap w$. □

Lemma 7.2. *Any two vectors v and w in a maximal clique K agree on at least one support place for K.*

Proof. $v \cap w$ lies in $J(K)$ and hence contains a minimal member of $J(K)$. By Lemma 7.1, this minimal member has the form $\{(p,a)\}$. Thus p is a support place by definition, and $v_p = a = w_p$ since the graphs of v and w each contain the "point" (p,a). □

Lemma 7.3. *If v lies in a maximal clique K and w agrees with v on the support of K, then w is in K.*

Proof. Let z be any vector in K. Then z and v agree on at least one support place, say p. Since v and w agree on the support of K, it follows that z and w also agree at p. Thus w is adjacent to every vector in K, so since K is a maximal clique, w must be in K. □

A support place p of a clique K is *conflicted* iff there are two tangent pairs of vectors in K, each of which agrees at p, but the two pairs do not take the same common value. That is, there are four vectors u, v, w, and z in K such that $u \cap v = \{(p,a)\}$ and $w \cap z = \{(p,b)\}$ but $a \neq b$. Let us say that K is *conflict-free* iff no support place of K is conflicted.

Theorem 7.4. *A maximal clique K is a Helly clique iff K is conflict-free.*

Proof. ($\rightarrow$) Suppose K is the Helly clique $K(0,\mathscr{F})$ where $\mathscr{F}$ is a Helly family in C. For any two vectors v and w in K, we then have

$$\emptyset \neq C(0,v) \cap C(0,w) \subseteq C(v,w).$$

The inclusion holds in general and the intersection is nonempty by the Helly property. In general $C(v,w)$ might be properly bigger than $C(0,v) \cap C(0,w)$. However, if v and

w are tangent, then $C(v,w)$ is singleton, so $C(0,v) \cap C(0,w) = C(v,w)$ is forced in that case. Thus, when v and w are tangent, both must assume the value 0 at the support place. This is independent of the tangent pair v and w, so the place where they are tangent is not conflicted. It follows that any Helly clique is conflict-free.

($\leftarrow$) Define a vector z as follows. At each support place p, choose a pair v and w tangent at p. Say, $v \cap w = \{(p, a_p)\}$. Assign the value a_p to z at the pth place. At the nonsupport places, assign arbitrary values for z. Now consider $\mathscr{F} := \{C(z,v)$: v is in $K\}$. We wish to show that $\mathscr{F}$ is a Helly family generating K.

By Lemma 7.1, for any two vectors x and y in K, there is a tangent pair s and t in K such that $s \cap t \subseteq x \cap y$. Now s and t agree at one place p in the support of K. Moreover, if $s \cap t = \{(p,b)\}$, the value b must be the same as a_p, the value assigned to z at p, because the place p is not conflicted. It follows that the ordered pair (p, a_p) belongs to the graph of all of the vectors s, t, x, y, and z. In particular, $p \in C(z,x) \cap C(z,y)$, so this intersection is nonempty. Hence $\mathscr{F}$ is a Helly family and $K(z, \mathscr{F})$ is a Helly clique. By definition, it contains K. Since K is maximal, it follows that $K = K(z, \mathscr{F})$, so K is a Helly clique as described. $\square$

In the setting of transitive edge-colorings, a pair of vertices in a clique is tangent iff the edge between them has a unique color. The support of a clique K is then the set of colors on all the monochromatic edges in K. However, the full structure of the meet-graph is needed to obtain the results above on maximal cliques. In general, maximal cliques may fail to have any support places.

8. Beating the Helly bound for nearly Boolean groups

Recall that a (finite) group is *Boolean* provided all of its (non-identity) elements are involutions—or equivalently, it is a direct product of copies of Z_2. Let G be the "nearly Boolean" factored group $G := A_1 \times A_2 \times \cdots \times A_{d-1} \times A_d$ where each A_i with $i \leqslant d-1$ has order 2 and A_d has order $s \geqslant 2$. Then G can be represented as a product $B \times A$ where B is the Boolean factored group $B := A_1 \times A_2 \times \cdots \times A_{d-1}$ and $A = A_d$. It will be convenient to think of the d-vectors which make up G as being of the form (v,a) where v is a $(d-1)$-tuple of 0's and 1's in B and a is an arbitrary element of A. We will refer to v as the *head* and a as the *tail* of (v,a). Let $2A$ denote the image of A under the doubling map $x \to 2x$. Notice that A is Boolean iff $|2A| = 1$. For G as above, we have $n := 2^{d-1}s$ and the Helly bound is given by $S_2(G) \geqslant n - 1 \cdot 1 \cdot \cdots \cdot 1 \cdot (s-1) = (2^{d-1}-1)s + 1$. If $A = A_d$ is also Boolean, then the Helly bound is best possible, as shown in Theorem 6.4. We now show that when A is not Boolean, the Helly bound can always be improved.

Theorem 8.1. *For $G = B \times A$, a nearly Boolean factored group as above with $d \geqslant 4$,*

$$S_2(G) = (2^{d-1} - 1)s + \min(2^{d-2} - 1, |2A|).$$

Proof. We begin by showing the inequality

$$S_2(G) \leqslant (2^{d-1}-1)s + \min(2^{d-2}-1,\ |2A|). \qquad (*)$$

Let u denote the vector of $d-1$ 1's in B, and partition G into $U:=\{u\}\times A$ and $Z:=G\backslash U$. Then $|U|=s$ and $|Z|=(2^{d-1}-1)s$. Now let K be any clique in G. We will divide our count of the distinct sums occuring in $K+K$ into two parts: those that lie in Z and those that lie in U. Notice that if $(u,c)=(v,a)+(w,b)$, then $w=u+v$, the Boolean complement of v. Thus (v,a) and (w,b) must disagree in the first $d-1$ coordinates. To be joined by an edge, they must agree in the last coordinate—that is, $a=b$ and $c=2a\in|2A|$. Let us call an edge *self-complementary* provided its endpoints have the form (v,a) and (w,b) where $v+w=u$. Then the above argument shows that in a clique K, the self-complementary edges form a matching and the sums in $(K+K)\cap U$ arise by adding the endpoints of self-complementary edges and have the form (u,x) for $x\in 2A$. Thus

$$|(K+K)\cap U| \leqslant \min(\mu(K),|2A|),$$

where $\mu(K)$ is the number of self-complementary edges in K. Since $|(K+K)\cap Z| \leqslant |Z|=(2^{d-1}-1)s$, inequality $(*)$ follows at once if either $\mu(K)\leqslant 2^{d-2}-1$ or $|2A|\leqslant 2^{d-2}-1$. Hence we now assume that $\mu(K)\geqslant 2^{d-2}$ and $s=|A|\geqslant|2A|\geqslant 2^{d-2}$.

Each self-complementary edge in K is determined by a pair of complementary head-vectors in the Boolean group B. Since B has order 2^{d-1}, it follows that $\mu(K)=2^{d-2}$ and K consists of 2^{d-1} vertices from G, all with distinct heads. The map $\gamma:(v,a)\to(v+u,a)$ is then an involutory automorphism of K with no fixed vertices and precisely the self-complementary edges left invariant. Let K^* denote the nonsimple graph obtained from K by deleting the self-complementary edges and adding a loop at each vertex. The sums of $K+K$ in Z are thus obtained by adding the endpoints of edges in K^*, where of course a loop corresponds to adding a vertex to itself. Notice that the sum of any edge (v,a) to (w,b) is the same as the sum of its image $(v+u,a)$ to $(w+u,b)$ under γ.

Since γ acts without fixed edges on K^*, the number of sums in $|(K+K)\cap Z|$ is at most half the number of edges in K^*. Now K itself has $2^{d-1}(2^{d-1}-1)/2$ edges; in forming K^*, 2^{d-2} self-complementary edges are removed and 2^{d-1} loops are added, yielding a total of 2^{2d-3} edges in K^*. Hence we get

$$|(K+K)\cap Z| \leqslant 2^{2d-4}\cdot < (2^{d-1}-1)2^{d-2}-1 \leqslant (2^{d-1}-1)s-1.$$

Therefore, the gain of an extra sum in U is (more than) balanced by the loss of sums from Z. Hence the desired bound $(*)$ holds.

It remains to construct a clique M achieving the bound $(*)$. Let $t=\min(2^{d-2}-1,\ |2A|)$, and let $a_1,a_2,\ldots,a_t$ be elements of A which have distinct doubles $2a_i$. Let $v_1,v_2,\ldots$ be a list of the *nonzero* vectors in B whose first coordinate is 0, and let z

denote the zero vector in B. Now let M consist of all vectors of the following three types:

(1) the s vectors in G with z as head;
(2) the $2t$ vectors (v_i, a_i) and $(v_i + u, a_i)$ for $i = 1, 2, \ldots, t$;
(3) the vectors $(v_i, 0)$ and $(v_i + u, 0)$ for $i = t + 1, \ldots, 2^{d-1} - 1$.

The t self-complementary pairs in (2) insure that $M + M$ will contain $t = \min(2^{d-2} - 1, |2A|)$ sums with head u. The vectors in (3) insure that all possible heads *except* u occur in M. Sums with head z can be obtained by adding type 1 vectors in M. Sums with head different from z and u can be obtained by adding type 1 vectors to vectors of types 2 and 3. Hence $M + M$ contains all of Z, and the bound $(*)$ is realized. □

When $d = 4$ and $s = 3$, the above theorem yields $S_2(Z_2 \times Z_2 \times Z_2 \times Z_3) = 24$, so $Z_2 \times Z_2 \times Z_2 \times Z_3$ has a singular sum cover. The results of [8] show that this is the only factored group of dimension $d = 4$ that has a singular sum cover. That is, this is the only case where $d = 4$ and $S_2(n) = n$.

If $d = 4$ and $s = 4$, there are two possibilities for A_4: either Z_4 or $Z_2 \times Z_2$. In the first case, $2A$ has two elements and in the second case only one. Hence Theorem 8.1 above gives different values for $S_2(G)$ for these cases. This is interesting because it is the first property of clique sums in factored groups thus far observed which depends on the actual structure of the groups and not just on their orders.

The construction in Theorem 8.1 depends so heavily on the Boolean part of the factored group that it seems plausible to make the following conjecture:

Conjecture. For $d = 4$, the Helly bound is best possible in the numeric case. That is, if n has exactly 4 distinct prime factors, then $S_2(n) = n - \phi(n)$.

However, this conjecture does not hold for $d = 5$. In fact, as will be shown elsewhere, the Helly bound can always be beaten when $d \geqslant 5$.

Note added in proof: Connections of 2-sum covers and the Helly bound to problems in finite geometry may be found in Mark Fitch and Robert E. Jamison, Minimal sum covers of small cyclic groups, Congr. Numer. 147 (2000) 65–81.

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Theorem 8.1 was originally given in a much weaker version. The author is grateful to the referees for suggesting that it could be extended.

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