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# Note

## Room Squares with Sub-squares

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Suppose there is a Room square of side $r$. Then there is a Room square of side $4r + 1$ with a sub-square of side $r$ which is isomorphic to the original square.

No example was known previously of a Room square of side $s$ with a sub-square of side $r$ where $s \leqslant 6r$.

### 1. ROOM SQUARES AND SUB-SQUARES

A *Room square* $\mathscr{R}$ of side $r$ based on $R$, where $R$ is a set of order $r + 1$, is an $r \times r$ array whose cells either are empty or contain an unordered pair of distinct elements of $R$, with the properties that $\mathscr{R}$ contains each such unordered pair precisely once, and that the entries in any given row or any given column of $\mathscr{R}$ contain every member of $R$ once. If $z$ is a distinguished member of $R$, we say $\mathscr{R}$ is *standardized with respect to* $z$ if the diagonal positions contain the entries $\{x, z\}$. It is possible to standardize any Room square by row and column permutations.

If there is a Room square of side $r$, then it is clear that $r$ must be an odd integer. Room squares of side $r$ have been constructed for every odd $r$ except 3, 5 and 257; sides 3 and 5 are impossible (see, for example, [5]), so that only side 257 remains in doubt.

Given a Room square $\mathscr{S}$ of side $s$ based on $S$, it may be that the entries common to some $r$ rows and $r$ columns form a Room square $\mathscr{R}$ of side $r$ based on a subset of $S$. We then say $\mathscr{R}$ is a subsquare of $\mathscr{S}$. To avoid a trivial case, we shall also demand that $r < s$. It is known [2] that, if there is a Room square of side $r_1$ and a Room square of side $r_2$ with a sub-square of side $r_3$, where $r_2 - r_3 > 6$, then there is a Room square of side $r_1(r_2 - r_3) + r_3$ with sub-squares of sides $r_1$, $r_2$ and $r_3$. Mullin [4] has extended this result to the case $r_2 - r_3 = 6$ for most values of $r_1$. Any Room square has sub-squares of sides 0 and 1.

The above results cannot be used to find a Room square of side $s$ with a sub-square of side $r$ unless $s \geqslant 6r + 1$. Lawless [3] has found a Room square of side 151 with a sub-square of side 25 using pairwise balanced designs. But no example seems to be known where $s \leqslant 6r$. On the other hand, the best-known lower bound was found in [1]: if there is a Room square of side $s$ with a sub-square of side $r$, then $s \geqslant 3r + 2$.

We shall prove the following theorem:

THEOREM 1. *If there is a Room square $\mathscr{R}$ of side $r$, $r > 1$, then there is a Room square of side $4r + 1$ with a sub-square of side $r$ isomorphic to $\mathscr{R}$.*

The proof of this Theorem is given in Section 2.

## 2. Proof of Theorem 1

Throughout this section we suppose $\mathscr{R}$ is a Room square of side $r$ based on the set $R = \{0, 1, 2,..., r\}$ and standardized with respect to 0. We further assume that $\mathscr{R}$ has $(x, x)$ entry $\{0, x\}$. (If necessary, this can be attained by permuting the labels $\{1, 2,..., r\}$.)

We shall denote by $R_i$ the set

$$R_i = \{1_i, 2_i, ..., r_i\},$$

so that $R_i$ has $r$ elements. We write $\mathscr{R}_{ij}$ for the array derived from $\mathscr{R}$ by first deleting all the diagonal entries from $\mathscr{R}$ and then replacing the pair $\{x, y\}$, where $x < y$, by $\{x_i, y_j\}$. It should be observed that the arrays $\mathscr{R}_{ij}$, where $i$ and $j$ each range from 1 to 4, contain between them all the unordered pairs of elements of $R_1 \cup R_2 \cup R_3 \cup R_4$ except for pairs of the form $\{x_i, x_j\}$; further, each row of the array

| | |
|---|---|
| $\mathscr{R}_{ij}$ | $\mathscr{R}_{jk}$ |

contains every member of $R_j$, except that $x_j$ is missing from the $j$-th row.

Assume $\mathscr{R}$ is of side $r > 1$. Then we can find permutations $\phi$ and $\psi$ on $\{1, 2,..., r\}$ with the properties that:

(i) the $(i, i\phi)$ and $(i, i\psi)$ cells of $\mathscr{R}$ are always empty;

(ii) $i\phi \neq i\psi$, $i\phi \neq i$ and $i\psi \neq i$ for any $i$.

(In fact, from [5], we can find a set of $\frac{1}{2}(r-1)$ permutations, any pair of which have these properties. Since $r > 1$ we know $r \geqslant 7$, so $\frac{1}{2}(r-1) \geqslant 3$.) Given two such permutations, consider the $4r \times 4r$ array

$$\mathcal{S} = \begin{array}{|c|c|c|c|} \hline \mathcal{R}_{22} & \mathcal{R}_{31} & \mathcal{R}_{13} & \mathcal{R}_{44} \\ \hline \mathcal{R}_{43}\phi & \mathcal{R}_{14} & \mathcal{R}_{32} & \mathcal{R}_{21} \\ \hline \mathcal{R}_{34}\phi & \mathcal{R}_{23} & \mathcal{R}_{41} & \mathcal{R}_{12} \\ \hline \mathcal{R}_{11}\psi & \mathcal{R}_{42} & \mathcal{R}_{24} & \mathcal{R}_{33} \\ \hline \end{array} .$$

($\mathcal{R}_{11}\psi$ means the array whose column $x\psi$ is column $x$ of $\mathcal{R}_{11}$; similarly for $\mathcal{R}_{34}\phi$ and $\mathcal{R}_{43}\phi$.) $\mathcal{S}$ contains all unordered pairs of entries from $R_1 \cup R_2 \cup R_3 \cup R_4$ except the $\{x_i, x_j\}$. Every member of the set appears once in each row and once in each column, except that each $x_i$ is missing from rows $x$, $r+x$, $2r+x$, and $3r+x$ and columns $r+x$, $2r+x$, and $3r+x$, and that

$x_2$ is missing from column $x$,

$x_1$ is missing from column $x\psi$,

$x_3$ and $x_4$ are missing from column $x\phi$.

To make $\mathcal{S}$ into a Room square of side $4r+1$ we must add entries to introduce the missing pairs and elements suitably, and also incorporate two new elements, 0 and $\infty$ say, and add an extra row and column to $\mathcal{S}$. To place the missing elements, we construct another array and superimpose it and $\mathcal{S}$. We shall write $\mathcal{D}_{ij}$ for the $r \times r$ array with entry $\{x_i, x_j\}$ in the $(x, x)$ position and all other cells empty. $\mathcal{B}_{ij}$ and $\mathcal{C}_{ij}$, respectively, shall be one-row and one-column arrays of size $r$ with $x$-th entry $\{x_i, x_j\}$. To avoid extra notation, we permit $i = 0$ or $i = \infty$ in $\mathcal{D}_{ij}$, with the understanding that $x_0 = 0$ and $x_\infty = \infty$ for every $x$. Finally we define an $r \times r$ array $\mathcal{A}$ with entries

$\{\infty, x_2\}$ in position $(x, x)$,

$\{0, x_1\}$ in position $(x, x\psi)$,

$\{x_3, x_4\}$ in position $(x, x\phi)$, for $x = 1, 2, \ldots, r$, and all other cells empty. If we write $\mathscr{T}$ for the $(4r + 1) \times (4r + 1)$ array

$\mathscr{T} =$

| | | | | |
|---|---|---|---|---|
| $\mathscr{A}$ | | | | |
| | $\mathscr{D}_{04}$ | | $\mathscr{D}_{\infty 1}$ | $\mathscr{C}_{23}$ |
| | $\mathscr{D}_{\infty 3}$ | $\mathscr{D}_{02}$ | | $\mathscr{C}_{14}$ |
| | $\mathscr{D}_{12}$ | $\mathscr{D}_{\infty 4}$ | $\mathscr{D}_{03}$ | |
| | | $\mathscr{B}_{13}$ | $\mathscr{B}_{24}$ | $\{0, \infty\}$ |

then the required Room square of side $4r + 1$ can be constructed by superimposing $\mathscr{S}$ on the first $4r$ rows and columns of $\mathscr{T}$. Because each $\mathscr{R}_{ij}$ has empty diagonal, and because the permutations $\phi$ and $\psi$ have properties (i) and (ii), this superimposition can be done without ever putting two pairs into the same cell. The entries common to rows $3r + 1$ to $4r$ and columns $3r + 1$ to $4r$ form a sub-square of side $r$ based on $\{0\} \cup R_3$, isomorphic to $\mathscr{R}$.

## References


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