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On the Construction of Handcuffed Designs

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A handcuffed design with parameters v, k, λ consists of a set of ordered k-subsets of a v-set, called handcuffed blocks; in a block $(a_1, a_2, ..., a_k)$ each element is assumed to be "handcuffed" to its neighbors. A block, therefore, contains $k - 1$ handcuffed pairs, the pairs being considered unordered. Each element of the v-set appears in exactly r blocks, and each pair of distinct elements of the v-set is handcuffed in exactly λ blocks of the design.

These designs have been studied recently by Hung and Mendelsohn [1], who construct a number of families of such designs by recursive methods. In this paper we show how difference methods can be applied to the construction of handcuffed designs. The methods are powerful, and a number of families of designs are constructed. A main new result is the determination of necessary and sufficient conditions for the existence of handcuffed designs for all parameter sets in which v is an odd prime power.

1. INTRODUCTION

The concept of handcuffed designs has recently been introduced by Hung and Mendelsohn [I], who have also proved a number of results concerning the existence of these designs. A handcuffed design with parameters v, k, λ consists of a set of ordered k-subsets of a v-set; the k-sets will be called handcuffed blocks. In a block (a, , a_2 ,..., a,) each element is said to be "handcuffed" to its neighbors, whence the block contains $k - 1$ handcuffed pairs $(a_1, a_2), (a_2, a_3), ..., (a_{k-1}, a_k)$. These pairs are considered unordered. A collection of b handcuffed blocks is said to form a handcuffed (v, k, λ) design if

(i) each element of the v-set appears in precisely r blocks (and at most once in a block), and

(ii) each (unordered) pair of distinct elements of the v-set is handcuffed in exactly λ of the blocks.

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It is readily shown [1] that the parameters of the handcuffed design satisfy

$$\lambda v(v-1)/2 = (k-1)b, \tag{1}$$

$$vr = bk. \tag{2}$$

In addition, it is easily shown [1] that any element of the u-set occurs in the interior (that is, not in the first or last position) of exactly u blocks, where

$$u + r = \lambda(v-1). \tag{3}$$

Given v, k, λ, (1), (2), and (3) are thus necessary conditions for the existence of a handcuffed design with these parameters. Solving for b, r, and u in terms of v, k, λ gives

$$b = \frac{\lambda v(v-1)}{2(k-1)}, \qquad r = \frac{k\lambda(v-1)}{2(k-1)}, \qquad u = \frac{\lambda(v-1)(k-2)}{2(k-1)}. \tag{4}$$

It is the purpose of this paper to show how difference methods can be applied to the construction of handcuffed (v, k, λ) designs. These methods are simple yet powerful; they produce somewhat simpler constructions of the designs constructed in [1], and yield many new designs besides. A new result proved in the paper is that, if v is an odd prime power, and v, k, λ satisfy relations (1), (2), (3), then a handcuffed (v, k, λ) design exists.

2. The Use of Difference Blocks in Constructing Handcuffed Designs

We present a simple theorem which allows the easy construction of many handcuffed designs. First we define one or two terms: Given an ordered sequence of elements $a_1, a_2, \ldots, a_k$ from a module M, we call the differences $a_2 - a_1, a_3 - a_2, \ldots, a_k - a_{k-1}$ *forward differences* and $a_1 - a_2, a_2 - a_3, \ldots, a_{k-1} - a_k$ ***backward differences***. We now present

Theorem 1. *Consider a module M with v elements. Suppose we can find m handcuffed blocks $B_1, \ldots, B_m$, each of size k, such that among the totality of $2m(k-1)$ forward and backward differences in the blocks each of the $v-1$ non-zero elements of M occurs exactly λ times. Then the set of $b = vm$ blocks of the form*

$$B_1 + \theta, B_2 + \theta, \ldots, B_m + \theta \qquad (\theta \in M)$$

is a handcuffed (v, k, λ) design.

Proof. Clearly each element of M will occur in exactly r = mk blocks. It only remains to check that each distinct pair of elements is handcuffed in exactly λ blocks of the design. Now x, y can be handcuffed together as either (x, y) or (y, x). A block will contain (x, y) if and only if some initial B_i contains (a_i, a_j) handcuffed (in this order), with x = $a_i + \theta$, $y = a_j + \theta$. This occurs for a unique $\theta \in M$ for a_i, a_j such that x $- y = a_i - a_j$. Similarly, a block will contain (y, x) if and only if some initial B_i contain (a_k, a_l) handcuffed with $y = a_k + \theta$, x = a, $+ \theta$; this occurs for a unique $\theta \in M$ for a,, a, such that y x = $a_k -$ a, . Hence, the total number of occurrences of (x, y) and $(y$, x) equals the number of times $\pm(x - y)$ appears as a forward difference among the initial blocks $B_1, \ldots, B_m$ or, equivalently, the number of times $(x - y)$ appears as a forward or backward difference. But this equals λ, and, hence, the theorem is proved.

COMMENT. A necessary condition for the application of the above theorem is evidently that

$$m = \frac{\lambda(v \quad 1)}{2(k - 1)} \tag{5}$$

be an integer. We will return to discuss this point below.

An example should make the construction clear.

EXAMPLE. We consider the case v = 10, k = 4, λ = 2. In this case we observe that m = (v 1)/2(k − 1) = 3. Consider the residue classes of integers 0, 1,..., 9 (mod 10), and the following three initial blocks:

$$B_1(0, 1, 9, 2), \qquad B_2(0, 6, 1, 5), \qquad B_3(0, 7, 9, 8).$$

The forward differences are +1, +8, +3, +6, +5, +4, +7, $2, $9, whence among the forward and backward differences each non-zero residue appears twice. The 30 blocks $(i, 1+i, 9+i, 2+i)$, $(i, 6+i, 1+i, 5+i)$, $(i, 7+i, 9+i, 8+i)$, $i = 0, \ldots, 9$, then form the desired handcuffed (10, 4, 2) design.

In the rest of the paper we employ the difference block method in constructing various families of handcuffed designs. First, we note one or two points concerning the parameters in a handcuffed (v, $k, A)$ design.

Since r $- u$ is an integer, we have from (4) that

$$n = r - u = \frac{\lambda(v - 1)}{k - 1} \tag{6}$$

must be integral. We can now write b, r, u as

$$b = \frac{nv}{2}, \qquad r = \frac{nk}{2}, \qquad u = \frac{n(k-2)}{2}. \tag{7}$$

In particular, we note that, when n is even, the parameters of the design satisfy the necessary relation (5) (with $m = n/2$) for the application of the difference method. When n is odd, such is not the case; nevertheless, as we show below, modifications of the difference method can be applied in this situation.

3. Handcuffed Designs with $\lambda = 1$

Hung and Mendelsohn show in [1] that, when $\lambda = 1$, the necessary conditions (1), (2), (3) for the existence of a handcuffed design are also sufficient. They give recursive methods of construction for such designs. In this section we give a somewhat shorter proof of this result via difference methods. We consider two cases, depending on whether k is odd or even:

Case 1. *k odd.* Let $k = 2h + 1$; the relations (1)–(4) imply that $v \equiv 1 \pmod{4h}$. In this case we write the parameters of the design as $v = 4hm + 1$, $k = 2h + 1$, $\lambda = 1$, $b = m(4hm + 1)$, $r = m(2h + 1)$. Noting that $\lambda(v-1)/2(k-1) = m$, we can attempt to find a handcuffed $(4hm + 1, 2h + 1, 1)$ design by constructing m initial difference blocks of size $2h + 1$. Consider m blocks $B_1, \ldots, B_m$, with the following $2h$ successive *forward differences* in each block:

$$B_1\colon +1, -2, +3, -4, \ldots, -2h$$
$$B_2\colon +(2h+1), -(2h+2), \ldots, -4h$$

$$B_m\colon +[2(m-1)h+1], -[2(m-1)h+2], \ldots, -2mh.$$

The actual blocks are to be formed by starting with an arbitrary integer (mod v) and then forming successive elements in the block from the given forward differences.

We first of all observe that among the totality of $4mh$ backward and forward differences (the backward differences just being the negatives of the forward differences) each integer among $1, 2, \ldots, 4mh$ appears exactly once. Hence, blocks $B_1, \ldots, B_m$ constructed from these differences will by Theorem 1 yield the desired handcuffed design, provided that $B_1, \ldots, B_m$ formed from the differences given are in fact proper blocks; that is, they must consist of $2h + 1$ *distinct* elements. This is easily seen to be the case for, if some element appears twice in a particular block B_i, this implies that some sum of x ($x \leqslant 2h$) consecutive forward differences in one of the B_i's must be zero. But this is clearly impossible; in fact,

$$a - (a + 1) + (a + 2) - \cdots *(a + x - 1) = -\frac{x}{2} \text{ (x even)}$$
$$= a + \frac{x-1}{2} \text{ (x odd).}$$

Hence, initial blocks $B_1, \ldots, B_m$ formed as described satisfy the requirements of Theorem I, and we have proved

THEOREM 2. *A handcuffed* $(v, k, 1)$ *design exists with* $k = 2h + 1$ *if and only if* $v \equiv 1 \pmod{4h}$.

This is also Theorem 2 of [1].

EXAMPLE. Consider $v = 17$, $k = 5$, $\lambda = 1$. According to the above procedure we form initial blocks B_1, B_2 from the forward differences $+1$, -2, $+3$, -4 for B_1, and \$5, -6, $+7$, -8, for B_2. The actual initial blocks can then conveniently be taken as $B_1(0, 1, 16, 2, 15)$, $B_2(0, 5, 16, 6, 15)$, and the 34 blocks of the design are obtained by developing B_1 and B_2 modulo 17.

Case 2. *k even.* Let $k = 2h$; then (l)-(4) imply that $v \equiv 1 \pmod{2h - 1}$. We, therefore, have $v = m(2h - 1) + 1$, $k = 2h$, $\lambda = 1$, $b = mv/2$, $r = mk/2$. We consider two subcases, according to whether m is odd or even:

(i) *m even* (v *odd*). In this case we can attempt to construct $m/2$ initial difference blocks of size $2h$, as required by Theorem 1. We form blocks $B_1, \ldots, B_{m/2}$, containing successive forward differences as follows:

$$B_1 : +1, -2, +3, \ldots, +(2h - 1)$$
$$B_2 : -2h, +(2h + 1), \ldots, -(4h - 2)$$
$$\vdots$$
$$B_{m/2} : \cdots \pm \frac{m}{2}(2h - 1).$$

In a manner similar to that employed in Case 1 above, we can readily show that blocks $B_1, \ldots, B_{m/2}$ constructed from the above forward differences satisfy the conditions of Theorem 1, and, hence, that these initial blocks yield the desired handcuffed design.

(ii) *m odd* (v even). In this case the necessary condition for the application of Theorem 1 does not hold, m/2 not being integral. However, a modification of the difference method can be used as follows: Let $m = 2t + 1$; then we have $v - 1 = 2t(2h - 1) + (2h - 1)$ and $b = vt + v/2$. We can construct $t + 1 = (m + 1)/2$ initial blocks, with the following successive forward differences:

B_1: $+1, -2, \ldots, +(2h - 1)$
B_2: $-2h, +(2h + 1), \ldots, -(4h - 2)$

B_t: $\pm[(t - 1)(2h - 1) + 1], \ldots, \pm(2h - 1)$
B_{t+1}^*: $+[t(2h - 1) + 1], -[t(2h - 1) + 2], \ldots, +(t + 1)(2h - 1)$.

Consider initial blocks $B_1, \ldots, B_t$, B_{t+1}^* formed to have these forward differences. Consider the $b = vt + v/2$ blocks of the form $B_1 + i$, $B_2 + i, \ldots, B_t + i$ $(i = 0, 1, \ldots, v - 1)$, $B_{t+1}^* + j$ $(j = 0, 1, \ldots, (v - 2)/2)$. That is, we use B_{t+1}^* to form only $v/2$ blocks. We now claim that the design so formed is a handcuffed design with the stated parameters.

First, it is clear from their method of construction that each of the initial blocks $B_1, \ldots, B_t$ contain $2h$ distinct elements. This is also seen to be true of B_{t+1}^*; note that the $(2h - 1)$ forward differences in B_{t+1}^* are in fact $[v/2 - (h - 1)], -[v/2 - (h - 2)], \ldots, [v/2 + (h - 1)]$. We now need to check that every element pair x, y is handcuffed exactly once in the design. Following the lines of proof of Theorem 1, we note that any two elements x, y which differ by $\pm 1, \pm 2, \ldots, \pm t(2h - 1)$ will be handcuffed exactly once, in one of the blocks derived from one of $B_1, \ldots, B_t$. Consider now two elements x, y such that $x - y = \pm(v/2 - i)$, where $0 \leqslant i \leqslant h - 1$. These differences do not appear in $B_1, \ldots, B_t$, but do appear in B_{t+1}^*; in fact, they each occur twice among the forward and backward differences in B_{t+1}^*, since $(v/2 - i) = -(v/2 + i)$. Let the actual elements in B_{t+1}^* be $(a_1, a_2, \ldots, a_k)$. If B_{t+1}^* were used in the usual way to give v blocks, x and y would appear handcuffed exactly twice. They would appear once in the order (x, y) in positions $(i, i + 1)$ of the θ-th block derived from B_{t+1}^*, where $a_{i+1} - a_i$ is the unique forward difference in B_{t+1}^* equal to $y - x$, and where $\theta = x - a_i$. **Also,** x and y would appear once in the order (y, x), in positions $(j, j + 1)$ of the θ'-th block, where $a_{j+1} - a_j$ is the unique forward difference equal to $x - y$, and $\theta' = y - a_j = x - a_{j+1}$. Now, from the construction of B_{t+1}^*, it can be noted that in this case $(a_j, a_{j+1}) = (a_{i+1} + v/2, a_i + v/2)$. This implies that $\theta' = x - a_{j+1} = x - a_i - v/2 = \theta - v/2$. Hence, if $0 \leqslant \theta \leqslant v/2 - 1$, then $v/2 \leqslant \theta' \leqslant v - 1$, and vice versa. Therefore, if we form only the $v/2$ blocks $B_{t+1}^* + l$ $(l = 0, 1, \ldots, v/2 - 1)$, x and y will occur in only one of the orders (x, y) or (y, x) and, hence, be handcuffed precisely once.

Hence, the design formed is, in fact, the desired type of handcuffed design.

An example should make the construction clear.

EXAMPLE. Consider $v = 10$, $k = 4$, $\lambda = 1$, whence $b = 15$ and $m = 3$, $h = 2$ in our above notation. We form initial blocks B_1 and $B_2{}^*$ with successive forward differences $+1, -2, +3$ and $+4, -5, \$6$, respectively. The actual initial blocks can be taken as $B_1(0, 1, 9, 2)$, $B_2{}^*(0, 4, 9, 5)$. The blocks $(i, 1 + i, 9 + i, 2 + i)$, $i = 0, \ldots, 9$ and $(j, 4 + j, 9 + j, 5 + j)$, $j = O, \ldots, 4$, then yield the desired handcuffed design.

The above construction allows us to state

THEOREM 3. *A handcuffed $(v, k, 1)$ design exists with $k = 2h$ if and only if $v \equiv 1 \pmod{2h - 1}$.*

This is also Theorem 3 of [I].

4. HANDCUFFED DESIGNS WITH $\lambda \geqslant 2$

We can also construct designs for the case $\lambda = 2$ in a straightforward manner, by using difference blocks. We consider only the case in which k is odd here, since it can be shown (see [I]) that designs with k even and $\lambda = 2$ can always be found by writing down twice a design with the same k values, and $\lambda = 1$.

Suppose then that $k = 2h + 1$; then (1)–(4) imply that $v \equiv 1 \pmod{2h}$. Hence, we have parameters $v = 2hm + 1$, $k = 2h + 1$, $\lambda = 2$, $b = mv$, $r = mk$, and this suggests the possible construction of the desired designs by forming m initial blocks of size k, satisfying the conditions of Theorem 1. Consider the m blocks, with the following successive forward differences:

$$B_1\colon +1, -2, +3, \ldots, -2h$$

$$B_2\colon (2h + I), -(2h + 2), \ldots, -4h$$

$$B_m\colon 2(m - 1)h + I, \ldots, -2mh.$$

It is clear that blocks $B_1, \ldots, B_m$ will each contain $2h + 1$ distinct elements. Further, the forward and backward differences taken over all of $B_1, \ldots, B_m$ include each non-zero residue, modulo v, exactly twice. Hence, by Theorem 1 we have proved

THEOREM 4. *A handcuffed $(v, k, 2)$ design with $k = 2h + 1$ exists if and only if $v \equiv I \pmod{2h}$.*

This is also Theorem 5 of [1].

Hence, the existence problem for handcuffed designs with $\lambda = 1$ or 2 is completely settled. We remark that results contained in Theorems 2 to 4, along with a little simple number theory, are used in [I] to also show that the necessary conditions (I), (2), for the existence of a handcuffed (v, k, λ) design are also sufficient in the following other cases: $k = 3$; k even and $(\lambda, k - 1) = 1$; k odd and $(\lambda, 2k - 2) = 2$.

It can be noted that, although the designs constructed in this and the preceding section have the same parameters as those constructed in [1], the designs here and in [1] are not generally isomorphic.

5. Handcuffed Designs with $v = p^e$, p Odd

We now consider the case in which v is an odd prime power, and show that the necessary conditions (1), (2), (3) for the existence of a handcuffed design are sufficient in this case as well.

Suppose v is an odd prime power. Since v is odd, we have from (7) that $n = \lambda(v \quad 1)/(k - 1)$ must be even, whence $m = \lambda(v \quad 1)/2(k \quad 1)$ is integral. This suggests the possibility of constructing a design through m initial blocks of size k. We now show that this can be done, regardless of the value of k and λ, as long as v, k, λ satisfy (1), (2), (3).

We assume in the construction below that $k <$ v. Let x be a primitive element in $GF(v)$. We form the following $n/2$ initial blocks:

$$B_1\colon 1, x, x^2, \ldots, x^{k-1}$$
$$B_2\colon x^{k-1}, x^k, \ldots, x^{2k-2}$$
$$B_3\colon x^{2k-2}, x^{2k-1}, \ldots, x^{3k-3}$$

$$B_{n/2}\colon x^{(k-1)(n/2-1)}, \ldots, x^{(k-1)n/2}.$$

Since $k <$ v, and 1, $x, \ldots, x^{v-2}$ give all the non-zero elements in $GF(v)$, it is clear no two elements in any block are the same. We need now show that among the forward and backward differences of $B_1, \ldots, B_{n/2}$, each non-zero element of $GF(v)$ occurs exactly λ times. To show this we distinguish two cases:

Case 1. λ even. In this case we note that $(k - 1)n/2 = (v - 1)\lambda/2$; hence, the final entry in $B_{n/2}$ is $x^{(v-1)\lambda/2} = 1$. The $(k - 1)n/2$ forward differences from the $n/2$ blocks are, therefore, $x^1 - x^0$, $x^2 - x^1, \ldots,$ $x^{v-1} - x^{v-2}$, h/2 times each. That is, the forward differences are (x — I), x(x $-$ 1), $x^2(x - 1),\ldots, x^{v-2}(x - 1)$, X/2 times each. These constitute all the non-zero elements of GF(v) h/2 times each, and hence the forward and backward differences together constitute all the non-zero elements of GF(v) λ times each.

Case (2). λ odd. In this case we have

$$\frac{n}{2}\,(\mathrm{k} - 1) = \left(\frac{\lambda - 1}{2}\right)(v - 1) + \left(\frac{v - 1}{2}\right).$$

Hence, the final entry in $B_{n/2}$ is $x^{(v-1)/2}$. The forward differences from the n/2 blocks, therefore, consist of

(a) (x $-$ I), x(x $-$ 1), $x^2(x - 1),\ldots, x^{v-2}(x - 1)$, $(\lambda - 1)/2$ times each, and

(b) (x $-$ 1), x(x $-$ 1) ,..., $x^{(v-3)/2}(x - 1)$, once each.

The differences in (a) and their negative values constitute all the non-zero elements of GF(v) $\lambda - 1$ times each. Further, we observe that among the values in (b) and their negatives each non-zero element of GF(v) occurs exactly once (note that if $x^i(x - 1) = -x^j(x - 1)$, with $0 \leqslant i \leqslant (v \quad 3)/2$, $0 \leqslant j \leqslant v - 2$, then since $j - i = (v - 1)/2$, *we* have $(v - 1)/2 \leqslant j \leqslant v - 2$). Hence, among the totality of forward and backward differences in the blocks, each non-zero element of GF(v) appears exactly λ times.

In virtue of Theorem 1, we now have proved

THEOREM 5. ***If v is an odd prime power and v > k, then a handcuffed (v, k, λ) design exists*** *for* ***all v, k, λ satisfying the basic relations (I),*** (2), ***(3).***

The condition $v > k$ in Theorem 5 is easily removed. Note that, if $v = k$, then from (4) we have $b = \lambda v/2 = r$; since v is odd, λ must be even. However, a design with the given value of v and having $\lambda = 2$ exists, by Theorem 4. Hence, such designs can be found for all even $\lambda = 2t$, by simply writing down the design for $\lambda = 2$ a total of t times.

6. HANDCUFFED DESIGNS WITH v = 2^t

The construction of the previous section can also be applied in some situations with $v = 2^t$. Suppose that v = 2^t and that $n = \lambda(v - 1)/(k - 1)$ is even. Since v   1 is odd, λ is, therefore, even as well. Assume $k < v$

and let x be a primitive element in $GF(2^t)$ and consider the n/2 initial blocks:

$$B_1\colon 1, x, x^2, \ldots, x^{k-1}$$
$$B_2\colon x^{k-1}, x^k, \ldots, x^{2k-2}$$

$$B_{n/2}\colon x^{(k-1)(n/2-1)}, \ldots, x^{(k-1)n/2}.$$

We note that $x^{(k-1)n/2} = x^{(v-1)\lambda/2} = 1$ whence among the forward differences of $B_1, \ldots, B_{n/2}$ each non-zero element of $GF(2^t)$ occurs exactly X/2 times. Hence, among the forward and backward differences each non-zero element of $GF(2^t)$ appears λ times, and by Theorem 1, these initial blocks lead to a handcuffed (v, k, λ) design. We have, therefore, proved

THEOREM 6. *If $v = 2^t > k$ and $\lambda(v-1)/(k-1)$ is even, then a handcuffed (v, k, λ) design exists for all v, k, λ satisfying* (1), (2), (3).

As in the case of Theorem 5, the restriction $v > k$ is easily removed. If $\lambda(v-1)/(k-1)$ is even, then if $v = k$, λ is even. The desired design exists for $\lambda = 2$ by Theorem 4, and the design for k, $\lambda = 2t$ can be obtained by writing down the design for k, $\lambda = 2$ a total of t times.

7. Related Problems and Final Comments

Some work has been done on a number of other problems similar to those discussed here, mainly in connection with the design of experiments. For example, Williams [5] considers the construction of designs with v elements in complete blocks of size $k = v$, such that each element precedes every other element the same number of times, say λ'. Patterson and Lucas [3] present similar designs, constructed mainly by trial and error for use in agricultural experiments, for cases in which $k < v$. It is clear that a design of this type with parameters v, k, λ' is also a handcuffed design with parameters $v, k, \lambda = 2\lambda'$.

Another type of design which has been of some use in experimental design is the so-called "neighbor" design. These are similar to handcuffed designs except that the blocks are considered to be circular. That is, in the block $(a_1, a_2, \ldots, a_k)$, a_k and a_1 are also considered to be handcuffed (or said to be neighbors), as well as a_1 and a_2, a_2 and a_3, etc. These designs have been considered by Rees [4], who provides difference-type solutions for small values of v and k, found mainly by trial and error. Lawless [2] has also considered this type of design, with the additional requirement that the design be a balanced incomplete block design. These designs are, like handcuffed designs, also of some interest in connection with the concept of a block design on a graph (see [1]).

As a final comment, we remark that it would be of great interest to try to determine precisely for what sets of parameter values handcuffed designs exist. A large number of these designs have now been constructed, and a reasonable attack might be made on the complete problem.

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