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When every projective module is a direct sum of finitely generated modules ☆

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Abstract

We characterize rings over which every projective module is a direct sum of finitely generated modules, and give various examples of rings with and without this property.

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1. Introduction

The study of finitely generated projective modules is a classical theme in module theory. These modules occur in Morita-type theorems, but also provide rich connections with K-theory, topology, and algebraic geometry. In contrast, non-finitely generated projective modules have drawn little attention. Nevertheless, many classical theorems have something to say about the general (not necessarily finitely generated) case. For instance, Kaplansky [27] showed that every

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projective module over a local ring is free. Later Bass [5] proved that every non-finitely generated projective module over an indecomposable commutative noetherian ring is free. Therefore (and partly due to Bass's discouraging remark [5, p. 24]) the theory of non-finitely generated projective modules has for a long time *'invited little interest.'*

It turns out that it is the opposite reason that may be somewhat discouraging the investigation of non-finitely generated projectives, namely the existence of 'very bad' (i.e., interesting!) examples of such modules, that is, examples that are far from being free. This can happen even when all finitely generated projectives are free. Indeed, this is the case over some classes of semilocal rings, see e.g. [39, Ex. 14.27] or [40]. Another example of this kind (due to Kaplansky) is given by the ring $C([0,1])$ of continuous real-valued functions on the interval $[0,1]$. Every finitely generated projective module over this ring is free, and yet there is a non-finitely generated indecomposable projective. See Section 9 for a complete decomposition theory of projective ideals of $C([0,1])$.

The other extreme is better known—when non-finitely generated projectives are, though possibly not free, yet more transparent than the finitely generated ones. For example, Levy and Robson [30] classified non-finitely generated projective modules over hereditary noetherian prime rings.

Note that even free modules can be, from a decomposition theory point of view, highly nontrivial. Indeed, passing from a module M to its endomorphism ring S, we obtain the free S-module S with the same decomposition theory as M. Since direct sum decompositions of M can be arbitrarily bad (see [14] for examples), it is a hopeless task to classify projective modules in general (see also [36] for a different approach).

In this paper we investigate the question of when every projective right module is a direct sum of finitely generated modules. An important result of this sort is due to Albrecht [3]: every projective right module over a right semihereditary ring is a direct sum of finitely generated modules each isomorphic to a right ideal of the ring (and this is an essential ingredient in the aforementioned Levy and Robson classification). Bass [6] proved the same for projective left modules over right semihereditary rings—with right ideals replaced by their duals. Later, Bergman [8] generalized both theorems by proving that every projective module over a weakly semihereditary ring is a direct sum of finitely generated modules.

Müller [35] showed that every projective module over a semiperfect ring is a direct sum of finitely generated modules. Generalizing this, Warfield [50] proved that every projective right module over an exchange ring is a direct sum of principal right ideals generated by idempotents, cf. Fact 3.6 below.

We give a precise criterion, Theorem 4.2, for every projective right module to be a direct sum of finitely generated modules. Roughly speaking, this criterion says that the set of idempotent matrices over the underlying ring is 'dense,' that is, 'meets' every *stable* sequence of rectangular matrices; see Section 4 for the precise formulations. Using this criterion we give a new proof of Bergman's aforementioned result, in Section 5. As another positive result, we prove, in Section 6, that every projective module over a principal right ideal ring is a direct sum of finitely generated modules.

All this rests on the fundamental observation (going back in part to Whitehead [51]) that every stable sequence $\{A\}$ of matrices leads to a countably generated projective module $P\{A\}$, and that, conversely, every countably generated projective module has this form.

In general, the matrices occurring in the criterion may have arbitrarily large size. Corollary 7.5 shows that this problem is essential. Curiously, to prove this negative fact we have to invoke some positive results like the following. If a ring is embeddable in the endomorphism ring of a finite length module, and $\{A\}$ is a stable sequence with uniformly bounded sizes of matrices, then $P\{A\}$ is a direct sum of finitely generated modules. The same is true, Proposition 7.4, for projective modules over rings with one-sided Krull dimension. (Note, this includes one-sided noetherian rings.)

Even for noetherian rings, our general question is widely open. We know the answer is positive in the commutative case and when the ring is simple, as can be derived from results of Bass, see Section 3. Generalizing the former, we draw, among other things, the following corollary of a result of Hinohara [23]: every projective module over a weakly noetherian commutative ring is a direct sum of finitely generated modules, Fact 3.1.

On the other hand, it follows from Akasaki [2] and Linnell [31] that over the integral group ring $\mathbb{Z}A_5$, there is a projective module with no finitely generated direct summands. Localizing this ring, we obtain a semilocal noetherian ring finite over its center with a projective module that is not a direct sum of finitely generated modules, Example 3.2. (It is this example that is later used to show that one cannot, in general, restrict the size of matrices in Theorem 4.2 (the criterion), see Corollary 7.5.)

Using a standard trick, the aforementioned example allows us to derive the existence of a cyclic artinian module M and a direct summand N of $M^{(\omega)}$ such that N has no finitely generated direct summands, Proposition 3.3.

In Section 8 we show that over a left Bézout ring one does not have to deal with the growing size of matrices: it suffices to consider ring elements in the criterion. In Corollary 8.2 we prove that every projective right module over a Bézout ring with one-sided Krull dimension is a direct sum of principal right ideals generated by idempotents. In Proposition 8.4, we identify the commutative Bézout rings for which every projective module is a direct sum of finitely generated modules as the so-called f-rings studied by Vasconcelos [49] and Jøndrup [25]. As a corollary we deduce that every projective module over a commutative Bézout ring of finite Goldie dimension is a direct sum of finitely generated modules. In contrast, we give an example of an indecomposable commutative Bézout ring of weak dimension 1 with a projective module without indecomposable direct summands, Example 9.21.

In general there are no implications (except the trivial ones) between the aforementioned theorems of Warfield, Bergman, Bass, Albrecht, and Hinohara. Restricted to the rings $C(X)$ of continuous real valued functions on a topological space X, however, these results fall into a linear hierarchy as follows.

Warfield's theorem turns out to be the most general one in this context: every projective $C(X)$-module is a direct sum of finitely generated modules iff $C(X)$ is an exchange ring iff X is strongly zero-dimensional. Bergman's result is less general: in Proposition 9.13 we prove that $C(X)$ is weakly semihereditary iff X is a strongly zero-dimensional F-space. In turn, Albrecht's and Bass' theorems are even more restrictive: a ring $C(X)$ is semihereditary iff X is basically disconnected. (For the special case of $C(X)$, this is due independently to Brookshear [9] and De Marco [13].) Finally, the least general one is Hinohara's theorem: the ring $C(X)$ is weakly noetherian iff X is a finite discrete space.

We set the scene by listing, in Section 2, a number of preliminaries needed in the sequel and by reviewing, in Section 3, some of the known results about our main question.

A number of open questions are scattered throughout the paper. For instance, we do not know if every projective module over a commutative domain is a direct sum of finitely generated modules, Question 3.10.

2. Preliminaries

Module, if not specified otherwise, means right module over a ring with unity. Endomorphisms are written on the opposite side of scalars. So, a (right) module M will also be considered as a left module over its endomorphism ring $\mathrm{End}(M)$.

A module is said to be *projective* if it is a direct summand of a free module. By Kaplansky [27, Theorem 1], every projective module is a direct sum of countably generated (projective) modules.

Let M be a right module over a ring R. Then $\mathrm{Add}(M)$ will denote the full subcategory of the category of right R-modules whose objects are direct summands of direct sums of copies of M. For instance, if $M = R_R$, then $\mathrm{Add}(M)$ is the category of projective right R-modules. The following fact shows how to convert $\mathrm{Add}(M)$ into a category of projective modules.

Fact 2.1. *(See [15, Theorem 4.27].) Let M be a finitely generated right module and $S = \mathrm{End}(M)$. Then there is a natural equivalence between* $\mathrm{Add}(M)$ *and the category of projective right S-modules.*

Applying this to the R-module eR, where e is an idempotent, we obtain an equivalence between $\mathrm{Add}(eR)$ and the category of projective right modules over $\mathrm{End}(eR) = eRe$.

Let P be a right R-module. The *trace* of P, $\mathrm{Tr}(P)$, is the sum of all images of morphisms $P \to R_R$. Clearly $\mathrm{Tr}(P)$ is a two-sided ideal of R. Note, if $e \in R$ is an idempotent, then eR is a projective right R-module, and $\mathrm{Tr}(eR) = ReR$ is the two-sided ideal of R generated by e, see e.g. [29, 2.41].

Fact 2.2. *(See [29, Proposition 2.40].) If P is a projective R-module, then $T = \mathrm{Tr}(P)$ is an idempotent ideal and $P = PT$.*

Whitehead proved a weak converse of this.

Fact 2.3. *(See [51].) Every idempotent ideal of a ring R which is finitely generated as a left ideal is the trace ideal of a countably generated projective right R-module.*

An R-module P is said to be a *generator* if $\mathrm{Tr}(P) = R$. This is the same as saying that, for some k, there is a morphism from P^k onto R_R, that is, R_R is a direct summand of P^k. It is immediate from what was said before Fact 2.2 that a right ideal eR generated by an idempotent e is a generator iff $ReR = R$.

A submodule M of a right module N is said to be *pure*, if for every (finitely presented) left module K, the induced map $M \otimes K \to N \otimes K$ is a monomorphism. Note, a right ideal I is pure in the ring R iff for every $r \in I$ there is $s \in I$ such that $sr = r$.

Fact 2.4. *Let R be a commutative ring.*

(1) [49, pp. 269–270] *If P is a projective R-module, then* $\mathrm{Tr}(P)$ *is a pure ideal.*
(2) [13, Proposition 1.14] *If P is a projective ideal of R, then* $\mathrm{Tr}(P)$ *is a projective and pure ideal.*

These statements are no longer true in the non-commutative setting. For instance, if R is the integral group ring of the alternating group A_5, then the augmentation ideal, I, of R is idempotent by [32]. Then Fact 2.3 shows that I is the trace of a countably generated projective right R-module. If I were pure, it would be projective, for I_R is finitely generated. Then it would be generated by an idempotent, which is impossible, as R has no non-trivial idempotents.

Following Vasconcelos [49, p. 274], a commutative ring R is said to be an *F-ring*, if every finitely generated flat R-module is projective. For instance, every commutative semilocal ring is an F-ring, and so is every commutative ring of finite Goldie dimension. Note, the F-rings are precisely the commutative S-rings from [41].

Fact 2.5. *(See [48, Theorem 2.1].) A commutative ring R is an F-ring if and only if every projective ideal of R is finitely generated.*

It is well known that the trace of a finitely generated projective module over a commutative ring is generated by an idempotent, see [29, 2.43]. Over F-rings the same is true for *arbitrary* projective modules, as the following lemma shows.

Lemma 2.6. *Let P be a projective module over a (commutative) F-ring R. Then* $\mathrm{Tr}(P) = eR$ *for some idempotent e.*

Proof. By Fact 2.4, $\mathrm{Tr}(P)$ is a pure ideal of R, hence $R/\mathrm{Tr}(P)$ is a flat cyclic module. By hypothesis, $R/\mathrm{Tr}(P)$ is projective, hence $\mathrm{Tr}(P)$ is a direct summand of R. $\square$

More on F-rings can be found at the end of Section 4.

A ring is said to be *semilocal* if the factor ring by its Jacobson radical is semisimple artinian. In the commutative case, this is equivalent to having finitely many maximal ideals. Recall that a ring is *semiperfect* iff it is semilocal and idempotents lift modulo the Jacobson radical. The following is some sort of folklore (though not necessarily easy to find in the literature).

Fact 2.7. *Every ring that is, as a right module, finitely generated over a semilocal subring, is itself semilocal.*

Proof. Let R be the ring in question and D a semilocal subring. By [15, Proposition 2.43], a ring is semilocal iff, as a one-sided module, it has finite dual Goldie dimension. Hence D_D has finite dual Goldie dimension. Then, using the hypothesis and [15, Proposition 2.42], one infers that R_D has finite dual Goldie dimension. Since the lattice of submodules of R_R is a sublattice of that of submodules of R_D, the module R_R has finite dual Goldie dimension all the more so. Consequently, R is semilocal, as desired. $\square$

Often we consider projective modules from a model-theoretic point of view. For basic notions in the model theory of modules, see [38] or [42]. In fact, all we need in this paper is the notion of divisibility formula and that of pp-type. A (right) *divisibility* formula (over the ring R) is a formula of the form $\exists \bar{y}\ (\bar{y}A = \bar{x})$, where A is an $l \times k$ matrix over R and $\bar{x} = (x_1, \ldots, x_k)$ and $\bar{y} = (y_1, \ldots, y_l)$ are matching tuples of variables. (The x_i are also known as the *free* variables while the y_i are the *bound* or *existentially quantified* variables of that formula.) We use the shorthand $A \mid \bar{x}$ ('A divides $\bar{x}$') for this formula. In particular, if $a \in R$, then $a \mid x$ is the formula

$\exists y\ (ya = x)$. It may suffice for the purpose of this paper to define the *pp-type* $\mathrm{pp}_M(\bar{m})$ of a tuple $\bar{m}$ in a module M as the collection of all divisibility formulae satisfied by $\bar{m}$ in M.

The following fact explains why only divisibility formulae matter.

Fact 2.8. *Let $\bar{m}$ be a finite tuple of elements of a projective module P. Then the pp-type of $\bar{m}$ in P is generated by a divisibility formula* (*whose matrix has as many columns as $\bar{m}$ has entries*).

Proof. Let $\bar{m}$ be a k-tuple and $P \oplus Q = F$, where F is a free module. Then there is a tuple $\bar{y} = (y_1, \dots, y_l)$ of basis elements of F and an $l \times k$ matrix A over R such that $\bar{m} = \bar{y}A$. By [42, F. 2.4], the pp-type of $\bar{m}$ in P is generated by $A \mid \bar{x}$. $\square$

Clearly, the solution set of a formula $A \mid \bar{x}$ as above in a module M forms (one says '$A \mid \bar{x}$ *defines*') a subgroup of the additive group of M^k. (So $a \mid x$ defines the subgroup Ma of M.) If $\bar{a}_i$ denotes the ith row of A, the subgroup defined by $A \mid \bar{x}$ is the sum of the subgroups defined by the formulae $\bar{a}_i \mid \bar{x}$, for $\bar{y}A = \sum_i y_i \bar{a}_i$. Another way of saying this is that $A \mid \bar{x}$ is equivalent to the sum of the formulae $\bar{a}_i \mid \bar{x}$, and one may write $\sum_i (\bar{a}_i \mid \bar{x})$ instead of $A \mid \bar{x}$. As an example, the formula $(a_1, \dots, a_l)^t \mid x$ (where 't' stands for transpose) is equivalent to the formula $\sum_i (a_i \mid x)$. Thus, passing to sums, one can diminish the size of the matrix in a divisibility formula.

Lemma 2.9. *Suppose every finitely generated submodule of the left R-module R^k is generated by n elements. Then every divisibility formula in k free variables $\bar{x} = (x_1, \dots, x_k)$ is equivalent to a formula of the form $B \mid \bar{x}$, where B is an $n \times k$ matrix over R.*

Consequently, if P is a projective right R-module and $\bar{m} = (m_1, \dots, m_k)$ is a tuple of elements of P, then the pp-type of $\bar{m}$ in P is generated by such a formula $B \mid \bar{x}$.

Proof. Let A be an $l \times k$ matrix over R. As mentioned, the formula $A \mid \bar{x}$ is equivalent to the formula $\sum_i \bar{a}_i \mid \bar{x}$, where $\bar{a}_i$ is the ith row of A. Consider each $\bar{a}_i$ as an element of the left R-module R^k.

By hypothesis, there are $\bar{b}_j \in R^k$, $j = 1, \dots, n$ such that $\sum_{i=1}^k R\bar{a}_i = \sum_{j=1}^n R\bar{b}_j$. Clearly, $\sum_i (\bar{a}_i \mid \bar{x})$ is equivalent to $\sum_j (\bar{b}_j \mid \bar{x})$. Let B be the $n \times k$ matrix over R whose jth row is $\bar{b}_j$. Then $A \mid \bar{x}$ is equivalent to $B \mid \bar{x}$. $\square$

A ring R is said to be *left Bézout* if every finitely generated left ideal of R is principal, i.e. for $k = 1$ in the previous lemma, n can be taken to be 1 as well. (Notice, every von Neumann regular ring is left Bézout.) As an immediate consequence of the preceding result we have

Corollary 2.10. *Let R be a left Bézout ring. Then every divisibility formula in one free variable x is equivalent to a formula of the form $b \mid x$, where $b \in R$.*

Consequently, if P is a projective right R-module and $m \in P$, then the pp-type of m in P is generated by such a formula $b \mid x$.

3. The question

When is every projective right module over a given ring a direct sum of finitely generated modules?

To address this question is the main objective of this study. Before we turn, in the next section, to a characterization of the rings where the answer is positive we have to state a few facts about these without proof.

One of the first results in this direction is Bass' theorem saying that non-finitely generated projectives over connected commutative noetherian rings are free, see [5, Corollary 4.5] (which contains also some generalizations of this). Immediately thereafter, Hinohara generalized this to connected weakly noetherian rings [23]. Here *connected* means that the ring has no non-trivial central idempotents, while *weakly noetherian* means that it is a commutative ring whose maximal spectrum has the d.c.c. on closed subsets (where the *maximal spectrum* is the set of all maximal ideals endowed with the Zariski (or hull–kernel) topology). First we draw an easy consequence of Hinohara's result generalizing all of these.

Fact 3.1. *Every projective module over a weakly noetherian* (*commutative*) *ring is a direct sum of finitely generated modules.*

Proof. As every decomposition of the ring given by an idempotent gives rise, in the obvious way, to a clopen partition of the maximal spectrum, the d.c.c. allows us to reduce the statement to a connected ring. Further, we may assume that the projective module in question is not finitely generated and therefore, by [23, Theorem], free. $\square$

This is no longer true even for semilocal noetherian rings module-finite over their center, as we show localizing $\mathbb{Z}A_5$. (For $\mathbb{Z}A_5$ itself, which is not semilocal, this follows from Akasaki [2] and independently Linnell [31].)

Example 3.2. Let R be the localization of the ring $\mathbb{Z}A_5$ at the multiplicative closed set $S = \mathbb{Z} \setminus (2\mathbb{Z} \cup 3\mathbb{Z} \cup 5\mathbb{Z})$. Then R is a semilocal noetherian ring module-finite over its center, and there exists a projective right (and left) module P without (non-zero) finitely generated direct summands.

Proof. If $D = \mathbb{Z}_{(S)}$, then D is a semilocal principal ideal domain and $R = DA_5$. Since DA_5 is a finitely generated D-module, R is semilocal by Fact 2.7.

As after Fact 2.4, one derives that the augmentation ideal of R is the trace of a countably generated projective right R-module P. Then P is not a generator and can therefore not have generators as direct summands either. So it suffices to prove that every finitely generated projective R-module Q is a generator.

Since $|A_5| = 60$, no prime dividing the order of A_5 is invertible in D. By Swan [46, Theorem 8.1], $\mathbb{Q} \otimes Q$ is a free $\mathbb{Q}A_5$-module. By Corollary 7.2 of the same paper, Q/pQ is a non-zero free $(D/pD)A_5 = (\mathbb{Z}/p\mathbb{Z})A_5$-module for every maximal ideal pD of D (that is, for $p = 2, 3, 5$).

Now, as in Akasaki [1, Corollary 14], we conclude that Q is a generator. $\square$

As a byproduct, we give an application of this example to artinian modules. For more on 'strange' behavior of direct sum decompositions of artinian modules see [15, Chapter 8].

Proposition 3.3. *There is a cyclic artinian module M and a direct summand N of $M^{(\omega)}$ such that N has no non-zero finitely generated direct summands.*

Proof. Let R be the ring from Example 3.2, in particular, R is module-finite over a semilocal commutative noetherian ring. By [15, Proposition 8.18], R is realized as the endomorphism ring of a cyclic artinian right module M (over an appropriate ring). By Fact 2.1, $\mathrm{Add}(M)$ is equivalent to the category of projective right R-modules.

If now N corresponds to a 'bad' projective R-module P as in Example 3.2, then N is a direct summand of $M^{(\omega)}$, but N has no non-zero finitely generated direct summands. □

In the same paper, Bass proved that also over simple right noetherian rings, non-finitely generated right projective modules are free. For purposes of reference we state this as

Fact 3.4. *(See [5].) Every projective right module over a simple right noetherian ring is a direct sum of finitely generated modules.*

Question 3.5. *Does the same hold for projective left modules* (*over a simple right noetherian ring*)?

Another class of rings with a positive answer to our main question is that of exchange rings, where a ring R is said to be an *exchange* ring, if for every $x \in R$ there exists an idempotent $e \in xR$ such that $1 - e \in (1 - x)R$. For instance, every semiperfect ring is an exchange ring. By Nicholson [37, Proposition 1.8], a commutative ring R is an exchange ring iff it is *clean*, that is, every element of R is the sum of a unit and an idempotent.

Recall from the introduction that Warfield's positive answer stated next generalizes an earlier result of Müller on projective modules over semiperfect rings.

Fact 3.6. *(See [50, Theorem 1].) Every projective module over an exchange ring R is isomorphic to a direct sum of modules $e_i R$, where the $e_i \in R$ are idempotents.*

Next we see that the rings in question (pun intended) are right *f-rings* in the sense that every pure right ideal is generated by idempotents. This class of rings was studied by Vasconcelos [49] in the commutative case, and by Jøndrup [25] in general. Notice, the argument in the proof of Lemma 2.6 shows that in a ring whose cyclic flat right modules are projective, every pure right ideal is generated by an idempotent (and vice versa), hence such a ring is a right f-ring. In particular, F-rings (and even right S-rings, cf. Section 2) are (right) f-rings.

Fact 3.7. *If every projective right R-module is a direct sum of finitely generated modules, then R is a right f-ring.*

Proof. Follows from [25, L. 2.1, Corollary 2.2]. □

For rings of continuous functions, the full converse holds, as will be proved in Section 9. There it will also become clear that we cannot strengthen f to F in the previous fact.

Jøndrup [25, Theorem 3.3] proves that, for commutative rings, being an f-ring can be characterized by a topological property of the prime spectrum of the ring, which shows that a commutative ring is an f-ring iff its factor ring modulo the prime radical is.

Here is a partial converse of Fact 3.7.

Fact 3.8. *(See [25, Theorem 2.5], [49, Theorem. 3.1].) A commutative ring is an f-ring if and only if every projective ideal is a direct sum of finitely generated ideals.*

More on f-rings can be found at the end of the next section.

Question 3.9. *Is there a commutative f-ring with a projective module that is not a direct sum of finitely generated modules?*

Fuchs and Salce [18, p. 246, Problem 16] asked if there exists a projective module over a commutative domain with no indecomposable direct summands. We do not even know the answer to the following weaker

Question 3.10. *Does there exist a projective module over a commutative domain which is not a direct sum of finitely generated modules?*

Among the commutative domains, Prüfer domains are ruled out by Albrecht's result, while semilocal domains and noetherian domains are by Hinohara's: for them all the main question has a positive answer.

4. The characterization

We use $\mathrm{M}(R)$ to denote the set of all finite rectangular matrices over R. Of course, matrix multiplication is defined only partially on that set. But we can still speak of ideals, etc. For instance, given $A \in \mathrm{M}(R)$, by $A\mathrm{M}(R)$ we mean the right ideal of $\mathrm{M}(R)$ generated by A, that is, the set of rectangular matrices AB, where $B \in \mathrm{M}(R)$ has matching size (so that the product AB is defined).

As usual, $\mathrm{M}_n(R)$ denotes the ring of all $n \times n$ matrices.

Let $\{A\} = A_1, A_2, \ldots$ be a sequence of rectangular R-matrices. We say that $\{A\}$ is *multiplicative*, if all consecutive products $A_{i+1}A_i$, $i \geqslant 1$ are defined. A multiplicative sequence $\{A\}$ is said to be *stable* if, for every $i \geqslant 1$, there exists a matrix C_i such that $C_i A_{i+1} A_i = A_i$. The term is motivated by the fact that, for a stable sequence as above, the descending chain of left ideals, $\mathrm{M}(R)A_1 \supseteq \mathrm{M}(R)A_2A_1 \supseteq \mathrm{M}(R)A_3A_2A_1 \supseteq \cdots$ stabilizes, even right at the first step, whence all those ideals are the same. Similarly, the chain of implications of divisibility formulas, $\cdots \to A_3A_2A_1 \mid \bar{x} \to A_2A_1 \mid \bar{x} \to A_1 \mid \bar{x}$, stabilizes as well, whence all those formulas are equivalent (provided the sequence $\{A\}$ is stable). The same is true when one starts from any i instead of 1, both for the left ideals and the right divisibility formulas, which means that for these one does not have to deal with products of terms in such sequences: single terms suffice.

Given a multiplicative sequence $\{A\}$, let $P\{A\}$ be the right R-module with generators $\bar{x}_1, \bar{x}_2, \ldots$ (of appropriate sizes) and relations $\bar{x}_{i+1}A_i = \bar{x}_i$. If A_i is an $n_{i+1} \times n_i$ matrix, then $P\{A\}$ is isomorphic to the direct limit of the directed system $R^{n_1} \xrightarrow{A_1\times} R^{n_2} \xrightarrow{A_2\times} \cdots$ (a chain). Here R^{n_i} is the free rank n_i right R-module (considered as a set of column vectors), and $A_i\times$ is the morphism given by left multiplication by A_i. Being a direct limit of free modules, $P\{A\}$ is flat. A module of this kind was used by Bass in his characterization of right perfect rings and also in the alternative proof given in [42, Theorem 4.2], to which the proof of the next lemma is very much related. The result is inspired by [51, Theorem 2.1].

Lemma 4.1.

(1) *If A is a stable sequence of matrices, then $P\{A\}$ is a countably generated projective right module.*
(2) *If P is a countably generated projective right module, then $P = P\{A\}$ for some stable sequence $\{A\}$.*

Proof. (1) As mentioned, $P\{A\}$ is flat. Since $\{A\}$ is stable, the pp-type of the tuple $\bar{x}_i$ in $P\{A\}$ is generated by $A_i \mid \bar{x}_i$ (see [42, L. 3.6]). So, by [42, Proposition 3.5], $P\{A\}$ is pure-projective, hence projective (as every pure-projective flat module is projective).

(2) Let $x_1, x_2, \ldots$ be a sequence of generators of P. Recursively we construct a new sequence of generators $\bar{x}_i \in P$ and a stable sequence of matrices $\{A\}$ such that $P = P\{A\}$.

Set $\bar{x}_1 = x_1$. As P is projective, by Fact 2.8, $p_1 = \mathrm{pp}_P(\bar{x}_1)$ is generated by a divisibility formula $A \mid \bar{x}$. This formula is already realized in a submodule P_1 of P generated by $x_1, \ldots, x_i$ for some $i > 1$. So there is $\bar{m} \in P_1$ such that $\bar{m}A = \bar{x}_1$. Set $\bar{x}_2 = (x_1, \ldots, x_i)$. Then there is another matrix B such that $\bar{m} = \bar{x}_2 B$, hence $\bar{x}_2 BA = \bar{x}_1$. Setting $A_1 = BA$, we have $\bar{x}_2 A_1 = \bar{x}_1$. Clearly, p_1 is generated also by $A_1 \mid \bar{x}$.

Similarly, p_2, the pp-type of $\bar{x}_2$ in P, is realized in some submodule of P generated by $x_1, \ldots, x_j$ for some $j > i$. Set $\bar{x}_3 = (x_1, \ldots, x_j)$ and write $\bar{x}_2 = \bar{x}_3 A_2$ for some matrix A_2, so that p_2 is generated by $A_2 \mid \bar{x}$.

From $\bar{x}_3 A_2 A_1 = \bar{x}_1$ it follows that $A_2 A_1 \mid \bar{x}$ is in p_1. Since p_1 is generated by $A_1 \mid \bar{x}$, we get $A_1 \mid \bar{x} \to A_2 A_1 \mid \bar{x}$, hence (by the special case of Prest's lemma [42, Remark 4.1]), there is a matrix C_1 such that $C_1 A_2 A_1 = A_1$.

Continuing in this manner, we obtain the desired representation of P. □

Note that the construction of (2) can be executed starting from any finite tuple of P instead of $\bar{x}_1$.

Now we are in a position to characterize the rings in question.

Theorem 4.2. *Given a ring R, the following are equivalent.*

(1) *Every projective right R-module is a direct sum of finitely generated modules.*
(2) *For every stable sequence of R-matrices, $A_1, A_2, \ldots$, there is an index j and an idempotent matrix E such that $A_j \cdot \ldots \cdot A_1 \mathrm{M}(R) \subseteq E\mathrm{M}(R) \subseteq A_j \mathrm{M}(R)$, i.e., there are matching matrices F and G such that $A_j \cdot \ldots \cdot A_1 = EF$ and $E = A_j G$.*

Proof. (1) ⇒ (2). Let $\{A\} = A_1, A_2, \ldots$ be a stable sequence of R-matrices. Consider the module $P = P\{A\}$ (with generators $\bar{x}_1, \bar{x}_2, \ldots$) as introduced before the previous lemma. As, by assumption, P is a direct sum of finitely generated modules, $\bar{x}_1$ is contained in a finitely generated direct summand P_1 of P. Let $\bar{m} \in P$ generate P_1. By construction, $\bar{m} = \bar{x}_j B$ for some matrix B.

Let p_j be the pp-type of $\bar{x}_j$ in P. Then p_j is generated by $A_j \mid \bar{x}$, see the previous proof. By [42, F. 2.4], the pp-type p of $\bar{m}$ in P is generated by $A_j B \mid \bar{x}$. In particular, $A_j B \mid \bar{m}$ holds in P. Since P_1 is a pure in P, it also holds in P_1, whence there is a matrix C such that $\bar{m} C A_j B = \bar{m}$, that is, $\bar{m}(CA_j B - 1) = 0$, where 1 stands for the (appropriate) identity matrix.

Because $A_j B \mid \bar{x}$ generates p, this formula implies $\bar{x}(CA_j B - 1) = 0$. By another special case of Prest's lemma, we obtain $A_j B(CA_j B - 1) = 0$,[1] that is, $A_j BCA_j B = A_j B$. It follows easily that $E = A_j BC$ is an idempotent matrix such that $E\mathrm{M}(R) = A_j B\mathrm{M}(R)$.

Now, as $\bar{x}_1$ is in P_1, there is a matrix D with $\bar{x}_1 = \bar{m}D$, hence $\bar{x}_1 = \bar{x}_j BD$. But also $\bar{x}_1 = \bar{x}_j A_{j-1} \cdot \ldots \cdot A_1$, which yields $\bar{x}_j(A_{j-1} \cdot \ldots \cdot A_1 - BD) = 0$. Since p_j is generated by $A_j \mid \bar{x}$, we conclude as before that $A_j(A_{j-1} \cdot \ldots \cdot A_1 - BD) = 0$, that is, $A_j \cdot \ldots \cdot A_1 = A_j BD$.

[1] Here is an easy proof of this: assuming that $A \mid \bar{x}$ implies $\bar{x}D = 0$, consider the module that is freely generated by $\bar{x}$; then $A \mid \bar{x}A$, hence, by assumption, $\bar{x}AD = 0$, and so $AD = 0$ by freeness.

Altogether we have $A_j \cdot \ldots \cdot A_1 \in A_j B\mathrm{M}(R) = E\mathrm{M}(R)$ and $E = A_j BC \in A_j \mathrm{M}(R)$, as desired.

(2) $\Rightarrow$ (1). Let P be a projective right R-module. By Kaplansky's theorem we may assume that P is countably generated. Then, by Lemma 4.1, $P = P\{A\}$ for some stable sequence of matrices $\{A\} = A_1, A_2, \ldots.$

We prove that P is a direct sum of finitely generated modules. By standard arguments, it suffices to prove that any finite tuple $\bar{m}$ in P can be included in a finitely generated direct summand of P. Further, since the $\bar{x}_i$ generate P, we may assume that $\bar{m} = \bar{x}_i$ for some i, and even that $i = 1$ (cf. remark after Lemma 4.1).

By assumption, there is an index j, an idempotent matrix E, and matrices F and G such that $E = A_j G$ and $A_j \cdot \ldots \cdot A_1 = EF$. Set $\bar{n} = \bar{x}_j G$ and let P_1 be the submodule of P generated by $\bar{n}$. We show that P_1 is a direct summand of P containing $\bar{x}_1$.

Indeed, as before, the pp-type of $\bar{n}$ in P is generated by $A_j G \mid \bar{x}$, that is, by $E \mid \bar{x}$. Since E is an idempotent matrix, we have $\bar{n} = \bar{n}E$, which is equally true in P as in P_1, whence the pp-type of $\bar{n}$ in P is the same as that of $\bar{n}$ in P_1. Since $\bar{n}$ generates P_1, this implies that P_1 is a pure submodule of the projective module P. Being also finitely generated, it is therefore a direct summand of P. Finally, $\bar{x}_1 = \bar{x}_{j+1} A_j \cdot \ldots \cdot A_1 = \bar{x}_{j+1} EF = \bar{x}_{j+1} A_j GF = \bar{x}_j GF = \bar{n}F$ shows that $\bar{x}_1$ is in P_1, as desired. $\square$

To conclude this section we describe f-rings (from Section 3) and F-rings (from Section 2) in terms similar to the ones introduced before.

A sequence $\{a\} = a_1, a_2, \ldots$ of elements of a ring R is said to be a (right) *a-sequence*, if $a_{i+1} a_i = a_i$ for every i. This leads to an ascending chain of right ideals, $a_1 R \subseteq a_2 R \subseteq \cdots$. We say that $\{a\}$ *converges*, if there exists an index k such that $a_{i+1} R = a_i R$ for every $i \geqslant k$. In this case, a_j is an idempotent for every $j \geqslant k + 1$, see [41, L. 2.3]. Clearly, every right $\{a\}$-sequence is stable, hence the countably generated right R-module $P\{a\}$ associated with $\{a\}$ is projective. In fact, $P\{a\}$ is isomorphic to the pure right ideal $\bigcup_i a_i R$. If $\{a\}$ converges, then $P\{a\}$ is finitely generated (by an idempotent).

Remark 4.3. By a result of Jøndrup [25], R is a right f-ring if and only if for every right a-sequence $\{a\}$, the (projective) right ideal $P\{a\}$ is generated by idempotents. By Mount [41, F. 7.1], a commutative ring R is an F-ring iff every a-sequence of elements of R converges iff $P\{a\}$ is finitely generated.

5. Weakly semihereditary rings

Let A and B be rectangular matrices over a ring R such that $AB = 0$. This zero relation is said to be *trivial* if there exists a (square) R-matrix X such that $AX = 0$ and $XB = B$. In this case we say that X *trivializes* the relation $AB = 0$.

Replacing X by $1 - X$ we see that the relation $AB = 0$ is trivial iff there exists a (square) matrix Y such that $AY = A$ and $YB = 0$.

The following lemma is well known (see [47, L. 2]).

Lemma 5.1. *A principal right ideal aR of a ring R is flat if and only if every zero relation $ab = 0$, where $b \in R$, is trivial.*

Proof. Suppose that aR is a flat module and $ab = 0$ for some $b \in R$. Since flat modules are torsion-free (see [29, Theorem 4.24]), there are $r_1, \ldots, r_n \in aR$ and $s_1, \ldots, s_n \in R$ such that $a = \sum_i r_i s_i$ and $s_i b = 0$ for every i. Write $r_i = at_i$ where $t_i \in R$ and set $y = \sum_i t_i s_i$. Then $a = ay$ and $yb = 0$, as desired.

The proof of the converse is similar. □

Next we characterize rings with trivial zero relations. Recall that the rings of weak dimension zero are exactly the von Neumann regular rings.

Proposition 5.2. *Every zero relation over a ring R is trivial if and only if R has weak dimension $\leqslant 1$.*

Proof. Suppose every zero relation over R is trivial. By [29, Ex. 5.62b] it suffices to prove that every (finitely generated) right ideal of R is flat. So let I be such a right ideal and n the number of its generators. The image of I via the Morita equivalence of categories of R and $\mathrm{M}_n(R)$-modules is I^n. Clearly I^n is a cyclic $\mathrm{M}_n(R)$-module isomorphic to a principal right ideal of $\mathrm{M}_n(R)$. By assumption and Lemma 5.1, this ideal is flat, hence I is flat.

The proof of the reverse implication is similar. □

Following Bergman (see [8, Definition 3.8] or Cohn [11, p. 13]) we say that a ring R is *n-weakly semihereditary* if every zero relation between $m \times n$ and $n \times k$ matrices over R is trivialized by an idempotent matrix. A ring is *weakly semihereditary*, if it is n-weakly semihereditary for every n. Thus, a ring R is weakly semihereditary if for every zero relation $AB = 0$ over R there exists an idempotent matrix E such that $AE = 0$ and $EB = B$. Switching from E to $1 - E$ we see that this is equivalent to the existence of an idempotent matrix F such that $AF = A$ and $FB = 0$. In particular (see [8, p. 44]), the notion of weakly semihereditary ring is left–right symmetric.

Corollary 5.3. *Every weakly semihereditary ring has weak dimension $\leqslant 1$. Consequently, every weakly semihereditary right noetherian ring is right hereditary.*

Proof. The first part follows from Proposition 5.2. If R is right noetherian, then, by Auslander's theorem (see [29, Theorem 5.60]), the weak dimension of R coincides with its right global dimension. Thus the right global dimension of R does not exceed 1, hence R is right hereditary. □

In Section 9 we will give an example of a commutative ring of weak dimension one with a projective module that is not a direct sum of finitely generated modules. By Proposition 5.4, this ring is not weakly semihereditary.

It is easily shown (using [11, Ex. 0.3.1]) that every one-sided semihereditary ring is weakly semihereditary. In Section 9 we will give an example of a commutative weakly semihereditary ring which is not semihereditary.

We can now give the promised new proof of Bergman's theorem. (As mentioned in the introduction, it is a generalization of Albrecht's theorem [3, Theorem] on right projective modules over right semihereditary rings, and a theorem of Bass [6, Proposition 4.1] on right projective modules over left semihereditary rings.)

Proposition 5.4. *(See Bergman [8], cf. [11, Theorem 3.7].) Every projective module over a weakly semihereditary ring R is a direct sum of finitely generated modules.*

Proof. Let $A_1, A_2, \dots$ be a stable sequence of R-matrices. By Theorem 4.2, it suffices to find an idempotent matrix E such that $A_2A_1\mathrm{M}(R) \subseteq E\mathrm{M}(R) \subseteq A_2\mathrm{M}(R)$.

From $C_1A_2A_1 = A_1$ it follows that $(1 - C_1A_2)A_1 = 0$. By hypothesis, there exists an idempotent matrix F such that $(1 - C_1A_2)F = 0$ (that is, $F = C_1A_2F$), and $FA_1 = A_1$. Then it is straightforward to check that $E = A_2FC_1 \in A_2\mathrm{M}(R)$ is an idempotent matrix. Further,

$$E \cdot A_2A_1 = A_2FC_1 \cdot A_2A_1 = A_2FC_1A_2FA_1 = A_2F^2A_1 = A_2A_1$$

shows that $A_2A_1 \in E\mathrm{M}(R)$, and so $A_2A_1\mathrm{M}(R) \subseteq E\mathrm{M}(R) \subseteq A_2\mathrm{M}(R)$, as desired. □

Since every one-sided semihereditary ring is weakly semihereditary, we obtain

Corollary 5.5. *Every projective module over a one-sided semihereditary ring is a direct sum of finitely generated modules.*

To be fair to Albrecht: his theorem says more, namely that all the finitely generated direct summands are isomorphic to finitely generated right *ideals*. Similarly, Bass's theorem shows that the finitely generated summands are in fact duals of finitely generated left ideals.

In the proof of Proposition 5.4 we were able to 'catch' the desired idempotent using just two terms of a stable sequence. It remains to be investigated how general this phenomenon is. In particular, we do not know the answer to the following

Question 5.6. *Is there a ring, all of whose right projective modules are direct sums of finitely generated modules, for which there is no uniform bound on the index j (of occurrence of an idempotent matrix in a stable sequence) in Theorem* 4.2*?*

6. Principal ideal rings

Recall that a *principal right ideal ring* is a ring all of whose right ideals are principal. This is equivalent to being a right noetherian right Bézout ring.

Non-trivial examples can be obtained from the first Weyl algebra A_1 over a field of characteristic zero. Indeed, although A_1 has a non-principal right (and left) ideal, every matrix ring $\mathrm{M}_n(A_1)$ with $n \geqslant 2$ is a prime principal ideal ring, cf. [33, 7.11.7–7.11.8].

For examples of (right uniserial) principal right ideal rings that are not principal left ideal rings, see [11, Section 8.8].

The next goal is to show that every projective (left or right) module over a one-sided principal ideal ring is a direct sum of finitely generated modules. For this we need some general auxiliary results that will also be used in the next section.

Lemma 6.1. *Let P be a projective right module over a ring R. Suppose that J is a nilpotent ideal of R such that P/PJ is a direct sum of finitely generated modules. Then P is a direct sum of finitely generated modules.*

Proof. By [28, Theorem 23.16], MJ is small in M for every right R-module M. In particular, PJ is small in P, hence P is a projective cover of $\bar{P} = P/PJ$. Further, $\bar{P}$ is a projective

$\bar{R} = R/J$-module. By hypothesis, $\bar{P} = \bigoplus_{i \in I} P_i'$, where P_i' are finitely generated projective $\bar{R}$-modules. Then each P_i' is a direct summand of a free $\bar{R}$-module $\bar{R}^n$ for some n, hence isomorphic to the module $E_i' \bar{R}^n$ for an idempotent $n \times n$ matrix E_i' over $\bar{R}$.

Since J is nilpotent, $\mathrm{M}_n(J)$ is a nilpotent ideal of $\mathrm{M}_n(R)$, hence one can lift idempotents modulo $\mathrm{M}_n(J)$, see [28, Theorem 21.28]. Thus, there exists an idempotent $n \times n$ matrix E_i over R such that $\bar{E}_i = E_i'$. Clearly $P_i = E_i R^n$ is a finitely generated projective right R-module such that $P_i' \cong \bar{P}_i = P_i / P_i J$, hence P_i is a projective cover of P_i'.

Then $P' = \bigoplus P_i$ is a projective cover of $\bar{P}$ (since $\bar{P} \cong P'/P'J$ and J is nilpotent), whence $P \cong P'$ by the uniqueness of projective covers. □

The main application of this lemma is when J is $N(R)$, the prime radical of R. By [15, Proposition 1.6], $N(R)$ is nil for any ring R. Also, if R has a right or left Krull dimension, then $N(R)$ is nilpotent, see [15, Theorem 7.21]. In particular, this applies to right or left noetherian rings.

Corollary 6.2. *Let R be a ring such that the prime radical $N(R)$ of R is nilpotent. Let P be a projective right R-module. If the R/N-module $\bar{P} = P/PN$ is a direct sum of finitely generated modules, then the same is true for P.*

If $P = P\{A\}$ in this corollary, where $\{A\}$ is a stable sequence of matrices, then $\bar{P} = P\{\bar{A}\}$, where $\{\bar{A}\}$ is the stable sequence consisting of images of matrices of $\{A\}$ in R/N.

Proposition 6.3. *Every projective module over a one-sided principal ideal ring is a direct sum of finitely generated modules.*

Proof. By Corollary 6.2 (factoring out the prime radical) we may assume that the ring is semiprime. As in [33, Example 5.2.11], one can show that it is in fact right hereditary (hence right semihereditary). It remains to apply Corollary 5.5. □

Recall that a ring is *connected* iff it has no non-trivial central idempotents. Smith [44, Corollary 4.9] shows that over connected principal right ideal rings non-finitely generated projective right modules are free. (And this yields, upon decomposing the ring into a finite direct sum of connected right principal ideal rings as before, another proof of the previous proposition for the case of projective right modules.) Next we extend this to left modules.

Recall, that if P is a finitely generated projective right R-module, then $P^* = \mathrm{Hom}(P, R_R)$, the *dual* of P, is a finitely generated projective left module. Moreover, if $e \in R$ is an idempotent, then $eR^* \cong Re$. Similarly, we can define the dual Q^* of a finitely generated projective left R-module Q. The operation * gives rise to a duality between categories of finitely generated projective right and left R-modules. In particular, if P is a finitely generated projective right or left module, then $P \cong P^{**}$.

Corollary 6.4. *Every non-finitely generated projective module over a connected one-sided principal ideal ring is free.*

Proof. Let R be a connected right principal ideal ring. As mentioned above, every non-finitely generated projective right R-module is free. Suppose now that P is a countably (but not finitely) generated projective left R-module. By Proposition 6.3, $P = \bigoplus_{i \in I} P_i$, where P_i are non-zero finitely generated projective left R-modules and I is infinite.

Using Eilenberg's trick (see [5, p. 24] or [29, Corollary 2.7]), one can easily show that P is free whenever it has ${}_R R^{(\omega)}$ as a direct summand. Partitioning I into infinitely many infinite sets, one sees that it suffices to prove that P has just one copy of ${}_R R$ as a direct summand.

Since $\bigoplus_{i \in I} P_i^*$ is a free right R-module, it contains a copy of R_R as a direct summand. Then already some finite sum $P_{i_1}^* \oplus \cdots \oplus P_{i_k}^*$ contains a free direct summand. Applying the above duality we see that $P_{i_1} \oplus \cdots \oplus P_{i_k}$ contains a copy of ${}_R R$ as a direct summand, as desired. $\square$

The question arises if something similar holds for higher uniform bounds on the number of generators of right ideals. For example, in any simple right noetherian ring of Krull dimension n, by [33, 6.7.8], every right ideal is generated by $n + 1$ elements. In this particular case, every projective right module *is* a direct sum of finitely generated modules, as follows from Fact 3.4. In general, however, this is not the case: we conclude this section with a counterexample. More precisely, we exhibit a right noetherian ring with a uniform bound on the number of generators of right ideals over which there is a projective right module which is not a direct sum of finitely generated modules. The authors thank the referee for pointing out such a possibility and to Larry Levy for explaining it to us.

Suppose Λ is an order of finite lattice type over a Dedekind domain D and let A be the Auslander lattice over Λ (that is, the sum of all indecomposable Λ-lattices, one for each isomorphism type). Let further $E = \mathrm{End}(A)$. It was proved in [10, Theorem 2.1] that the category of projective right E-modules is equivalent to the category of generalized Λ-lattices, that is, Λ-modules which are projective as D-modules. Thus our original question for E-modules (when every projective right E-module is a direct sum of finitely generated modules) is equivalent to the question whether every generalized Λ-lattice is a direct sum of Λ-lattices.

If Λ is an order in a separable K-algebra, where K is the field of quotients of D, the latter question was completely answered in [43] in terms of the hypergraph of Λ. For instance, while, as follows from [10], the answer is positive when $\Lambda = \mathbb{Z}C_p$, the group ring of the cyclic group of prime order, this is not the case for the $\mathbb{Z}$-order $\mathbb{Z} \oplus \mathbb{Z} \cdot 5i$ in $\mathbb{Q}[i]$, see [43, Example 3]. In the latter case, $(E, +)$ is a free abelian group of finite rank. Therefore, there is a uniform bound on the number of generators of one-sided ideals of E.

7. Restricting the size of matrices

We addressed the issue of bounds on the index j in Theorem 4.2, cf. Question 5.6. Another such issue is that of bounds on the size of the matrices figuring in that theorem. For ease of reference, we restate what the criterion says about stable sequences of matrices of uniformly bounded size.

Remark 7.1. Let R be a ring such that every projective right R-module is a direct sum of finitely generated modules. Suppose that $A_1, A_2, \ldots$ is a stable sequence of $n \times n$ matrices over R (that is, for all i there is some $n \times n$ matrix C_i with $C_i A_{i+1} A_i = A_i$). Then there exists an idempotent $n \times n$ matrix E such that $A_j \cdot \ldots \cdot A_1 \mathrm{M}_n(R) \subseteq E\mathrm{M}_n(R) \subseteq A_j \mathrm{M}_n(R)$ for some j.

Considering the case $n = 1$ in this, let $a \in R$ be such that $ba^2 = a$ for some $b \in R$. Then the constant sequence $a, a, a, \ldots$ is stable, and we obtain that the corresponding module $P\{a\}$ is countably generated projective.

Corollary 7.2. *Let $a, b \in R$ with $ba^2 = a$. If the projective module $P\{a\}$ is a direct sum of finitely generated modules, then there is an idempotent $e \in R$ such that $a^n R \subseteq eR \subseteq aR$ for some n.*

Note that the conclusion may fail even for semilocal rings. Indeed, let S be the (semilocal) endomorphism ring of a uniserial module M as described in [40, Section 6]. Let $a \in S$ be a monomorphism of M which is not an epimorphism. Then $Sa = Sa^2$, hence $a = ba^2$ for some $b \in S$. Suppose there is an idempotent $e \in S$ as in the corollary. Since S has no non-trivial idempotents, $e = 0$ or $e = 1$. If $e = 0$, then $a^n \in eS$ implies $a^n = 0$, a contradiction. If $e = 1$, then $e \in aS$ implies that a is a unit, a contradiction again.

The question arises if it suffices to consider stable sequences of matrices of uniformly bounded size in the criterion. To answer this question, we first show that for such a bounded sequence of matrices, over certain rings, the module $P\{A\}$ is automatically a direct sum of finitely generated modules. Then we point out an example of such a ring over which there are, nevertheless, projectives that do not so decompose, which shows in turn that in general we may not restrict ourselves to sequences of bounded size, Corollary 7.5.

Proposition 7.3. *Let M be a finite length right module over an arbitrary ring, and let R be a subring of the endomorphism ring of M acting on M on the left. Then, for every stable sequence of $n \times n$ matrices, $\{A\} = A_1, A_2, \ldots$ over R, there exists an idempotent $n \times n$ matrix E such that $A_j \cdot \ldots \cdot A_1 \mathrm{M}_n(R) \subseteq E\mathrm{M}_n(R) \subseteq A_j \mathrm{M}_n(R)$ for some j. Thus the corresponding projective right R-module $P\{A\}$ is a direct sum of finitely generated modules.*

Proof. Since M is of finite length, the same is true for M^n. Further, $\mathrm{M}_n(R)$ is a subring of $\mathrm{End}(M^n) = \mathrm{M}_n(\mathrm{End}(M))$. So we may assume $n = 1$, that is, we have to consider only stable sequences of elements of R. Let $\{a\} = a_1, a_2, \ldots$ be such a sequence. We may also assume that $a_i \neq 0$ for every i, for otherwise, taking $e = 0$, we are done.

Let $c_i \in R$ be such that $c_i a_{i+1} a_i = a_i$ for every i. Setting $b_i = c_i a_{i+1}$, we have $b_i a_i = a_i$. Then b_i acts on $\mathrm{im}(a_i)$ as the identity, in particular, $\mathrm{im}(a_i) \subseteq \mathrm{im}(b_i)$.

Now choose i such that $\mathrm{im}(a_i)$ has maximal length. By the choice of b_i, the length of $\mathrm{im}(b_i)$ does not exceed that of $\mathrm{im}(a_{i+1})$, which, in turn, does not exceed that of $\mathrm{im}(a_i)$. Therefore we must have $\mathrm{im}(a_i) = \mathrm{im}(b_i)$. Then b_i acts on its image as the identity, whence b_i is an idempotent.

Note that $(a_{i+1}c_i)^3 = a_{i+1} b_i^2 c_i = a_{i+1} b_i c_i = (a_{i+1}c_i)^2$. It follows readily that $e = (a_{i+1}c_i)^2$ is an idempotent. Since $e \in a_{i+1}R$, it suffices to check that $a_{i+1} \cdot \ldots \cdot a_1 \in eR$. But as $b_i a_i = a_i$, we obtain this from $e \cdot a_{i+1} \cdot \ldots \cdot a_1 = a_{i+1} b_i^2 a_i \cdot \ldots \cdot a_1 = a_{i+1} \cdot \ldots \cdot a_1$, as desired. □

In particular, over a right noetherian ring that is embeddable in a right artinian ring (for instance, over a semiprime noetherian ring), by Proposition 7.3, any projective module of the form $P\{A\}$ for $\{A\} = A_1, A_2, \ldots$ a stable sequence over R with uniformly bounded sizes of matrices is a direct sum of finitely generated modules. The next goal is to extend this to *any* one-sided noetherian (and even much more general) rings. The previous proposition seems not to yield this, because there are noetherian rings that are not embeddable in a right artinian ring, e.g., there is an example of a noetherian algebra over a field that is not embeddable into an artinian ring (see [12]), which is therefore not embeddable in a right artinian ring either, cf. [45, Theorem 7.13]. But using Lemma 6.1 and Corollary 6.2, we may factor out the prime radical and then apply Proposition 7.3 as follows.

Proposition 7.4. *Let R be a ring with right or left Krull dimension (e.g. a one-sided noetherian ring). If $\{A\} = A_1, A_2, \dots$ is a stable sequence of $n \times n$ matrices over R, then there exists an idempotent $n \times n$ matrix E such that $A_i \cdot \ldots \cdot A_1 \mathrm{M}_n(R) \subseteq E\mathrm{M}_n(R) \subseteq A_i \mathrm{M}_n(R)$ for some i. Thus the countably generated projective right R-module $P\{A\}$ is a direct sum of finitely generated modules.*

Proof. By Corollary 6.2, we may assume that R is semiprime. By [15, Corollary 7.19], R is a right (or left) Goldie ring, that is, R has finite right (left) Goldie dimension and the a.c.c. on right (left) annihilators.

Then, by Goldie's theorem, R is a subring of its right (left) quotient ring $Q(R)$ which is a semisimple artinian ring. It remains to apply Proposition 7.3. $\square$

So, to construct a 'bad' example of a projective module of the form $P\{A\}$ over a ring with one-sided Krull dimension, the size of the corresponding matrices has to go to infinity.

Corollary 7.5. *Let R be the ring of Example* 3.2*. Then every projective right R-module of the form $P\{A\}$, for $\{A\}$ a stable sequence of matrices of bounded size, is a direct sum of finitely generated modules, and yet there is a projective right R-module that is not a direct sum of finitely generated modules.*

Consequently, one cannot, in general, restrict the criterion in Theorem 4.2 *to stable sequences whose matrices have uniformly bounded size.*

Using [15, Propositions 10.6, 10.7], the same proof shows that Proposition 7.4 is true if R is the endomorphism ring of an artinian module (over any ring). Again, Example 3.2 serves as an example of such a ring over which not every projective module is a direct sum of finitely generated modules.

We conclude with an application to idempotent ideals (see Section 2 about the role of idempotent ideals in connection with traces). Note, first of all, that any idempotent $e \in R$ gives rise to an idempotent ideal, ReR. But there are others. For instance, the augmentation ideal of the universal enveloping algebra $Usl_2(\Bbbk)$ over a field $\Bbbk$ of characteristic zero is idempotent (and being a domain, this ring has no idempotents). Note that this idempotent ideal is 2-generated both as a right ideal and as left ideal. As an application of the previous proposition, we show that '2' is best possible here.

Corollary 7.6. *Let R be a ring with one-sided Krull dimension and with no non-trivial idempotents. Then no non-zero proper idempotent ideal of R is cyclic as a left ideal.*

Proof. Suppose that $I = Ra$ is an idempotent ideal of R. Then $a \in I^2$ implies $a = \sum_i r_i a s_i a$ for some $r_i, s_i \in R$. From $as_i \in I$ it follows that $as_i = t_i a$ for some $t_i \in R$. If $b = \sum_i r_i t_i$, then $ba^2 = a$. By Proposition 7.4 and Corollary 7.2, there is an idempotent $e \in R$ such that $a^n R \subseteq eR \subseteq aR$. By hypothesis, $e = 0$ or $e = 1$, the latter of which leads to the contradiction $I = R$. If, on the other hand, $e = 0$, then $a^n = 0$, which upon successive application of $ba^2 = a$ yields the contradiction $a = 0$. $\square$

8. Bézout rings

Recall that a ring R is *left Bézout* if every finitely generated left ideal of R is principal. It may not be all that surprising that over such rings one may indeed restrict the size of matrices in the criterion of Theorem 4.2, even to just ring elements, as we show first. (For left Bézout domains, this follows from Albrecht's theorem: every projective right (and left) module is free in that case.)

Proposition 8.1. *Given a left Bézout ring R, the following are equivalent.*

(1) *Every projective right R-module is a direct sum of finitely generated modules.*
(2) *If $a_1, a_2, \dots$ is a stable sequence of elements of R, then there is an idempotent $e \in R$ such that $a_i \cdot \ldots \cdot a_1 R \subseteq eR \subseteq a_i R$ for some i.*
(3) *Every projective right R-module is isomorphic to a direct sum of right ideals $e_i R$, where the $e_i \in R$ are idempotents.*

Proof. (3) $\Rightarrow$ (1) is trivial.

(1) $\Rightarrow$ (2) follows from Remark 7.1.

(2) $\Rightarrow$ (3). Let P be a projective right R-module. By Kaplansky's theorem, we may assume that P is countably generated. It suffices to prove that every $m \in P$ is contained in a direct summand of P isomorphic to eR, for some idempotent $e \in R$.

The following construction is similar to the proof (2) $\Rightarrow$ (1) of Theorem 4.2.

Since P is projective and R is left Bézout, the pp-type p_1 of $m = m_1$ in P is generated by a divisibility formula $a_1 \mid x$ for some $a_1 \in R$, see Corollary 2.10. Take $m_2 \in P$ such that $m_2 a_1 = m_1$. The pp-type of m_2 in P is generated by $a_2 \mid x$ for some $a_2 \in R$. Continuing this way, we obtain a sequence $\{a\} = a_1, a_2, \dots$ of elements of R, and a sequence $m_1, m_2, \dots$ of elements of P such that $m_{i+1} a_i = m_i$ for every $i \geqslant 1$.

As in Lemma 4.1, one can check that the sequence $\{a\}$ is stable. By assumption, there is an idempotent $e \in R$ such that $e = a_i f$ and $a_i \cdot \ldots \cdot a_1 = eg$ for some index i and some $f, g \in R$. Now, similarly to the proof of Theorem 4.2, the pp-type of $n = m_i f$ is generated by $e \mid x$, and $m_1 \in nR$. Thus, nR is a direct summand of P isomorphic to eR, as desired. $\square$

Corollary 8.2. *Let R be a left Bézout ring which has one-sided Krull dimension or is embeddable into the endomorphism ring of a finite length module. Then every projective right R-module is a direct sum of right ideals $e_i R$, where the $e_i \in R$ are idempotents.*

Proof. By Propositions 8.1, 7.3, and 7.4. $\square$

Corollary 8.2 applies, for instance, to semiprime left Bézout right Goldie rings, since they have semisimple artinian quotient rings. (Recall that *right Goldie* means finite right Goldie dimension and a.c.c. on right annihilators.)

Question 8.3. *Is every projective right module over a left Bézout right Goldie ring a direct sum of finitely generated modules?*

The ring S mentioned after Corollary 7.2 is (left and right) Bézout, has (left and right) Goldie dimension 1, and, see [40, Section 6], possesses indecomposable non-finitely generated projective modules on both sides. But it fails to have the a.c.c. on right or left annihilators, and is thus neither right nor left Goldie.

For commutative rings, the aforementioned criterion admits further simplification.

Proposition 8.4. *Given a commutative Bézout ring R, the following are equivalent.*

(1) *Every projective R-module is a direct sum of finitely generated modules.*
(2) *If $b_1, b_2, \ldots$ is an a-sequence (that is, $b_{k+1}b_k = b_k$ for every k), then there exists an idempotent $e \in R$ such that $b_1R \subseteq eR \subseteq b_iR$ for some i.*
(3) *R is an f-ring, that is, every pure ideal of R is generated by idempotents.*

Proof. (1) $\Rightarrow$ (2). By Proposition 8.1, there is an idempotent e such that $b_i \cdot \ldots \cdot b_1R \subseteq eR \subseteq b_iR$. But, by the definition of a-sequence, $b_i \cdot \ldots \cdot b_1 = b_1$.

(2) $\Rightarrow$ (1). Let $a_1, a_2, \ldots$ be a stable sequence of elements of R. By Proposition 8.1, it suffices to find an idempotent $e \in R$ such that $a_i \cdot \ldots \cdot a_1R \subseteq eR \subseteq a_iR$ for some i. Let $b_1 = a_1$, and set $b_i = c_{i-1}a_i$ for $i \geqslant 2$, where the c_i witness stability, that is, $c_ia_{i+1}a_i = a_i$. We claim that $b_1, b_2, \ldots$ is an a-sequence. Indeed, $b_2b_1 = c_1a_2a_1 = a_1 = b_1$. Also, if $i \geqslant 2$, then $b_{i+1}b_i = c_ia_{i+1}c_{i-1}a_i = c_{i-1}c_ia_{i+1}a_i = c_{i-1}a_i = b_i$.

By assumption, there is an idempotent e, and elements $f, g \in R$ such that $eg = b_1$ and $e = b_if$ for some i. If $i = 1$, then $b_1R = eR$, hence $a_1R = b_1R = eR$, as desired. Otherwise, $i > 1$. Then $e = b_if = c_{i-1}a_if \in a_iR$, and $a_1 = b_1 = eg$ implies $a_i \cdot \ldots \cdot a_1 = a_i \cdot \ldots \cdot a_2eg \in eR$, as desired.

The equivalence (2) $\Leftrightarrow$ (3) follows from [25, L. 2.1]. □

The first equivalence says that every projective module over a commutative Bézout ring is a direct sum of finitely generated modules iff idempotents are 'dense' in a-sequences. We now see what all this means when the ring has only trivial idempotents.

Corollary 8.5. *Given a commutative Bézout ring R without non-trivial idempotents, the following are equivalent.*

(1) *Every projective R-module is a direct sum of finitely generated modules.*
(2) *Every projective module is free.*
(3) *R is an F-ring, that is, every cyclic (or finitely generated) flat R-module is projective.*

Proof. (2) $\Rightarrow$ (1) is obvious, and (3) $\Rightarrow$ (1) follows from Proposition 8.4 (since every F-ring is an f-ring).

(1) $\Rightarrow$ (2). By Proposition 8.1, every projective R-module is a direct sum of modules e_iR, where $e_i \in R$ are idempotents. Since R has no non-trivial idempotents, $e_i = 1$, and so $e_iR = R$.

(1) $\Rightarrow$ (3). Suppose that R/I is a flat non-zero R-module. Then I is a pure ideal of R, and so there is an a-sequence $\{a\}$ consisting of elements of I. Further, the corresponding projective ideal $P\{a\}$ is pure in R. By hypothesis, it contains a finitely generated direct summand P_1. Then also P_1 is pure in, hence a direct summand of R. Since R has no non-trivial idempotents, $P_1 = R$, and so $I = R$, contrary to its choice. □

Since the existence of Krull dimension implies finite Goldie dimension (but not vice versa), the following is a partial generalization of Corollary 8.2 for commutative Bézout rings.

Proposition 8.6. *If R is a commutative Bézout ring of finite Goldie dimension, then every projective R-module is isomorphic to a direct sum of modules $e_i R$, where $e_i \in R$ are idempotents.*

Proof. Since R has finite Goldie dimension, we may assume that R has no non-trivial idempotents. Then, by [26, Remarks], R is an F-ring. It remains to apply Corollary 8.5. □

The boolean ring corresponding to an atomless boolean algebra is von Neumann regular (hence Bézout) and has superdecomposable projective modules (e.g. the ring as a module over itself). Below we will give such an example without idempotents, Example 9.21.

9. Rings of continuous functions

In this section we consider the ring $C(X)$, that is, the ring of all real-valued continuous functions on a topological space X. As is customary we assume that X is a Tychonoff space, i.e., X is completely regular and Hausdorff. For more information on rings of continuous functions the reader is referred to [20].

The contents of this final section can (up to some more equivalences that we derive) be summarized by Fig. 1.

Given $f \in C(X)$, the *zero set* of f, denoted $\mathrm{zer}(f)$, is the closed set $\{x \in X \mid f(x) = 0\}$, while the *cozero set* of f, $\mathrm{coz}(f)$, is the open set $\{x \in X \mid f(x) \neq 0\}$. Since X is Tychonoff, cozero sets form a base for the open topology on X. As is easily seen, $\mathrm{pos}(f) = \{x \in X \mid f(x) > 0\}$ and $\mathrm{neg}(f) = \{x \in X \mid f(x) < 0\}$ are cozero sets. Further, every cozero set can be represented as $\mathrm{neg}(g)$ and as $\mathrm{pos}(h)$ for appropriate $g, h \in C(X)$.

We are going to characterize the ring theoretic properties of $C(X)$ that have arisen above by topological properties of the underlying spaces.

A topological space X is said to be *strongly zero-dimensional* if disjoint zero sets of X can be separated by disjoint clopen sets. Every strongly zero-dimensional space is *zero-dimensional* in the sense that it has a base of clopen sets. By [20, Theorem 16.17], X is strongly zero-dimensional iff the Stone–Čech compactification of X, $\beta(X)$, is zero-dimensional. (The reader should be warned that in the older literature, including [20], the term zero-dimensional is used for what is here called strongly zero-dimensional.)

First we give a complete answer to our original question for $C(X)$.

Proposition 9.1. *The following are equivalent.*

(1) *Every projective $C(X)$-module is a direct sum of finitely generated modules.*
(2) *$C(X)$ is an f-ring.*
(3) *X is strongly zero-dimensional.*
(4) *$C(X)$ is an exchange ring.*
(5) *Every projective $C(X)$-module is a direct sum of modules $e_i R$, where $e_i \in R$ are idempotents.*

Proof. Trivially, (5) implies (1). (2) is equivalent to (3) by De Marco [13, Corollary 2.4]. Further, (1) $\Rightarrow$ (2) follows from Fact 3.7, and (3) $\Leftrightarrow$ (4) follows from [34, Theorem 13] and [4]. The rest follows from Fact 3.6. □

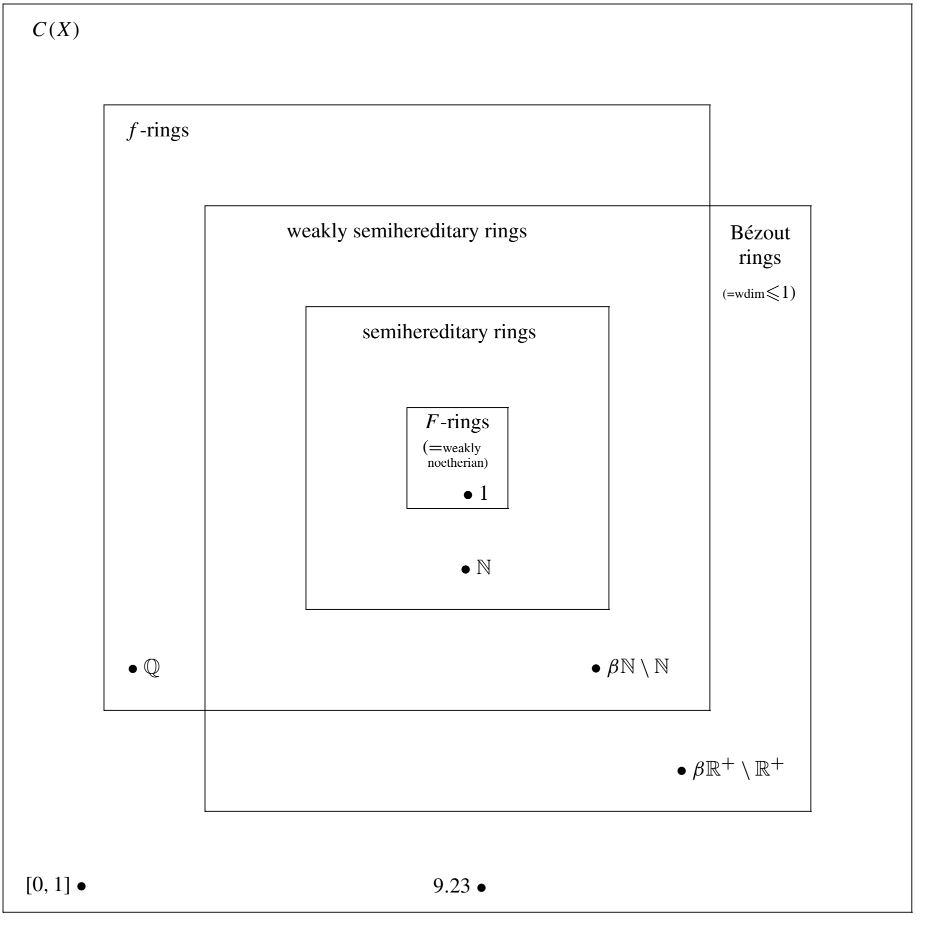

Fig. 1.

Example 9.2. Being a countable zero-dimensional space, $\mathbb{Q}$ with the usual topology is a strongly zero-dimensional space, hence $C(\mathbb{Q})$ is an exchange (and an f-) ring. Thus, all projective $C(\mathbb{Q})$-modules are direct sums of finitely generated modules.

More on this ring will be said in Corollary 9.15.

Example 9.3. Another extreme can be observed over the ring $C([0,1])$. As the interval $[0,1]$ is connected, it has no non-trivial clopen sets. But there are many disjoint zero sets (e.g. $\{0\}$ and $\{1\}$), which can therefore not be separated by clopen sets. Thus, $C([0,1])$ is not an exchange ring, and hence some projective $C([0,1])$-module does not decompose into a direct sum of finitely generated modules. In fact, the ideal P of all $f \in C([0,1])$ vanishing on some open neighborhood of 0 is an indecomposable countably, but not finitely generated projective and pure ideal, which is of the form $P\{a\}$ for some a-sequence $\{a\}$, see [29, Example 2.12.D]. More on projectives over $C([0,1])$ can be found in Proposition 9.6 below and in [36].

We are going to have to elaborate somewhat on [13]. Suppose that X is a compact topological space. Then there is a one-to-one correspondence between pure ideals of $C(X)$ and the closed subsets of X. Namely, if I is a pure ideal, then $\operatorname{zer}(I) = \bigcap_{f \in I} \operatorname{zer}(f)$ is a closed subset of X, and conversely, if Y is a closed subset of X, then the set, O_Y, of functions vanishing on some open neighborhood of Y is a pure ideal of $C(X)$. There is also a description of direct sum decompositions of pure ideals. Indeed, since X is compact, we can identify X with the maximal spectrum of $C(X)$. Then the direct sum decompositions of I are in one-to-one correspondence with open partitions of $X \setminus \operatorname{zer}(I)$:

Fact 9.4. *(See [13, Proposition 1.9 and Theorem 1.13].) Let I be a pure ideal of $C(X)$ with X compact.*

(1) *If $I = \oplus I_i$, then $X \setminus \operatorname{zer}(I) = \bigcup U_i$ is an open partition, where $U_i = X \setminus \operatorname{zer}(I_i)$.*
(2) *If $X \setminus \operatorname{zer}(I) = \bigcup U_i$ is an open partition, then $I = \bigoplus I_i$, where I_i is the pure ideal $O_{X \setminus U_i}$ corresponding to the closed set $X \setminus U_i$.*
(3) *In particular, any two direct sum decompositions of I have a common refinement, for this is obvious on the side of open partitions.*
(4) *If I is, in addition, projective, any direct sum decomposition of I has a refinement of the form $I = \bigoplus O_{\operatorname{zer}(f_i)}$ with $f_i \in C(X)$.*

Suppose P is a projective ideal of $C(X)$. Then, by Fact 2.4, its trace I is projective and pure, so the above applies. Furthermore, there is one-to-one correspondence between decompositions of I and P.

Fact 9.5. *(See [13, Proposition 1.14].) Let P be a projective ideal of $C(X)$ and $I = \operatorname{Tr}(P)$.*

(1) *If $P = \bigoplus P_i$, then $I = \bigoplus \operatorname{Tr}(P_i)$.*
(2) *If $I = \bigoplus I_i$, then $P = \bigoplus P I_i$.*
(3) *P is finitely generated if and only if I is finitely generated, and P and I have the same (infinite) cardinality of generators otherwise (and the same applies to the P_i and I_i, respectively).*

Note that $f \in C(X)$ is a not zero divisor if and only if $\operatorname{coz}(f)$ is dense in X.
We can now describe direct sum decompositions of projective ideals of $C([0,1])$.

Proposition 9.6. *Let P be a projective ideal of $C([0,1])$.*

(1) *If P is finitely generated, then it is free of rank 1 with $P = fC([0,1])$ where $\operatorname{coz}(f)$ is a dense subset of $[0,1]$.*
(2) *If P is not finitely generated, then it uniquely decomposes into a direct sum of countably many indecomposable countably, but not finitely generated (projective) ideals.*

Proof. (1) Since $[0,1]$ is contractible, every finitely generated projective $C([0,1])$-module is free. In particular, P is free, hence $P = \bigoplus_{i=1}^{n} P_i$ where P_i are free rank 1 ideals of $C([0,1])$. Clearly $P_i = f_i C([0,1])$ where f_i is a non-zero divisor (hence $\operatorname{coz}(f_i)$ is dense in $[0,1]$). If now $n > 1$, then $0 \neq f_i f_j \in P_i \cap P_j$ for some $i \neq j$, a contradiction.

(2) Assume P is not finitely generated. Then $I = \mathrm{Tr}(P)$ is a pure and projective ideal of $C([0,1])$ that is not finitely generated. Fact 9.4 shows that I is uniquely determined by the closed subset $\mathrm{zer}(I)$ of $[0,1]$, and the direct sum decompositions of I are in one-to-one correspondence with open partitions of the open set $[0,1] \setminus \mathrm{zer}(I)$. But every open subset of $[0,1]$ is uniquely represented as a disjoint union of at most countably many open intervals (a_i, b_i). Thus I uniquely decomposes into a direct sum of ideals $I_i = O_{[0,1]\setminus(a_i,b_i)}$ that are pure and projective. Since (a_i, b_i) is connected, I_i is indecomposable. As I_i is pure and proper, it cannot be finitely generated (otherwise this would give rise to non-trivial idempotents). It remains to apply Fact 9.5 and Kaplansky's theorem to get the desired decomposition of P. □

If $f = x - x^2$, then $\mathrm{coz}(f) = (0,1)$ is dense in $[0,1]$, hence $fC([0,1])$ is an example of a free rank 1 proper ideal of $C([0,1])$, which is not pure and a non-trivial example of part (1) above.

It seems there should be an abundance of non-finitely generated projective ideals of $C([0,1])$ that are not pure. However, we do not know any particular example.

If X is a compact metric space, by [13, Corollary 3.4], every pure ideal of $C(X)$ is projective and countably generated. This applies in particular to $C([0,1])$.

A module M is said to be *superdecomposable* if M has no indecomposable direct summands. (We consider 0 as a decomposable module!)

Question 9.7. *Is there a non-zero superdecomposable projective $C([0,1])$-module?*

After the next result we will see that $C([0,1])$ is not a Bézout ring.

The next task is to topologically describe weakly semihereditary rings of continuous functions. For this we have to prepare by a detour via a larger class of rings (respectively spaces) whose intersection with that of f-rings (respectively strongly zero-dimensional spaces) will yield the desired class of weakly semihereditary rings, see Proposition 9.13.

Two subsets A and B of a topological space X are *completely separated*, if there is an $f \in C(X)$ such that $f(A) = 0$ and $f(B) = 1$. Observe that completely separated sets have disjoint closures. The space X is called an *F-space*, if for every $f \in C(X)$, the sets $\mathrm{neg}(f)$ and $\mathrm{pos}(f)$ are completely separated. It follows that X is an F-space precisely when every pair of disjoint cozero sets is completely separated.

Fact 9.8. *(See [20, Theorem 14.25].) The ring $C(X)$ is Bézout if and only if X is an F-space.*

Remark 9.9. There are connected F-spaces, but $[0,1]$ is not an F-space. Indeed, if $f(x) = x - \frac{1}{2}$, then $\mathrm{neg}(f) = [0, \frac{1}{2})$ and $\mathrm{pos}(f) = (\frac{1}{2}, 1]$ are disjoint cozero sets which are not completely separated. In fact, a metric space is an F-space exactly when it is discrete (see [20, Ex. 14M3]). It follows that $C([0,1])$ is not a Bézout ring.

Recall, a commutative ring R is a *valuation ring*, if the lattice of ideals of R is a chain, and R is *arithmetical*, if the lattice of ideals of R is distributive. By Jensen [24, L. 1], a commutative ring is arithmetical iff every localization of R at a prime ideal (equivalently, maximal ideal) is a valuation ring.

Fact 9.10. *(See [24, Theorem].) A commutative ring has weak dimension $\leqslant 1$ if and only if R is arithmetical and reduced (that is, has no nilpotent elements).*

For the convenience of the reader we include the following known result with a self-contained proof.

Lemma 9.11. *(See [17].) The following are equivalent.*

(1) $C(X)$ *has weak dimension* $\leqslant 1$.
(2) $C(X)$ *is Bézout.*
(3) $C(X)$ *is arithmetical.*
(4) X *is an* F*-space.*

Proof. (1) $\Rightarrow$ (4). We need to show that disjoint cozero sets are completely separated. Suppose that $\mathrm{coz}(g) \cap \mathrm{coz}(h) = \emptyset$. This means that $gh = 0$. Since $C(X)$ has weak dimension $\leqslant 1$, the principal ideal $gC(X)$ is flat. Proposition 5.2 yields $u \in C(X)$ with $gu = g$ and $uh = 0$. Then $u|_{\mathrm{coz}(g)} = 1$ and $u|_{\mathrm{coz}(h)} = 0$, whence $\mathrm{coz}(g)$ and $\mathrm{coz}(h)$ are completely separated.

(2) and (4) are equivalent by Fact 9.8 above. (2) implies (3), because every Bézout ring is arithmetical. Finally, (3) $\Rightarrow$ (1) follows from Fact 9.10, for $C(X)$ is always reduced. $\square$

A topological space X is said to be a *P-space*, if every zero set of X is open. By [20, Theorem 14.29], $C(X)$ is a von Neumann regular ring iff X is a P-space. Since von Neumann regular rings are exactly the rings of weak dimension zero, we derive the following

Corollary 9.12. *A ring* $C(X)$ *has weak dimension* 1 *if and only if* X *is an* F*-space that is not a* P*-space.*

Now we are able to characterize weakly semihereditary rings of continuous functions.

Proposition 9.13. *The following are equivalent.*

(1) $C(X)$ *is weakly semihereditary.*
(2) $C(X)$ *is a Bézout* f*-ring.*
(3) X *is a strongly zero-dimensional* F*-space.*

Proof. (2) and (3) are equivalent by Proposition 9.1 and Fact 9.8.

(1) $\Rightarrow$ (2). That $C(X)$ is Bézout follows from Corollary 5.3 and Lemma 9.11; that it is an f-ring, from Propositions 5.4 and 9.1.

(2) $\Rightarrow$ (1). To prove that $C(X)$ is weakly semihereditary, by Bergman [7, Proposition 6.2 and thereafter], it suffices to prove that $C(X)$ is 1-weakly semihereditary and that every localization of $C(X)$ at a maximal ideal is a valuation domain. That every such localization is a valuation *ring* follows from the Bézout property. But $C(X)$ is reduced, so none of these contain zero-divisors.

To show that $C(X)$ is 1-weakly semihereditary, let $f_1, \ldots, f_n, g_1, \ldots, g_m \in C(X)$ be such that $f_i g_j = 0$ for all i, j, that is, $AB = 0$, where A is the column $(f_1, \ldots, f_n)^t$ and B is the row $g_1, \ldots, g_m$. We have to find an idempotent $e \in C(X)$ such that $f_i e = f_i$ and $eg_i = 0$ for every i.

First of all, $C(X)$ being Bézout allows us to reduce the matter to the case $n = m = 1$ as follows. Write $\sum_i f_i R = fR$ and $\sum_j g_j R = gR$ for appropriate $f, g \in R$. Then $f_i g_j = 0$ for all i, j iff $fg = 0$ (for $C(X)$ is commutative). Further, $fe = f$ and $eg = 0$ would imply $f_i e = f_i$ and $eg_j = 0$ for all i, j.

From $f_1 g_1 = 0$ we get $\operatorname{coz}(f_1) \cap \operatorname{coz}(g_1) = \emptyset$. Since X is an F-space, these cozero sets are completely separated, that is, they have disjoint zero set neighborhoods. But X is also strongly zero-dimensional, hence two disjoint zero sets have disjoint clopen neighborhoods U and V. If e is the projection onto U, then e is an idempotent, and $f_1 e = f_1$ and $e g_1 = 0$. □

Corollary 9.14. *If X is connected with more than one point, then $C(X)$ is not weakly semihereditary.*

Corollary 9.15. *The exchange ring $C(\mathbb{Q})$ from Example* 9.2 *is not weakly semihereditary.* (*Hence one can apply Warfield's, but not Bergman's theorem to conclude the decomposition of projective $C(\mathbb{Q})$-modules as direct sums of finitely generated modules.*)

Proof. Since a metric F-space is discrete it follows that $C(\mathbb{Q})$ is not Bézout, and thus not weakly semihereditary. □

Other interesting Tychonoff spaces are *basically disconnected* spaces, that is, spaces all of whose cozero sets have clopen closure. It turns out that these give rise to semihereditary rings of continuous functions, and vice versa. This was first proved, independently, by Brookshear [9] and De Marco [13].

Fact 9.16. *The ring $C(X)$ is semihereditary if and only if X is basically disconnected.*

Example 9.17. Let $\mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers carrying the discrete topology. Then $C(\mathbb{N}^*)$ is a weakly semihereditary ring which is not semihereditary. (Hence one can apply Bergman's, but neither Albrecht's nor Bass' theorems to conclude the decomposition in question.)

Proof. As $\mathbb{N}$ is locally compact and σ-compact, $\mathbb{N}^*$ is a compact F-space by [20, Theorem 14.27]. By [20, Exercise 6S 4], $\mathbb{N}^*$ is also (strongly) zero-dimensional. So $C(\mathbb{N}^*)$ is weakly semihereditary. On the other hand, by [20, Exercise 6W], $\mathbb{N}^*$ is not basically disconnected, hence $C(\mathbb{N}^*)$ is not semihereditary. □

Finally, we characterize weakly noetherian rings of continuous functions topologically.

Proposition 9.18. *The following are equivalent.*

(1) *$C(X)$ is* (*weakly*) *noetherian.*
(2) *$C(X)$ is hereditary.*
(3) *$C(X)$ is an F-ring.*
(4) *X is a finite discrete space.*

Proof. (1) $\Rightarrow$ (4). If $C(X)$ is weakly noetherian, $\operatorname{Max}(C(X))$ has the d.c.c. on closed subsets. Being homeomorphic to this space, $\beta(X)$ has the d.c.c. on closed subsets too. But $\beta(X)$ is compact, Hausdorff, and completely regular. It is not difficult to derive that $\beta(X)$, and hence also X, is a finite discrete space.

The converse being trivial, (1) and (4) are equivalent. By [13, Corollaries 3.4(a) and 2.3], (4) is also equivalent to (2) and to (3). □

Example 9.19. If X is an infinite discrete space ($\mathbb{N}$, for example), then $C(X) = \mathbb{R}^X$ is a von Neumann regular (and thus semihereditary) ring that is not weakly noetherian. (Hence one can apply Abrecht's and Bass's, but not Hinohara's theorem to conclude the decomposition in question.)

Next we give an example of a Bézout ring $C(X)$ whose projective ideals show a decomposition theory completely opposite to that of $C([0,1])$.

Recall that a commutative ring R is an *elementary divisor ring* if, for every $n \times n$ matrix A over R, there are invertible R-matrices U and V such that $UAV = \operatorname{diag}(a_1, \dots, a_n)$ with $a_{i+1} \in a_i R$ for every i. Notice, every elementary divisor ring is Bézout.

Remark 9.20. Let R be an elementary divisor ring. Then all finitely presented R-modules are direct sums of cyclically presented modules, i.e. modules of the form R/rR. If such a cyclically presented module is projective, rR must split in R_R, and so rR is generated by an idempotent. Hence, finitely generated projective R-modules are direct sums of principal ideals generated by idempotents. Consequently, if R has no non-trivial idempotents, finitely generated projective R-modules are free.

Example 9.21. (Gillman–Henriksen) Let $\mathbb{H} = [0, \infty)$ be the half-line and set $\mathbb{H}^* = \beta\mathbb{H} \setminus \mathbb{H}$. Then $C(\mathbb{H}^*)$ is a non-weakly semihereditary elementary divisor ring of weak dimension 1 without non-trivial idempotents, all of whose proper projective ideals are superdecomposable.

Proof. By [20, Theorem 14.27], $\mathbb{H}^*$ is a compact F-space and so, by Lemma 9.11, $C(\mathbb{H}^*)$ is a Bézout ring of weak dimension $\leqslant 1$. Clearly $\mathbb{H}^*$ is not a P-space, hence the weak dimension of $C(\mathbb{H}^*)$ is exactly 1 (Corollary 9.12). The fact that $C(\mathbb{H}^*)$ is an elementary divisor ring can be derived from [19, Example 4.9]. Furthermore, by [19, p. 92], $\mathbb{H}^*$ is connected, hence $C(\mathbb{H}^*)$ is not weakly semihereditary, by Corollary 9.14, and has no non-trivial idempotents (as idempotents in $C(X)$ correspond to clopen subsets of X).

Then, by Remark 9.20, every finitely generated projective $C(\mathbb{H}^*)$-module is free. Arguing as in Proposition 9.6 we see that, if P is a finitely generated projective ideal of $C(\mathbb{H}^*)$, then $P = fC(\mathbb{H}^*)$, where $\operatorname{coz}(f)$ is dense. But it is well known (see [16]) that $\mathbb{H}^*$ has no proper dense cozero sets, hence $P = C(\mathbb{H}^*)$.

It remains to verify that $C(\mathbb{H}^*)$ has no non-zero proper indecomposable projective ideal. If there were such an ideal, P, it would have to be non-finitely generated by what we have already proved. Then $I = \operatorname{Tr}(P)$ is a pure and projective ideal of $C(\mathbb{H}^*)$ which is not finitely generated, in particular I is proper and non-zero. By Fact 9.5, it suffices to prove that I is not indecomposable.

Otherwise, since $\mathbb{H}^*$ is compact, using part (4) of Fact 9.4 we see that $I = O_Z$, where $Z = \operatorname{zer}(f)$ for some $f \in C(\mathbb{H}^*)$. Clearly $\operatorname{zer}(f)$ is a non-empty proper subset of $\mathbb{H}^*$, hence the same is true for $\operatorname{coz}(f)$. Since direct sum decompositions of I correspond to open partitions of $\operatorname{coz}(f)$, it suffices to prove that $\operatorname{coz}(f)$ is not connected. By way of contradiction, suppose $\operatorname{coz}(f)$ is connected. Then its closure V in $\mathbb{H}^*$ is a connected closed, hence compact subspace of $\mathbb{H}^*$.

Theorem 4.2 of [21] states that $\mathbb{H}^*$ is an indecomposable continuum. This means that every proper compact, connected subspace is nowhere dense (hence has empty interior). Since $\operatorname{coz}(f) \subseteq V$ is open and non-empty, we conclude that $V = \mathbb{H}^*$, hence $\operatorname{coz}(f)$ is dense in $\mathbb{H}^*$. But we have already noticed that $\mathbb{H}^*$ has no proper dense cozero sets.

(Interestingly, [20, Ex. 6L4] shows that every proper zero set of $\mathbb{H}^*$ is also disconnected.) $\square$

Question 9.22. *Is there an indecomposable non-finitely generated projective $C(\mathbb{H}^*)$-module?*

We conclude with another curious example of a ring of continuous functions, one that is in none of the classes under investigation. For an example of a zero-dimensional space that is not strongly zero-dimensional, see [20, 16M].

Example 9.23. If X is a zero-dimensional space that is not strongly zero-dimensional, then every non-zero ideal of $C(X)$ contains an idempotent, but $C(X)$ has a projective and pure ideal which is not a direct sum of finitely generated ideals and not generated by idempotents.

Proof. By Proposition 9.1, $C(X)$ is not an f-ring, and by [34, Proposition 18] each of its non-zero ideals contains an idempotent. By Fact 3.8, it has a countably generated projective ideal which is not a direct sum of finitely generated ideals. By Fact 9.5, the same is true for its trace ideal. So there is a countably generated projective and pure ideal of the same kind. If it were generated by idempotents, it would split into a direct sum of principal ideals generated by those idempotents. □

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