Poles, zeros, and sheaf cohomology

Bostwick F. Wyman

Department of Mathematics, The Ohio State University, 231 West 18th Avenue, Columbus, OH 43210-1101, USA

Received 21 December 2000; accepted 31 July 2001

Submitted by J. Rosenthal

Abstract

The Fundamental Pole–Zero Exact Sequence for a transfer function matrix, first enunciated in 1989, supplies a structural meaning to the hope that “the number of zeros of a transfer function is the same as the number of poles”. This principle requires the inclusion of generic zeros counted using Wedderburn–Forney spaces which measure the kernel and cokernel of the transfer function. In this brief note we identify the Wedderburn–Forney space as a cohomology group and give a new approach to the Fundamental Pole–Zero Exact Sequence in terms of exact sequences of sheaf cohomology groups. © 2002 Elsevier Science Inc. All rights reserved.

Keywords: Pole module; Zero-module; Wedderburn–Forney space transfer function; Sheaf cohomology

1. Introduction

A rational function has just as many poles as zeros, if multiplicities are counted correctly and the point at infinity is included. The same result holds for square invertible matrices of rational functions, but not for rectangular or singular matrices. For the classical “fudge terms” to fix the hope that “a transfer function has just as many zeros as poles,” see, for example [4].

Kalman introduced the “pole module” or minimal realization space in [5] to give an algebraic structure to the set of poles of a transfer function. Subsequently Wyman–Sain [13] defined a module of zeros, and all this stuff was extended to include the point at infinity [1]. For an exposition and history, see [10,14,15]. The main goal of [14] was to give a structural interpretation of the statement “number of poles =
number of zeros” which included singular and rectangular transfer function matrices. The result, called the Fundamental Pole–Zero Exact Sequence, is

\[ 0 \to \mathcal{Z}(G) \to \mathcal{Z}(G) \to \mathcal{W}(\ker G) \to \mathcal{W}(\text{im } G) \to 0. \]

Here \( \mathcal{W}(\ker G) \) and \( \mathcal{W}(\text{im } G) \) denote Wedderburn–Forney spaces reviewed in Section 3 below. These spaces give a rather mystical finite measure of a rational vector space, more precise than the dimension. The ideas have roots in old work of Wedderburn [11], and a version first appeared in lost pages written by Forney [2]. See [15, Section 8], and for related material [2].

In our original work, we asked about the real meaning of Wedderburn–Forney spaces and an appropriate abstract definition. The present brief and rather formal note achieves this goal, at least if you believe that cohomology of sheaves can supply a real meaning. The Wedderburn–Forney spaces are identified as 1-cohomology groups of some natural sheaves, and the Fundamental Pole–Zero Exact Sequence follows quickly from the exactness of some cohomology sequences.

2. Sheaves, cohomology, poles, and zeros

In his papers [6–8], Lomadze introduced a concrete description of sheaves on the projective line \( \mathbb{P}^1 \) which is very well suited for computations in algebraic system theory. We begin here by briefly summarizing Lomadze’s point of view. In what follows \( k \) is a field of scalars. A sheaf is given by

\[ \mathcal{E} = (E, L, M, i, j) \]
or \( \mathcal{E} = (E, L, M) \) for short, where \( E \) is a vector space over the field of rational functions \( k(z) \), \( L \) is a module over the polynomial ring \( k[z] \), and \( M \) is a module over the ring \( \mathcal{O}_\infty \) of proper rational functions. The maps \( i \) and \( j \) map \( L \) and \( M \) into \( E \) with \( k(z) \otimes L \cong E \) and \( k(z) \otimes M \cong E \).

The rank of \( \mathcal{E} \) is the vector space dimension of \( E \).

If the map \( \alpha : L \oplus M \to E \) is given by \( \alpha(\ell, m) = i(\ell) - j(m) \), then the sheaf cohomology groups are given by \( H^0(\mathcal{E}) = \ker \alpha \) and \( H^1(\mathcal{E}) = \text{coker} \alpha \). These groups are exactly the standard cohomology groups in algebraic geometry, computed concretely here for \( \mathbb{P}^1 \), and they are all finite-dimensional vector spaces over \( k \). The sheaf \( \mathcal{E} \) is called a “vector bundle” when \( i \) and \( j \) are injective, and in this case we usually identify \( L \) and \( M \) as submodules of \( E \). For vector bundles the cohomology groups are computed simply by

\[ H^0(\mathcal{E}) = L \cap M, \]
\[ H^1(\mathcal{E}) = \frac{E}{L + M}. \]

If \( E = 0 \), then both \( L \) and \( M \) are finitely generated torsion modules. In this case \( H^0(\mathcal{E}) = L \oplus M \) and \( H^1(\mathcal{E}) = 0 \).
A short exact sequence of sheaves

\[ 0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0 \]

yields a long (six-term) exact sequence of cohomology spaces

\[ 0 \to H^0(\mathcal{E}_1) \to H^0(\mathcal{E}) \to H^0(\mathcal{E}_2) \to H^1(\mathcal{E}_1) \to H^1(\mathcal{E}) \to H^1(\mathcal{E}_2) \to 0. \]

The proof of exactness uses the snake lemma applied to a diagram containing various \( \alpha \)-mappings.

2.1. Twisting

Suppose given a sheaf \( \mathcal{E} = (E, L, M) \).

The “twist by an integer \( n \)”, \( \mathcal{E}(n) \), is given by

\[ \mathcal{E}(n) = (E, L, z^n M). \]

In these notes twisting is used primarily with \( n = -1 \). Most frequently \( M \) is a module of proper vectors such as \( \mathcal{O}_{\infty}^n \), and \( z^{-1} M \) is the corresponding module of strictly proper vectors such as \( z^{-1} \mathcal{O}_{\infty}^n \).

2.2. The standard structure sheaf

The standard structure sheaf \( \mathcal{O} \) on \( \mathbb{P}^1 \) is given by

\[ \mathcal{O} = (k(z), k[z], \mathcal{O}_{\infty}) \]

and its twists are

\[ \mathcal{O}(n) = (k(z), k[z], z^n \mathcal{O}_{\infty}). \]

The sheaf \( \mathcal{O}(n) \) has Chern number \( n \). For \( n \geq 0 \) we have \( \dim H^0(\mathcal{O}(n)) = n + 1 \) and \( H^1(\mathcal{O}(n)) = 0 \). For \( n < 0 \), \( H^0(\mathcal{O}(n)) = 0 \) and \( \dim H^1(\mathcal{O}(n)) = -n - 1 \).

2.3. Sheaves of poles and zeros for transfer functions

We consider vector spaces \( U \) and \( Y \) over the field of scalars, with \( \dim U = m \) and \( \dim V = p \). The corresponding rational vector spaces, obtained by tensoring up with \( k(z) \), are denoted by \( U(z) \) and \( Y(z) \). We have free polynomial submodules \( U[z] \) and \( Y[z] \), and free modules \( U(z)_{\text{pr}} \) and \( Y(z)_{\text{pr}} \) over the ring \( \mathcal{O}_{\infty} \). It will not hurt to think of \( U(z) \) as a space of column vectors of rational functions, \( U[z] \) as polynomial vectors, and \( U(z)_{\text{pr}} \) as vectors of proper rational functions. We will also use the notation \( U(z)_{\text{sp}} = z^{-1}U(z)_{\text{pr}} \) for the appropriate \( \mathcal{O}_{\infty} \)-module of vectors of strictly proper rational functions. A transfer function \( G : U(z) \to Y(z) \) is a \( k(z) \)-linear transformation, or (thinking concretely) a \( p \times m \) matrix over \( k(z) \). The pole sheaf associated to \( G \) is the finite-dimensional torsion sheaf
\[ \mathcal{X}(G) = \left( 0, \frac{GU[z] + Y[z]}{Y[z]}, \frac{GU(z)_{sp} + Y(z)_{sp}}{Y(z)_{sp}} \right). \]

The stalk of the pole sheaf at a finite point \( z_0 \) or at \( \infty \) is exactly the local pole module there in the sense of [12]. The polynomial part of the sheaf is just the minimal state space. Note that since the modules are torsion, the corresponding field is zero.

The sheaf of zeros associated to the transfer function \( G \) is the finite-dimensional torsion sheaf

\[ \mathcal{Y}(G) = \left( 0, \frac{GU(z) \cap Y[z]}{GU[z] \cap Y[z]}, \frac{GU(z)_{sp} \cap Y(z)_{sp}}{GU(z)_{sp} \cap Y(z)_{sp}} \right). \]

The stalk of \( \mathcal{Y}(G) \) at a finite point or at \( \infty \) is just the local zero module there. The polynomial part is the Zero Module studied in [13].

3. Cohomology and Wedderburn–Forney spaces

In [14] we introduced the notion of the Wedderburn–Forney Space of a rational vector space. Suppose given \( V \subset k(z)^n \), a space of column vectors of rational functions. If \( v \) is a column vector in \( V \), let \( \pi_+ v \) be the polynomial part of \( v \) and let \( \pi_- v \) be the strictly proper part of \( v \). Then the Wedderburn–Forney space of \( V \) is just the space of polynomial parts of \( V \) modulo the wholly polynomial vectors in \( V \):

\[ W(V) = \frac{\pi_+ V}{V \cap k[z]^n}. \]

Equivalently, since applying \( \pi_+ \) is the same as factoring out strictly proper parts, we can write

\[ W(V) = \frac{V}{V \cap k[z]^n + V \cap k(z)^n_{sp}}. \]

The Wedderburn–Forney spaces are finite dimensional over the scalar field. These spaces measure “zero structures” and fit into the “Fundamental Pole–Zero Exact Sequence” discussed in Section 4 below. We left unresolved the problem of finding a persuasive natural definition of the Wedderburn–Forney space. In this section, we identify the Wedderburn–Forney as a 1-cohomology group of an attractive sheaf.

Given \( V \subset k(z)^n \), define a sheaf

\[ \mathcal{V} = (V, V \cap k[z]^n, V \cap k(z)^n_{sp}). \]

Note that \( \mathcal{V} \) is not a coordinate-free construction based on the abstract vector space \( V \) but depends essentially on the imbedding of \( V \) in \( k(z)^n \).

**Theorem 1.** The twisted sheaf

\[ \mathcal{V}(-1) = (V, V \cap k[z]^n, V \cap k(z)^n_{sp}) \]
has the following cohomology groups:

\[ H^0(\mathcal{V}(-1)) = 0, \quad (3.1) \]
\[ H^1(\mathcal{V}(-1)) = \mathcal{W}(V). \quad (3.2) \]

**Proof.** Since \( k[z] \cap k(z)_{sp} = 0 \), (3.1) is immediate. Result (3.2) follows from the second definition of \( \mathcal{W}(V) \) (after “equivalently”) and the definition of cohomology groups. A kind reader will refrain from asking why we did not know this result in 1989. □

According to Grothendieck’s Theorem on vector bundles on \( \mathbf{P}^1 \) (see, for example [3,7]), each vector bundle can be written as a direct sum of line bundles of the form \( \mathcal{O}(m) \). Write

\[ \mathcal{V} = \mathcal{O}(-n_1) \oplus \mathcal{O}(-n_2) \oplus \cdots \oplus \mathcal{O}(-n_k), \]

where \( k = \dim \mathcal{V} \) and each \( n_i \geq 0 \). Twisting one more time, we obtain

\[ \mathcal{V}(-1) = \mathcal{O}(-1 - n_1) \oplus \mathcal{O}(-1 - n_2) \oplus \cdots \oplus \mathcal{O}(-1 - n_k). \]

We have by computations in Section 2,

\[ \dim \mathcal{W}(\mathcal{V}) = \dim H^1(\mathcal{V}(-1)) = n_1 + n_2 + \cdots + n_k. \]

The \( n_i \) are known variously as Wedderburn numbers, Kronecker indices, Forney indices, or (in some contexts) controllability indices. The study of these numbers is an industry which will be ignored here.

### 4. The Fundamental Pole–Zero Exact Sequence

In this section we identify the Fundamental Pole–Zero Exact Sequence, first introduced in [14] and recalled in the introduction above, as a cohomology exact sequence.

Consider the transfer function \( G : U(z) \rightarrow Y(z) \). We need to consider general \( G \), not necessarily injective or surjective. We present three sheaves, each with rational section space \( GU(z) \), but with different polynomial and proper module structures

\[ \mathcal{G} = (GU(z), GU(z) \cap Y[z], GU(z) \cap Y(z)_{pr}), \]
\[ \mathcal{G}_1 = (GU(z), GU[z] \cap Y[z], GU(z)_{pr} \cap Y(z)_{pr}), \]
\[ \mathcal{G}_2 = (GU(z), GU[z], GU(z)_{pr}), \]
\[ \mathcal{G}_3 = (\ker G, \ker G \cap U[z], \ker G \cap k(z)_{sp}). \]

The sheaves \( \mathcal{G} \) and \( \mathcal{G}_3 \) use the construction from Section 3, while the sheaves \( \mathcal{G}_1 \) and \( \mathcal{G}_2 \) are special sheaves whose usefulness will appear soon. We need to compute a bunch of cohomology groups, and the following lemma summarizes results. Some proofs are routine, but others are difficult and are central to our development.
Theorem 2. The twisted versions of three sheaves above have the following cohomology groups:

\[ H^0(G(-1)) = 0, \]  

\[ H^1(G(-1)) = \mathcal{W}(\text{im } G), \]  

\[ H^0(G_1(-1)) = 0, \]  

\[ H^1(G_1(-1)) = \frac{\mathcal{X}}{\mathcal{W}(\ker G)}, \]  

\[ H^0(G_2(-1)) = \mathcal{W}((\ker G), \]  

\[ H^1(G_2(-1)) = 0. \]

Proof. Results (4.1), (4.3), and (4.6) are routine. Eq. (4.2) follows immediately from Section 3. The two results involving assertions (4.4) and (4.5) are more difficult and will be proved below.

Recall first the definition of zeros given in Section 2

\[ \mathcal{Z}(G) = \left(0, \frac{GU(z) \cap Y[z]}{GU(z) \cap Y(z)_{sp}}, \frac{GU(z) \cap Y(z)_{sp}}{GU(z)_{sp} \cap Y(z)_{sp}}\right). \]

The sheaf of zeros fits into the short exact sequence of sheaves

\[ 0 \rightarrow G_1(-1) \rightarrow G(-1) \rightarrow \mathcal{Z}(G) \rightarrow 0. \]

By cohomology

\[ 0 \rightarrow H^0(\mathcal{Z}(G)) \rightarrow H^1(G_1(-1)) \rightarrow H^1(G(-1)) \rightarrow 0 \]

is exact. Since (like the pole sheaf) \( \mathcal{Z}(G) \) is finite, we have the \( H^0(\mathcal{Z}(G)) \) is essentially \( \mathcal{Z}(G) \). We saw above that \( H^1(G(-1)) = \mathcal{W}(\text{im } G) \). (Assertion (4.2) of the theorem.) According to assertion (4.4), still unproved,

\[ H^1(G_1(-1)) = \frac{\mathcal{X}}{\mathcal{W}(\ker G)}. \]

Putting all this together, we obtain our main theorem:

Main Theorem (Fundamental Pole–Zero Exact Sequence). The following sequence of finite-dimensional vector spaces over the field \( k \) of scalars is exact:

\[ 0 \rightarrow \mathcal{Z}(G) \rightarrow \frac{\mathcal{X}(G)}{\mathcal{W}(\ker G)} \rightarrow \mathcal{W}(\text{im } G) \rightarrow 0, \]

which is nothing but the Fundamental Pole–Zero Exact Sequence.
Proof. All that remains is to prove assertion (4.4) of the previous theorem.

Consider the short exact sequence of sheaves with natural morphisms

\[ 0 \to \mathcal{G}_1(-1) \to \mathcal{G}(-1) \to \mathcal{Y} \to 0, \]

where \( \mathcal{Y} \) is the sheaf of zeros of \( G \). By cohomology

\[ 0 \to H^0(\mathcal{Y}) \to H^1(\mathcal{G}_1(-1)) \to H^1(\mathcal{G}(-1)) \to 0 \]

is exact. Since (like the pole sheaf) \( \mathcal{Y} \) is finite, we have the \( H^0(\mathcal{Y}) \) is essentially \( \mathcal{Y} \).

Just as in Section 3, \( H^1(\mathcal{G}(-1)) = \mathcal{W}(\operatorname{im} G) \). Furthermore, we claim

\[ H^1(\mathcal{G}_1(-1)) \cong \frac{\mathcal{X}(G)}{\mathcal{W}(\ker G)}. \]  

(4.4)

Assuming this last claim, we can rewrite the sequence above as

\[ 0 \to \mathcal{Y}(G) \to \frac{\mathcal{X}(G)}{\mathcal{W}(\ker G)} \to \mathcal{W}(\operatorname{im} G) \to 0, \]

which is nothing but the Fundamental Pole–Zero Exact Sequence.

To prove (4.4) we express the global pole sheaf slightly differently. We need to check the exactness of

\[ 0 \to \mathcal{G}_1(-1) \to \mathcal{G}_2(-1) \to \mathcal{X} \to 0. \]

The vector space parts of the first and middle terms are identical, producing zero as expected for the vector space in the cokernel. To check the polynomial module isomorphism, look at the definition of poles in Section 1 and note that

\[ \frac{GU[z] + Y[z]}{Y[z]} \cong \frac{GU[z]}{GU[z] \cap Y[z]}. \]

The stalk at infinity can be handled in the same way.

Cohomology yields

\[ 0 \to H^0(\mathcal{G}_2(-1)) \to H^0(\mathcal{X}) \to H^1(\mathcal{G}_1(-1)) \to 0. \]

We are trying to compute \( H^1(\mathcal{G}_1(-1)) \). If we believe that \( H^0(\mathcal{G}_2(-1)) = \mathcal{W}(\ker G) \), then we are done, since we have the short exact sequence

\[ 0 \to \mathcal{W}(\ker G) \to \mathcal{X} \to H^1(\mathcal{G}_1(-1)) \to 0. \]

To finish this (not quite infinite) regression, we use the sheaf-theoretic kernel \( \mathcal{G}_3 \) of the transfer function \( G : U(z) \to Y(z) \) defined in Section 4 which fits into an exact sequence

\[ 0 \to \mathcal{G}_3 \to \mathcal{O} \otimes U \to \mathcal{G}_2 \to 0. \]

Twisting, we find that

\[ \mathcal{G}_3(-1) = (\ker G, \ker G \cap U[z], \ker G \cap k(z)_{\text{sp}}) \]

so that the usual calculation gives \( H^1(\mathcal{G}_3(-1)) = \mathcal{W}(\ker G) \).
Finally, cohomology (after twisting by $-1$) gives
\[ 0 \to H^0(G_2(-1)) \to H^1(G_3(-1)) \to 0, \]
or $H^0(G_2(-1)) \cong \mathcal{W}(\ker G)$, as required. □

5. The global Hermann–Martin Sheaf

We conclude this paper with a modest generalization of the idea of the Hermann–Martin Sheaf of a transfer function introduced in [6].

Define the “global” Hermann–Martin Sheaf of a transfer function $G : U(z) \to Y(z)$ by
\[ \mathcal{H}\mathcal{M}(G) = (Y(z), Y[z] + GU[z], Y(z)_{pr} + GU(z)_{pr}) \]
with a twisted counterpart
\[ \mathcal{H}\mathcal{M}(G)(-1) = (Y(z), Y[z] + GU[z], Y(z)_{sp} + GU(z)_{sp}). \]

The cohomology group $H^1(\mathcal{H}\mathcal{M}(G)(-1)) = 0$. To see this, note that
\[ \frac{Y(z)}{Y[z] + Y(z)_{sp}} = 0. \]

The remainder of this section is devoted to showing that the cohomology group $H^0(\mathcal{H}\mathcal{M}(G)(-1))$ is essentially the pole sheaf.

Recall that the pole sheaf of $G$ is
\[ \mathcal{X}(G) = \left(0, \frac{GU[z] + Y[z]}{Y[z]}, \frac{GU(z)_{sp} + Y(z)_{sp}}{Y(z)_{sp}}\right). \]

Also, the structure sheaf on $Y$ is given by $\mathcal{O}(Y) = (Y(z), Y[z], Y(z)_{pr})$ so that the pole sheaf fits into the short exact sequence of sheaves
\[ 0 \to \mathcal{O}(Y)(-1) \to \mathcal{H}\mathcal{M}(G)(-1) \to \mathcal{X}(G) \to 0. \]

The morphisms are the natural ones suggested by the sheaf definitions. The corresponding exact sequence of cohomology groups collapses to
\[ 0 \to H^0(\mathcal{H}\mathcal{M}(G)(-1)) \to H^0(\mathcal{X}) \to 0. \]

Since $\mathcal{X}$ is finite dimensional, we have that $H^0(\mathcal{X})$ and $\mathcal{X}$ are both given by the sum of the polynomial pole space and the pole space at infinity. On the other hand, $\mathcal{X}$ is a sheaf while $H^0(\mathcal{X})$ is only a vector space, so perhaps they are not quite the same object. Close enough, perhaps, so that we can say the “The global sections of the Hermann–Martin sheaf give the poles of the transfer function”.

Acknowledgements

I would like to thank the referees for an important correction and many useful comments.

References