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NICHE GRAPHS

Charles CABLE
Allegheny College, Meadville, PA 16335, USA

Kathryn F. JONES and J. Richard LUNDGREN
Department of Mathematics, University of Colorado at Denver, 1200 Larimer Street, Campus Box 170, Denver, CO 80204, USA

Suzanne SEAGER
Carleton University, Ottawa, Ontario, Canada K1S 5B6

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If $D=(V,A)$ is an acyclic digraph and $G=(V,E)$ is a graph such that two vertices x and y are adjacent in G if and only if they have a common predator vertex or prey vertex in D, then G is called a niche graph. It is easy to show that not all graphs are niche graphs. However in many cases it is possible to adjoin a finite set of vertices, say I_m, to the vertex set V of both G and D, and also some additional arcs to the arc set to obtain G' and D' respectively where G' is a niche graph, $V'=V\cup I_m$ and $E'=E$. The smallest number of vertices that one must adjoin to G to obtain a niche graph is called the niche number of G. Some classes of niche graphs are investigated, including paths and cycles. We also calculate the niche number of some other graphs. An infinite class of graphs is exhibited in which none of the graphs in that class has a niche number and a characterization of niche graphs is given.

1. Introduction

Competition graphs were introduced by Cohen [3] in 1968 in connection with a problem in ecology. Since then, several authors, including Cohen [1–4], Dutton and Brigham [5], Lundgren and Maybee [8, 9], Opsut [10], Roberts [12, 14, 15], Roberts and Steif [16], Steif [18], and Sugihara [19, 20] have studied competition graphs. Recently, Raychaudhari and Roberts [11] investigated applications to communications, radio and television transmission, and large modeling problems by considering generalized competition graphs. Here we study another natural extension of competition graphs.

Lundgren and Maybee [8] showed that the competition graph of an acyclic digraph D is the row graph, RG(A), of the adjacency matrix A of D. They introduced the dual of this graph, the common enemy graph of D [9], which is the column graph of A, CG(A). This led Scott [17] to introduce the competition-common-enemy graph (CCE graph) of D. This graph is essentially the intersection of the competition graph and the common enemy graph. That is, two vertices are adjacent if and only if they have both a common prey and a common enemy in D. The niche graph, which we study here, is the union of the competition graph and the common enemy graph.

If $D=(V,A)$ is a digraph, the niche graph corresponding to D is the undirected graph $G=(V,E)$ with an edge between two distinct vertices x and y of V if and only if for some $z \in V$, there are arcs $[x,z]$ and $[y,z]$ in D or there are arcs $[z,x]$ and $[z,y]$ in D.

For a digraph D, let C(D) be the competition graph of D, CE(D) be the common enemy graph, CCE(D) the CCE graph, and N(D) the niche graph. From Fig. 1, we can see the relationship between these graphs. In particular, CCE(D) $\subseteq$ C(D) $\subseteq$ N(D)

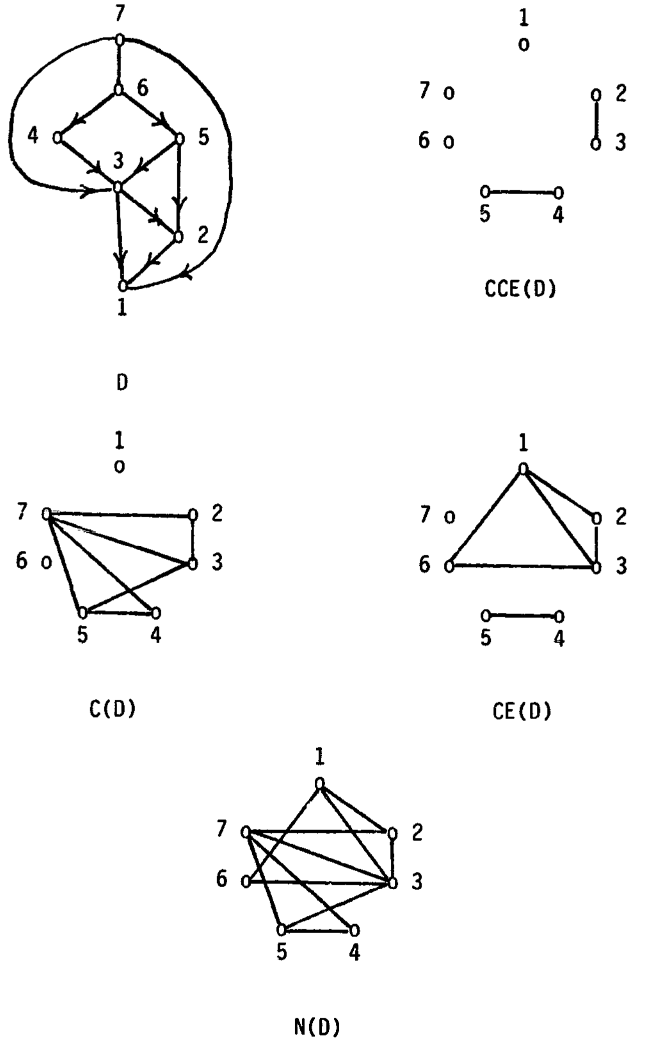

Fig. 1.

and $CCE(D) \subseteq CE(D) \subseteq N(D)$. One reason ecologists have studied competition graphs is to determine the dimension of trophic niche space. Most food webs occurring in nature have competition graphs that are interval. However, determining which acyclic digraphs have interval competition graphs remains an open problem. Using niche graphs may be an alternative for determining the dimension of trophic niche space.

Many of the properties of these graphs are illustrated in Fig. 1. Ecologists generally assume that the digraphs modeling food webs are acyclic, so we will investigate the problem of determining which graphs are niche graphs of acyclic digraphs. In $C(D)$, two vertices are adjacent if they have a common prey, and in $CE(D)$ two vertices are adjacent if they have a common predator. $C(D)$ and $CE(D)$ each have at least one isolated vertex in Fig. 1, and since D is acyclic, it is easy to see that this is always true for competition graphs and common enemy graphs. This follows from the result that the vertices of an acyclic digraph can be labeled so that (i,j) an arc implies $i<j$ (see Roberts [13, Theorem 11.13]). Roberts [14] showed that every graph G could be made into a competition graph of an acyclic digraph by adding a finite number of isolated vertices. He defined the competition number $k(G)$ to be the least k such that G together with k isolated vertices, $G \cup I_k$, is the competition graph of an acyclic digraph. Similarly, since D is acyclic, $CCE(D)$ always has at least two isolated vertices, and the double competition number $dk(G)$ is defined in a similar manner (see Scott [17] and Jones et al. [7]). Following this pattern, we define the niche number $n(G)$ to be the smallest number of isolated vertices k such that $G \cup I_k$ is the niche graph of an acyclic digraph. However, there are two significant differences. From Fig. 1, we see that $n(G)=0$ is possible for a connected graph. In the next section we will describe a class of graphs that cannot be made into niche graphs by the addition of isolated points. For such graphs we say that the niche number is infinite and write $n(G)=\infty$.

In this paper we find niche numbers for various classes of graphs. Surprisingly, all graphs we have studied either have $n(G) \leq 2$ or $n(G)=\infty$. While we give a characterization of niche graphs in Section 4, it is difficult to use. Indeed, finding $n(G)$ is not easy and is most likely an NP-complete problem as Opsut [10] established for $k(G)$.

2. Graphs with infinite niche number

In contrast to the situation for competition graphs and CCE graphs, there are graphs that cannot be made into niche graphs by the addition of isolated points. An infinite class of such graphs is presented in this section.

Let $K_{1,n}$ be the star consisting of one central vertex and n neighboring vertices with no other edges. We define a *nova* to be a graph obtained by replacing each edge of the star $K_{1,n}$ where $n \geq 3$ by a clique on at least two vertices. The graph in Fig. 2 is a nova obtained by replacing the edges of a $K_{1,3}$ by cliques of size 2, 3, and 4.

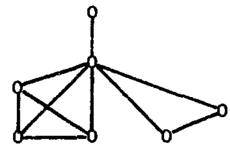

Fig. 2.

Lemma 2.1. *If $G=(V,E)$ is a nova with center x and $G \cup I_m$ is the niche graph of an acyclic digraph D, then each maximal clique C of G contains a vertex $y \neq x$ so that y and x have in D a common predator or prey that is not a vertex of C.*

Proof. If C is a K_2, then the result is immediate. So we assume that C is a K_n where $n \geq 3$. Since $D=(V,A)$ is acyclic, V may be labeled so that the arc $(i,j) \in A$ only if $i<j$. Let u be the vertex of C that has the smallest label among the vertices of C and v the vertex of C with the largest label. Since the edge $[u,v] \in E$, u and v have a common predator or prey in D. If they have a common predator, then its label is smaller than that of either u or v and hence it is not in C. Similarly, a common prey cannot be in C. Hence, if either u or v is the center vertex x, we are done.

So we may assume that neither u nor v is x. Without loss of generality, we suppose that u and v have a common prey $y \notin C$. Since $[u,x] \in E$, u and x have a common predator or prey. If they have a common predator, then it is not in C, since u has the smallest label in C. If u and x have a common prey z, then either (1) $z=y$ or (2) $z \neq y$ and then, since z and y have the common predator u, $[z,y] \in E$. But since $y \notin C$ and $z \neq x$, then $z \notin C$. □

Theorem 2.2. *If G is a nova, then $n(G)=\infty$.*

Proof. Let G be a nova with maximal cliques $C_1, C_2, \ldots, C_j$, where $j \geq 3$ and suppose for some integer $m \geq 0$, $G \cup I_m$ is the niche graph of an acyclic digraph D. Let x be the center of G. By Lemma 2.1, each maximal clique C_i contains a vertex $v_i \neq x$ such that v_i and x have a common predator or prey which is not in C_i. Since there are at least three maximal cliques, at least two of these vertices in different cliques have common predators with x or two have common prey with x. By symmetry, we may assume that there are vertices b and c adjacent to x such that $b \in C_2$ and $c \in C_3$, where C_2 and C_3 are distinct cliques and vertices a_1 and a_2 such that $a_1 \notin C_2$, $a_2 \notin C_3$, a_1 preys on b and x and a_2 preys on c and x. Now $a_1 \neq a_2$ else there is an edge between b and c. Since a_1 and a_2 both prey on x, $[a_1,a_2] \in E$, so that a_1 and a_2 belong to the same clique C_1, distinct from C_2 and C_3.

By Lemma 2.1, there is a vertex $u \in C_1$ such that u and x have a common predator or prey which is not in C_1. But all predators of x are in C_1 since any such predator must be adjacent to a_1 and a_2. Thus x has a prey in $C' \neq C_1$. Now C' may be C_2, C_3, or distinct from these, but by symmetry we may assume that $C' \neq C_3$.

The edge $[a_2, x] \in E$, so a_2 and x must have a common predator or prey. Since all prey of a_2 are in C_3 and all prey of x are in $C' \neq C_3$, a_2 and x must have a common predator. But all predators of x are in C_1, so there is $a_3 \in C_1$ such that a_3 preys on x and a_2. Now $a_3 \neq a_1$, since all prey of a_1 must be in C_2. Thus a_3 is distinct from a_1 and a_2. In a similar way, using the edge $[a_3, x]$, we can find a vertex $a_4 \in C_1$ that preys on a_3 and x. Then $a_4 \neq a_1$ since all prey of a_1 are in C_2, and $a_4 \neq a_2$ since then we would have a cycle $\langle a_2, a_3, a_2 \rangle$ in D. So the sequence $\{a_4, a_3, a_2\}$ determines a path in D. We can continue in this way so that given the path $(a_k, \dots, a_2)$ in D with $a_i \in C_1$, we can use the edge $[a_k, x]$ to find a new vertex $a_{k+1} \in C_1$ that preys on a_k and x. But C_1 is finite, so this is a contradiction. Hence, $n(G) = \infty$. □

Since $K_{1,n}$ is a nova for $n \geq 3$, we have $n(K_{1,3}) = \infty$. However, $K_{1,3}$ is not a forbidden subgraph for niche graphs. To see this, let G be the graph in Fig. 3(a) and D the digraph in Fig. 3(b). Then G is the niche graph of D.

Fig. 3.

It is easy to see that if G has components $C_1, C_2, \dots, C_k$, then $n(G) \leq n(C_1) + n(C_2) + \cdots + n(C_k)$. In particular, if all of the components of G are niche graphs, then so is G. But the converse can fail quite spectacularly: $n(K_{1,3}) = \infty$, but $n(K_{1,3} \cup K_{1,3}) = 0$. To see this, if D is the digraph in Fig. 4(a), then $K_{1,3} \cup K_{1,3}$ is the niche graph of D as illustrated in Fig. 4(b).

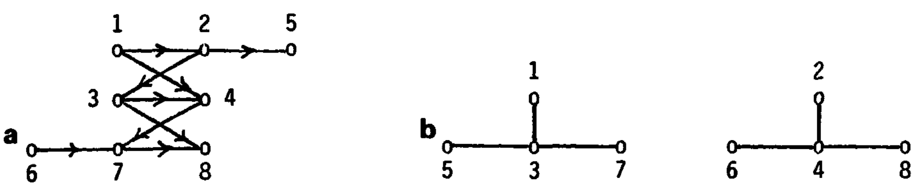

Fig. 4.

3. Niche numbers of some classes of graphs

In this section we calculate the niche numbers of various classes of graphs including paths, cycles, and complete graphs.

Theorem 3.1. *If K_n is the complete graph on n vertices with $n \geq 2$, then $n(K_n)=1$.*

Proof. If we let D be the digraph in Fig. 5, then the niche graph of D is $K_n \cup \{a\}$. Hence, $n(K_n) \leq 1$. On the other hand, if D is any acyclic digraph on n vertices, then D has a vertex x with an indegree of 0 and a vertex $y \neq x$ having outdegree 0. So x is not adjacent to y in the niche graph of D. Hence, K_n is not the niche graph of D and $n(K_n)>0$. □

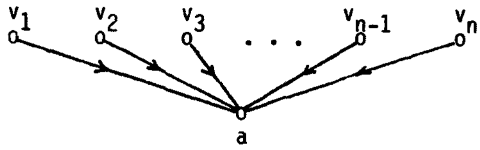

Fig. 5.

The next class of graphs that we consider are noncomplete graphs on $n+1$ vertices that contain a clique on n vertices.

Theorem 3.2. *If G is the complete graph K_n together with an additional vertex that is adjacent to k of the vertices of K_n and $0 \leq k < n$, then G is a niche graph.*

Proof. Let D be the digraph of Fig. 6. Clearly, vertices $v_1, \ldots, v_n$ form a complete graph on n vertices in the corresponding niche graph. Moreover, v_{n+1} is adjacent to $v_2, v_3, \ldots, v_{k+1}$. □

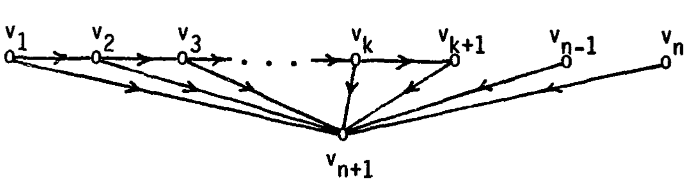

Fig. 6.

If P_n is a path on n vertices, then $n(P_2)=1$ by Theorem 3.1. However, for $n \geq 3$, the digraph D in Fig. 7 has P_n as its niche graph. We have proved the following theorem.

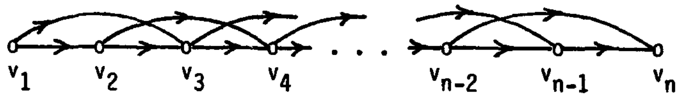

Fig. 7.

Theorem 3.3. *If P_n is a path on n vertices, then $n(P_n)=0$ for $n \geq 3$ and $n(P_2)=1$.*

The situation for cycles C_n is more complicated than for paths, but for all n sufficiently large we get that $n(C_n)=0$.

Theorem 3.4. *Let C_n be a cycle on n vertices. The following holds:*

(i) $n(C_n)=0$ *for* $n=7$ *and* $n\geq 9$.
(ii) $n(C_n)=1$ *for* $n=3$ *and* $n=8$.
(iii) $n(C_n)=2$ *for* $n=4$, 5, *and* 6.

Proof. By Theorem 3.1, $n(C_3)=1$. In the following, let the vertices of the cycle C_n be labeled consecutively $v_1, v_2, \ldots, v_n$. For $n=4,5,6$, we construct an acyclic digraph D_n as follows: let $v_1, v_2, \ldots, v_{n-1}$ induce the digraph of Fig. 7. Then let a and b be vertices such that a preys on v_1 and v_n, and v_{n-1} and v_n prey on b. The niche graph of D_n is $C_n \cup I_2$, thus establishing that $n(C_n)\leq 2$ for $n=4,5,6$. That $n(C_n)\geq 2$ for $n=4,5,6$ has been verified by exhaustive computer search and can be established by tedious case arguments. We include the case argument proof for $n=4$.

Suppose first that C_4 is the niche graph of an acyclic digraph D. In order that $[v_1, v_2]\in E$, v_1 and v_2 must have common predator or prey in D. Suppose v_1 and v_2 have common prey v_3. (The case that they have common prey v_4 is entirely symmetric.) Then v_3 can have no additional predators, since this would produce a triangle in the niche graph. Thus in order that $[v_3, v_4]\in E$, v_3 and v_4 must have common prey in D, which is either v_1 or v_2. But this produces a cycle $\langle v_1, v_3, v_1\rangle$ or $\langle v_2, v_3, v_2\rangle$ in D. Thus $n(C_4)\geq 1$.

Next suppose $G=C_4 \cup I_1$ is the niche graph of an acyclic digraph D and let u be the isolated vertex. Since there are no triangles in G, the isolated vertex u has exactly two predators or preys in D. Suppose without loss of generality that v_1 and v_2 prey on u. Then since u is isolated, v_1 and v_2 can have no other prey. Now in order that $[v_4, v_1]\in E$ and $[v_2, v_3]\in E$, there exists a vertex a that preys on v_1 and v_4 and a vertex b that preys on v_2 and v_3. It must be the case that $a=v_3$ and $b=v_4$. But then $\langle v_3, v_4, v_3\rangle$ is a 2-cycle, a contradiction. Thus $n(C_4)\geq 2$.

For $n=7$, we let D be the digraph of Fig. 8. This digraph has niche graph C_7, thus establishing that $n(C_7)=0$.

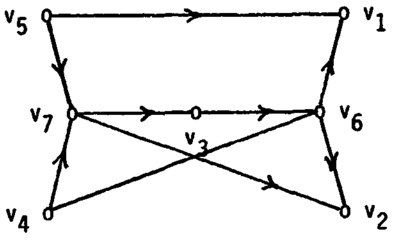

Fig. 8.

To establish that $n(C_8) \leq 1$, we use the digraph in Fig. 9, which has $C_8 \cup I_1$ as its niche graph. That $n(C_8) = 1$ has been verified by exhaustive computer search.

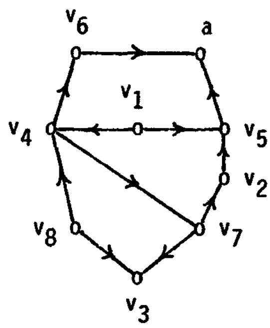

Fig. 9.

Finally, the digraph of Fig. 10 has C_n, $n \geq 9$, as its niche graph. This proves then $n(C_n) = 0$ for $n \geq 9$. □

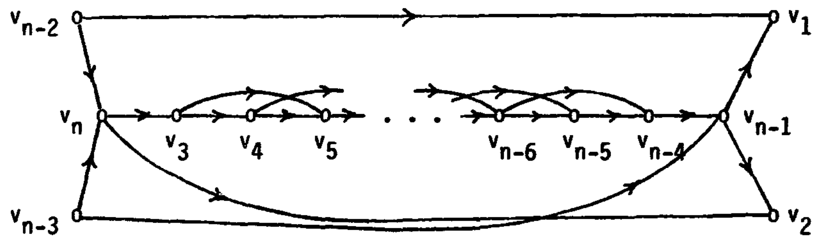

Fig. 10.

As a consequence of the previous theorems, it follows that all graphs on four or fewer vertices are niche graphs with the following exceptions: the complete graphs, which have niche number 1; the cycle C_4, which has niche number 2; and the star $K_{1,3}$ which is the smallest graph without finite niche number.

4. Bounds and characterization of niche graphs

In this section we present an upper bound for the niche number of those graphs which have a finite niche number. We also prove a theorem which gives a sufficient condition for a graph to have infinite niche number. Finally, we give a clique characterization of niche graphs.

Theorem 4.1. *If $G = (V, E)$ is a graph and $n(G) < \infty$, then $n(G) \leq |V|$.*

Proof. Suppose that $n(G) = m$ and $D = (V \cup I_m, A)$ is an acyclic digraph with niche graph $G \cup I_m$. Let $b \in I_m$. Then there are at least two arcs of the digraph D leading from b to the vertex set V or at least two arcs of D leading from V into b. (Otherwise, this isolated vertex would not be necessary and we could delete it.)

In D there cannot be two arcs leading from a vertex v of V to the set I_m of isolated vertices. Otherwise, there would be a pair of vertices of I_m which are adjacent in G. Similarly, there cannot be two arcs in D leading from the set I_m into a single vertex v of V. Hence, if $v \in V$, there can be at most one arc of D leading from v to a vertex of I_m and at most one arc of D leading from the set I_m to the vertex v.

If we let S denote the set of arcs of D connecting the set of isolated vertices I_m to the vertex set V, then we have shown that $2n(G) \le |S| \le 2|V|$. Hence, $n(G) \le |V|$. □

The following theorem gives a sufficient condition for a graph to have an infinite niche number.

Theorem 4.2. *If G is K_{m+1}-free and $n(G) < \infty$, then G has maximum degree at most $2m(m-1)$.*

Proof. Suppose G contains no subgraph isomorphic to K_{m+1} and $G \cup I_k$ is the niche graph of an acyclic digraph D. Since there is no K_{m+1} in $G \cup I_k$, both the indegree and the outdegree of any vertex v of D is at most m. Let v be any vertex of G and let X be the set of neighbors of v in G which have prey in common with v in D. At most $m-1$ neighbors of v may share a common prey with v. Thus, since v has outdegree at most m, $|X| \le m(m-1)$. If Y is the set of neighbors of v which have a predator in common with v, a similar argument yields $|Y| \le m(m-1)$. Since $d(v)$, the degree of v in G, equals $|X| + |Y|$, we have $d(v) \le 2m(m-1)$. □

From this theorem we see that if G has a vertex of degree greater than $2m(m-1)$ and G is K_{m+1}-free, then $n(G) = \infty$. We point out that while this condition is sufficient to get $n(G) = \infty$, it is not necessary. For example, $n(K_{1,3}) = \infty$, and while $K_{1,3}$ is K_3-free, its maximum degree is less than 4.

By applying this result with $m = 2$, we can deduce that a tree which contains a vertex of degree 5 or more must have infinite niche number.

Dutton and Brigham [5] and Lundgren and Maybee [8] gave a clique cover characterization of competition graphs. Scott [17] gave a clique cover characterization of competition-common enemy graphs. Here we give a similar type of characterization for niche graphs.

Theorem 4.3. *G is a niche graph if and only if G has subgraphs H and K, $G = H \cup K$, and H has an edge clique cover $\mathscr{C} = \{C_1, \ldots, C_n\}$ such that $i \in C_j$ implies $i > j$ and $[i, k]$ is an edge in K if and only if $C_i \cap C_k \ne \emptyset$.*

Proof. Suppose G is a niche graph for an acyclic digraph D. Then D can be labeled so that its adjacency matrix $A(D)$ is strictly lower triangular. If we let H and K be the row graph and column graph, respectively, of $A(D)$, then by a result of Lundgren and Maybee [9], H is the competition graph of D and K is the common enemy graph of D. Hence, $G=H\cup K$. Furthermore, by a result of Greenberg, Lundgren, and Maybee [6], the columns of $A(D)$ determine an edge clique cover $\mathscr{C}=\{C_1,\ldots,C_n\}$ of H and $[i,k]$ is an edge in K if and only if $C_i\cap C_k\neq\emptyset$. Since $A(D)$ is strictly lower triangular, $i\in C_j$ implies $i>j$.

Now suppose G has subgraphs H and K, $G=H\cup K$, and H has an edge clique cover $\mathscr{C}=\{C_1,\ldots,C_n\}$ such that $i\in C_j$ implies $i>j$ and $[i,k]$ is an edge in K if and only if $C_i\cap C_k\neq\emptyset$. Using the graph inversion method of [6], we construct an $n\times n$ $\{0,1\}$-matrix A where the columns of A correspond to the cliques of $\mathscr{C}$. Then A is strictly lower triangular, so that A is the adjacency matrix for an acyclic digraph D and H is the competition graph of D. Furthermore, by [6], the column graph of A is isomorphic to K, so K is the common enemy graph of D. Hence, G is the niche graph of D. □

Because it is difficult to find the subgraphs H and K in the above theorem, this theorem is not particularly useful in determining whether or not a graph is a niche graph. So it would be nice to find a more useful characterization. Other open problems include finding a graph with finite niche number greater than two, determining which digraphs have interval niche graphs, and investigating the structure of niche graphs for known food webs.

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