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Approximate analytical solutions of reaction–diffusion equations with exponential source term: Homotopy perturbation method (HPM)

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ABSTRACT

In this letter, the solutions of some nonlinear differential equations have been obtained by means of the homotopy perturbation method (HPM). Applications of the homotopy method to some nonlinear reaction–diffusion equations with exponential source term show rapid convergence of the sequence constructed by this method to the exact solutions.

1. Introduction

In reality, closed-form solutions for nonlinear problems are difficult to come-by and this has resulted in the development of numerical and other approximate solutions [1]. Recently, the application of the homotopy perturbation method (HPM) to nonlinear initial boundary value problems shows that there is a rapid convergence of the sequence constructed by this method to the exact solutions [2,3]. In this paper, we consider the initial value problem (IVP)

$$\frac{d\theta}{dx} + \delta e^{\theta} = 0, \tag{1}$$

$$\theta(0) = 0, \tag{2}$$

which describes the temperature equation in a pressure driven porous media combustion with weak internal thermal diffusion [4]. In the absence of reactant consumption, the general heat balance equation for the one-step reaction system can be written as [1]

$$\frac{\partial\theta}{\partial t} = \Delta\theta + \delta e^{\theta} = 0, \quad \text{in } \Omega \tag{3}$$

$$\frac{\partial\theta}{\partial n} + Bi\,\theta = 0, \quad \partial\Omega, \tag{4}$$

where Δ is an operator and δ is related to the characteristic chemical reaction, while $\frac{\partial}{\partial n}$ is the outward normal derivative on the boundary and Bi is the Biot number, a parameter which determines whether or not the temperatures inside a body will vary significantly in space.

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2. Homotopy perturbation technique

In line with [5,6,3,7–11], we illustrate the homotopy perturbation method, we consider the nonlinear equation

$$A(u) - f(r) = 0, \quad r \in \Omega, \tag{5}$$

with the boundary conditions:

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad (r \in \partial\Omega), \tag{6}$$

where A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytic function, and $\partial\Omega$ is the boundary of domain Ω. The operator A is generally divided into two parts; L and N, where L and N are linear and nonlinear parts of A, respectively. Therefore, (3) may be written as

$$L(u) + N(u) - f(r) = 0. \tag{7}$$

We construct a homotopy $v(r, p) : \Omega \times [0, 1] \to \Re$ which satisfies

$$H(v, p) = [L(v) - L(u_0)] + p[L(u_0)] + p[N(v) - f(r)] = 0, \tag{8}$$

or

$$H(v, p) = [L(u) - L(u_0)] + p[A(u) - f(r)] = 0 \tag{9}$$

where $p \in [0, 1]$ is called the homotopy parameter and u_0 is an initial approximation of (5) which satisfies the specified boundary conditions. When $p = 0$ or $p = 1$, we have

$$H(v, 0) = L(u) - L(u_0) = 0, \qquad H(v, 1) = A(u) - f(r) = 0. \tag{10}$$

On the other hand, if $p \in (0, 1)$, then the homotopy $H(v, p)$ deforms from $L(u) - L(u_0)$ to $A(u) - f(r)$. Thus, the solution of (5)–(6) may be expressed as

$$v = v_0 + pv_1 + p^2 v_2 + p^3 v_3 + \cdots. \tag{11}$$

Eventually, at $p = 1$, the system takes the original form of the equation and the final stage of deformation gives the desired solution. Thus taking limit

$$u = \lim_{p\to 1} v = v_0 + v_1 + v_2 + . \tag{12}$$

3. Some particular examples

3.1. Example I

We start by applying the homotopy technique to an initial value problem of the form

$$\frac{d\theta}{dx} + \delta e^{\theta} = 0. \tag{13}$$

$$\theta(0) = 0. \tag{14}$$

In line with [1], we define homotopy as

$$\frac{d\theta}{dx} - \frac{dy_0}{dx} + p\left(\frac{dy_0}{dx} + \delta e^{\theta}\right) = 0. \tag{15}$$

Suppose that the solution of (13)–(14) takes the form

$$\theta = v_0 + pv_1 + p^2 v_2 + p^3 v_3 + p^4 v_4 + p^5 v_5 \cdots, \tag{16}$$

with an initial approximation

$$v_0(x) = y_0(x) = c, \tag{17}$$

where c is to be determined. Eq. (15) may be expressed as [7]

$$\frac{d\theta}{dx} - \frac{dy_0}{dx} + p\left[\frac{dy_0}{dx} + \delta e^{v_0}\left(1 + \theta_1 + \frac{\theta_1^2}{2!} + \frac{\theta_1^3}{3!} \cdots\right)\right] = 0, \tag{18}$$

where $\theta_1 = pv_1 + p^2v_2 + p^3v_3 + p^4v_4 + p^5v_5 \cdots$. After substituting (16) into (18), and collecting terms in powers of p, we obtain

$$\frac{dv_1}{dx} + \frac{dy_0}{dx} + \delta e^{v_0} = 0, \tag{19}$$

$$\frac{dv_2}{dx} + \delta e^{v_0} v_1 = 0, \tag{20}$$

$$\frac{dv_3}{dx} + \delta e^{v_0}\left(v_2 + \frac{v_1^2}{2}\right) = 0, \tag{21}$$

$$\frac{dv_4}{dx} + \delta e^{v_0}\left(v_3 + v_1v_2 + \frac{v_1^3}{6}\right) = 0, \tag{22}$$

$$\frac{dv_5}{dx} + \delta e^{v_0}\left(v_4 + v_1v_3 + \frac{v_1^2v_2}{2} + \frac{v_2^2}{2} + \frac{v_1^4}{24}\right) = 0, \tag{23}$$

$$v_1(0) = v_2(0) = v_3(0) = v_4(0) = v_5(0) \cdots = 0. \tag{24}$$

The solutions of (19)–(23) are

$$v_1 = -\delta e^c x, \qquad v_2 = \frac{\delta^2}{2}e^{2c}x^2, \qquad v_3 = -\frac{\delta^3}{3}e^{3c}x^3, \qquad v_4 = \frac{\delta^4}{4}e^{4c}x^4, \quad \text{and} \quad v_5 = -\frac{\delta^5}{5}e^{5c}x^5. \tag{25}$$

The fifth order approximation is given by

$$\theta = v_0 + v_1 + v_2 + v_3 + v_4 + v_5 = c - \delta e^c x + \frac{\delta^2}{2}e^{2c}x^2 - \frac{\delta^3}{3}e^{3c}x^3 + \frac{\delta^4}{4}e^{4c}x^4 - \frac{\delta^5}{5}e^{5c}x^5. \tag{26}$$

By using (14) on (26), we obtain $c = 0$, and the solution corresponds to the exact solution

$$\theta = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}\frac{\delta^n x^n}{n!} = -\ln(1 + \delta x). \tag{27}$$

3.2. Example II

The steady state formulations of (3)–(4) due to Frank-Kamenetskii have been developed and studied by many others [12,1,13,14]

$$\frac{d^2\theta}{dx^2} + \frac{j}{x}\frac{d\theta}{dx} + \delta x^{-\beta}e^{\theta} = 0, \tag{28}$$

$$\frac{d\theta}{dx}(0) = 0, \qquad \theta(1) = 0, \tag{29}$$

where j is related to the geometry and β is a numerical exponent. Thus after substituting (16) into (28)–(29) and collecting terms in p,

$$\frac{d^2v_1}{dx^2} + \frac{j}{x}\frac{dv_1}{dx} + \frac{d^2y_0}{dx^2} + \delta x^{-\beta}e^{v_0} = 0, \tag{30}$$

$$\frac{d^2v_2}{dx^2} + \frac{j}{x}\frac{dv_2}{dx} + \delta x^{-\beta}e^{v_0}v_1 = 0, \tag{31}$$

$$\frac{d^2v_3}{dx^2} + \frac{j}{x}\frac{dv_3}{dx} + \delta x^{-\beta}e^{v_0}\left(v_2 + \frac{v_1^2}{2!}\right) = 0, \tag{32}$$

$$\frac{d^2v_4}{dx^2} + \frac{j}{x}\frac{dv_4}{dx} + \delta x^{-\beta}e^{v_0}\left(v_3 + v_1v_2 + \frac{v_1^3}{3!}\right) = 0, \tag{33}$$

$$\frac{d^2v_5}{dx^2} + \frac{j}{x}\frac{dv_5}{dx} + \delta x^{-\beta}e^{v_0}\left(v_4 + v_1v_3 + \frac{v_2^2}{2!} + \frac{v_1^2v_2}{2!} + \frac{v_1^4}{4!}\right) = 0. \tag{34}$$

The solutions of (30)–(34) are

$$v_0 = c, \qquad v_1 = -\frac{\delta e^c x^{2-\beta}}{(2-\beta)(j+1-\beta)}, \qquad v_2 = \frac{\delta^2 e^{2c} x^{4-2\beta}}{2(2-\beta)^2(j+1-\beta)(j+3-2\beta)},$$
$$v_3 = -\frac{\delta^3 e^{3c}(2j+4-3\beta)}{2(2-\beta)^2(j+1-\beta)^2(j+3-2\beta)(j+5-3\beta)} x^{6-3\beta},$$
$$v_4 = \frac{\delta^4 e^{4c}(3j^2+16j-11j\beta-25\beta+17+9\beta^2)}{12(2-\beta)^4(j+1-\beta)^3(j+3-2\beta)(j+5-3\beta)(j+7-4\beta)} x^{8-4\beta}, \tag{35}$$
$$v_5 = \frac{-\delta^5 e^{5c}\begin{pmatrix} 24j^4+316j^3-206j^2\beta+609j^2\beta^2-1890j^2\beta \\ +1416j^2-745j\beta^3+3468j\beta^2-5190j\beta+2516j \\ +324\beta^4-1991\beta^3+4419\beta^2-4234\beta+1488 \end{pmatrix} x^{10-5\beta}}{120(2-\beta)^5(j+1-\beta)^4(j+3-2\beta)^2(j+5-3\beta)(j+7-4\beta)(j+9-5\beta)},$$

where $\beta \neq 2$, which is also satisfied by the corresponding closed-form solution [14]. The value of c is determined by using the Dirichlet condition in the boundary conditions (29). For example, when $j = \beta = 0$ and $\delta = 0.86$, $c = 0.8363$.

3.3. Example III

An unsteady 1D form of (3)–(4) is

$$\frac{\partial\theta}{\partial t} = \frac{\partial^2\theta}{\partial x^2} + \delta e^{\theta} = 0, \quad (x,t) \in \Omega \subset \Re^2, \tag{36}$$
$$\theta(x,0) = 1/2x(1-x), \tag{37}$$
$$\frac{d\theta}{dx}(0,t) = 0, \qquad \theta(1,t) = 0. \tag{38}$$

In line with [3], we construct homotopy in the form

$$\frac{\partial\theta}{\partial t} - \frac{\partial v_0}{\partial t} = p\left(\frac{\partial^2\theta}{\partial x^2} + e^{\theta} - \frac{\partial v_0}{\partial t}\right), \quad (x,t) \in \Omega \subset \Re^2. \tag{39}$$

After substituting Eq. (16) into Eq. (39) and collecting terms in p,

$$\frac{\partial v_1}{\partial t} = \frac{\partial^2 v_0}{\partial x^2} - \frac{dv_0}{dt} + e^{v_0}, \tag{40}$$
$$\frac{\partial v_2}{\partial t} = \frac{\partial^2 v_1}{\partial x^2} + e^{v_0} v_1, \tag{41}$$
$$\frac{\partial v_3}{\partial t} = \frac{\partial^2 v_2}{\partial x^2} + e^{v_0}\left(v_2 + \frac{v_1^2}{2!}\right), \tag{42}$$
$$\frac{\partial v_4}{\partial t} = \frac{\partial^2 v_3}{\partial x^2} + e^{v_0}\left(v_3 + v_1 v_2 + \frac{v_1^3}{3!}\right), \tag{43}$$
$$\frac{\partial v_5}{\partial t} = \frac{\partial^2 v_4}{\partial x^2} + e^{v_0}\left(v_4 + v_1 v_3 + \frac{v_2^2}{2!} + \frac{v_1^2 v_2}{2!} + \frac{v_1^4}{4!}\right). \tag{44}$$

In line with [1], it is convenient to choose $v_0 = 1/2x(1-x)$, then

$$v_1 = \left(e^{\frac{1}{2}x(1-x)} - 1\right)t, \tag{45}$$
$$v_2 = \frac{t^2}{2!}\left[e^{\frac{1}{2}x(1-x)} + \left(\frac{1}{2} - x\right)^2 - 2\right]e^{\frac{1}{2}x(1-x)}, \tag{46}$$
$$v_3 = \frac{t^3}{3!}\left[5 - 6e^{\frac{1}{2}x(1-x)} + \left(5e^{\frac{1}{2}x(1-x)} - 7\right)\left(\frac{1}{2} - x\right)^2 + 2e^{x(1-x)} + \left(\frac{1}{2} - x\right)^4\right]e^{\frac{1}{2}x(1-x)}, \tag{47}$$

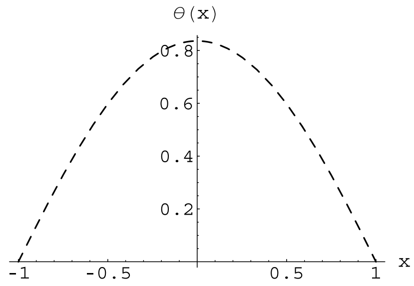

Fig. 1. Plot of $\theta(x)$ against x (HPM).

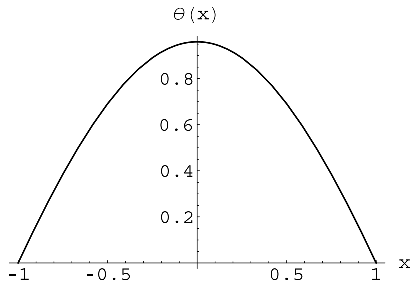

Fig. 2. Plot of $\theta(x)$ against x (Exact).

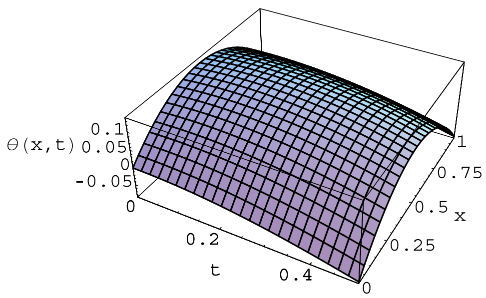

Fig. 3. 3D plot of $\theta(x, t)$ against x (HPM).

$$\begin{aligned} v_4 = \frac{t^4}{4!}\Bigg[& -20 + 6e^{\frac{3}{2}x(1-x)} - 20e^{x(1-x)} + 31e^{\frac{1}{2}x(1-x)} + 34\left(\frac{1}{2}-x\right)^2 e^{x(1-x)} \\ & - 59\left(\frac{1}{2}-x\right)^2 e^{\frac{1}{2}x(1-x)} + 21\left(\frac{1}{2}-x\right)^4 e^{\frac{1}{2}x(1-x)} + 52\left(\frac{1}{2}-x\right)^2 \\ & - 8\left(\frac{1}{2}-x\right)^4 + \left(\frac{1}{2}-x\right)^6 - 20\left(\frac{1}{2}-x\right) e^{\frac{1}{2}x(1-x)}\Bigg] e^{\frac{1}{2}x(1-x)}. \end{aligned} \tag{48}$$

$$v_5 = \frac{t^5}{5!}\left[\frac{\partial^2 v_4^*}{\partial x^2} + e^{\frac{1}{2}x(1-x)}\left(v_4^* + 4v_1^* v_3^* + \frac{30}{2!}v_2^{*2} + \frac{15}{2!}v_1^{*2}v_2^* + \frac{1}{4!}v_1^{*4}\right)\right] e^{\frac{1}{2}x(1-x)}, \tag{49}$$

where

$$v_1^* = v_1/t, \qquad v_2^* = \frac{2!}{t^2}e^{\frac{-1}{2}x(1-x)}v_2, \qquad v_3^* = \frac{3!}{t^3}3!e^{\frac{-1}{2}x(1-x)}v_3, \qquad v_4^* = \frac{4!}{t^4}3!e^{\frac{-1}{2}x(1-x)}v_4. \tag{50}$$

4. Conclusion

It is observed that Fig. 1. obtained by the fifth order approximation (HPM) converges to the profiles given by the exact solution (Fig. 2). Similarly, in the unsteady state, Figs. 3 and 4 show that the HPM solution is reliable as it confirms the numerical solutions obtained by NDSolve in the MATHEMATICA package. Although the approximate homotopy perturbation

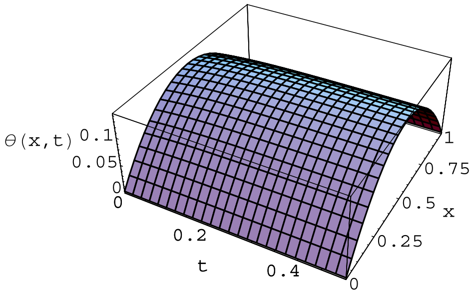

Fig. 4. 3D plot of $\theta(x, t)$ against x (NDSolve).

method is found to work extremely well in the examples considered, the approach may be less effective and accurate in the presence of more complicated nonlinear source terms.

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