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Some Inequalities for Mappings whose Derivatives are Bounded and Applications to Special Means of Real Numbers

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Abstract—In this paper, we shall introduce two new inequalities of Hermite-Hadamard type for convex functions with bounded derivatives. Some applications to special means of real numbers are also included.

Keywords—Hermite-Hadamard inequality, Convex functions, Special means.

1. INTRODUCTION

Let $f : I \subset R \to R$ be a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$. The inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

is known as Hermite-Hadamard's inequality for convex functions [1].

Recently, Dragomir and Agarwal [2] presented some results connected with the right part of (1.1). Furthermore, some inequalities for convex functions whose derivatives are bounded were given in [3]. The first author [4] gave some results connected with the left part of (1.1).

In [5], Abramovic and Pecaric introduced a function

$$M(a,b,h(s)) = \frac{1}{M(a,b)}h^{-1}\left(\int_0^1 h(s)\,dt\right),$$

where $s = s(a,b,t)$ and $M(a,b)$ is a given mean value as a function which satisfies the following conditions:

(i) $M : R^+ \times R^+ \to R$,
(ii) $M(a,b) = M(b,a)$,
(iii) $M(a,a) = a$,
(iv) $0 < a < M(a,b) < b$, for $1 < b/a$.

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They showed that $0 < a \leq M(a,b,h(s)) \leq b$ for $1 \leq b/a \leq T$, $T > 1$ are on a limited interval, where $h : R^+ \to R$ is a strictly monotonic function in R^+ with a continuous derivative.

We deal with the left part of (1.1). In Section 2, we give two new inequalities for convex functions with bounded derivatives. In Section 3, we note some consequent applications to special means.

2. THE RESULTS

We first prove the following lemma.

LEMMA 2.1. *Let $f : I \subset R \to R$ be a differentiable mapping on I and $a, r = M(a,b,h(s)) \in I$. If $f' \in L[a,r]$, $(a < r)$, then we have*

$$\frac{1}{r-a}\int_a^r f(x)\,dx - f\left(\frac{a+r}{2}\right) = (r-a)\left[\int_0^{1/2} tf'(r+(a-r)t)\,dt + \int_{1/2}^1 (t-1)f'(r+(a-r)t)\,dt\right].$$

PROOF. By integration by parts, we deduce

$$\begin{aligned}\Delta &= \int_0^{1/2} tf'(r+(a-r)t)\,dt + \int_{1/2}^1 (t-1)f'(r+(a-r)t)\,dt \\ &= \frac{f(r+(a-r)t)}{a-r}t\bigg|_0^{1/2} - \int_0^{1/2}\frac{f(r+(a-r)t)}{a-r}\,dt + \frac{f(r+(a-r)t)}{a-r}(t-1)\bigg|_{1/2}^1 \\ &\quad - \int_{1/2}^1 \frac{f(r+(a-r)t)}{a-r}\,dt \\ &= \frac{1}{a-r}f\left(\frac{a+r}{2}\right) + \frac{1}{(a-r)^2}\int_a^r f(x)\,dx,\end{aligned}$$

where we have used the change of the variable $x = r + (a-r)t$, $t \in [0,1]$. Hence, we have the conclusion.

Now, we prove two theorems.

THEOREM 2.2. *Let $f : I \subset R \to R$ be a differentiable mapping on I, $a, r \in I$, and its derivative $f' : (a,b) \to R$ is bounded in (a,b), that is, $\|f'\|_\infty = \sup_{t\in(a,b)} |f'(x)| < \infty$ with $a < r = M(a,b,h(s)) < b$. If $|f'|$ is convex on $[a,b]$, then we have*

$$\left|\frac{1}{r-a}\int_a^r f(x)\,dx - f\left(\frac{a+r}{2}\right)\right| \leq \left(\frac{r-a}{8}\right)(|f'(a)| + |f'(r)|) \leq \left(\frac{r-a}{4}\right)K.$$

PROOF. Using Lemma 2.1, it follows that

$$\begin{aligned}\left|\frac{1}{r-a}\int_a^r f(x)\,dx - f\left(\frac{a+r}{2}\right)\right| &= \left|(r-a)\left[\int_0^{1/2} tf'(r+(a-r)t)\,dt \right.\right.\\ &\quad \left.\left. + \int_{1/2}^1 (t-1)f'(r+(a-r)t)\,dt\right]\right| \\ &\leq (r-a)\left[\int_0^{1/2} t\,|f'(r+(a-r)t)|\,dt \right.\\ &\quad \left. + \int_{1/2}^1 |t-1|\,|f'(r+(a-r)t)|\,dt\right].\end{aligned}$$

Using the convexity of $|f'|$, we obtain

$$|f'(r+(a-r)t)| = |f'(ta+(1-t)r)| \le t|f'(a)| + (1-t)|f'(r)|, \qquad t \in (0,1).$$

Hence,

$$\begin{aligned}\left|\frac{1}{r-a}\int_a^r f(x)\,dx - f\left(\frac{a+r}{2}\right)\right| &\le (r-a)\left[\int_0^{1/2} \left(t^2\,|f'(a)| + t(1-t)\,|f'(r)|\right)\,dt\right.\\ &\quad \left. + \int_{1/2}^1 \left((1-t)t\,|f'(a)| + (1-t)^2\,|f'(r)|\right)\,dt\right]\\ &\le \left(\frac{r-a}{8}\right)\left(|f'(a)| + |f'(r)|\right),\end{aligned}$$

where we have used the facts that

$$\int_0^{1/2} t^2\,dt = \int_{1/2}^1 (1-t)^2\,dt = \frac{1}{24}, \qquad \int_0^{1/2} (1-t)t\,dt = \int_{1/2}^1 (1-t)t\,dt = \frac{1}{12}.$$

Since the function f' is bounded (i.e., $|f'(x)| \le K$, where K is a constant), we write

$$\begin{aligned}\left|\frac{1}{r-a}\int_a^r f(x)\,dx - f\left(\frac{a+r}{2}\right)\right| &\le \left(\frac{r-a}{8}\right)\left(|f'(a)| + |f'(r)|\right)\\ &\le \left(\frac{r-a}{8}\right)(K+K) = \left(\frac{r-a}{4}\right)K.\end{aligned}$$

This concludes the proof.

THEOREM 2.3. *Let $f : I \subset R \to R$ be a differentiable mapping on I with $a < r = M(a,b,h(s)) < b$ and its derivative $f' : (a,b) \to R$ is bounded in (a,b), that is, $\|f'\|_\infty = \sup_{t\in(a,b)} |f'(x)| < \infty$. If $|f'|^{p/(p-1)}$ is convex on $[a,b]$, then we have, for all $p > 1$,*

$$\begin{aligned}\left|\frac{1}{r-a}\int_a^r f(x)\,dx - f\left(\frac{a+r}{2}\right)\right| &\le \left(\frac{r-a}{4}\right)\left(\frac{4}{p+1}\right)^{1/p}\left(|f'(a)| + |f'(r)|\right)\\ &\le \left(\frac{r-a}{2}\right)\left(\frac{4}{p+1}\right)^{1/p} K.\end{aligned}$$

PROOF. Using Lemma 2.1 and Hölder's integral inequality, we find

$$\begin{aligned}\left|\frac{1}{r-a}\int_a^r f(x)\,dx - f\left(\frac{a+b}{2}\right)\right| &\le (r-a)\left[\int_0^{1/2} t\,|f'(r+(a-r)t)|\,dt\right.\\ &\quad \left. + \int_{1/2}^1 (t-1)\,|f'(r+(a-r)t)|\,dt\right]\\ &\le (r-a)\left[\left(\int_0^{1/2} t^p\,dt\right)^{1/p}\left(\int_0^{1/2} |f'(r+(a-r)t)|^q\,dt\right)^{1/q}\right.\\ &\quad \left. + \left(\int_{1/2}^1 |t-1|^p\,dt\right)^{1/p}\left(\int_{1/2}^1 |f'(r+(a-r)t)|^q\,dt\right)^{1/q}\right],\end{aligned}$$

where $1/p + 1/q = 1$.

Since

$$|f'(r+(a-r)t)|^q = |f'(ta+(1-t)r)|^q \le t\,|f'(a)|^q + (1-t)\,|f'(r)|^q, \qquad t \in (0,1),$$

and the function $|f'|^{p/(p-1)}$ is convex on $[a,b]$, we have

$$\int_0^{1/2} |f'(r+(a-r)t)|^q \, dt \le \int_0^{1/2} \left[t\,|f'(a)|^q + (1-t)\,|f'(r)|^q\right] dt = \frac{|f'(a)|^q + 3|f'(r)|^q}{8},$$

$$\int_{1/2}^{1} |f'(r+(a-r)t)|^q \, dt \le \int_{1/2}^{1} \left[t\,|f'(a)|^q + (1-t)\,|f'(r)|^q\right] dt = \frac{3|f'(a)|^q + |f'(r)|^q}{8},$$

$$\int_0^{1/2} t^p \, dt = \int_{1/2}^{1} |t-1|^p \, dt = \int_{1/2}^{1} (1-t)^p \, dt = \frac{1}{(p+1)2^{p+1}},$$

where we have used the facts that

$$\int_0^{1/2} t\,dt = \int_{1/2}^{1} (1-t)\,dt = \frac{1}{8}, \qquad \int_0^{1/2} (1-t)\,dt = \int_{1/2}^{1} t\,dt = \frac{3}{8}.$$

Let $a_1 = |f'(a)|^q$, $b_1 = 3|f'(r)|^q$, $a_2 = 3|f'(a)|^q$, $b_2 = |f'(r)|^q$. Here $0 < (p-1)/p < 1$, for $p > 1$. Using the fact that

$$\sum_{k=1}^{n} (a_k + b_k)^s \le \sum_{k=1}^{n} a_k^s + \sum_{k=1}^{n} b_k^s,$$

for $(0 \le s < 1)$, $a_1, a_2, \ldots, a_n \ge 0$, $b_1, b_2, \ldots, b_n \ge 0$, we obtain

$$\begin{aligned} \left|\frac{1}{r-a}\int_a^r f(x)\,dx - f\left(\frac{a+r}{2}\right)\right| &\le \left(\frac{r-a}{16}\right)\left(\frac{4}{p+1}\right)^{1/p} 4\left(|f'(a)| + |f'(r)|\right) \\ &\le \left(\frac{r-a}{4}\right)\left(\frac{4}{p+1}\right)^{1/p} \left(|f'(a)| + |f'(r)|\right). \end{aligned}$$

Since the function f' is bounded in (a,b) (i.e., $|f'(x)| \le K$, where K is a constant), we deduce

$$\left|\frac{1}{r-a}\int_a^r f(x)\,dx - f\left(\frac{a+r}{2}\right)\right| \le \left(\frac{r-a}{2}\right)\left(\frac{4}{p+1}\right)^{1/p} K.$$

Hence, we have the conclusion.

3. APPLICATIONS TO SPECIAL MEANS

We shall consider arithmetic, geometric, and generalized logarithmic means for real numbers. We take

$$\begin{aligned} A(\alpha,\beta) &= \frac{\alpha+\beta}{2}, && \alpha,\beta \in R, \\ G(\alpha,\beta) &= \sqrt{\alpha\beta}, && \alpha,\beta \in R, \\ L_n(\alpha,\beta) &= \left[\frac{\beta^{n+1} - \alpha^{n+1}}{(n+1)(\beta-\alpha)}\right]^{1/n}, && n \in Z \setminus \{-1,0\}, \quad \alpha,\beta \in R, \quad \alpha \ne \beta. \end{aligned}$$

PROPOSITION 3.1. *Let $a, r, b \in R$, $0 \notin [a,b]$. If $a < r < b$, then we have*

$$\left|[G(a,r)]^{-2} - A^{-2}(a,r)\right| \le \left(\frac{r-a}{2}\right) A\left(|a|^{-3}, |r|^{-3}\right).$$

PROOF. The assertion follows from Theorem 2.2 applied for $f(x) = 1/x^2$, $x \in [a,b]$.

PROPOSITION 3.2. *Let $a, r, b \in R$, $0 \notin [a, b]$, $n \in Z \setminus \{-1, 0\}$. If $a < r < b$, then we have, for all $p > 1$,*

$$\left| L_{-n}^{-n}(a,r) - A^{-n}(a,r) \right| \leq |n| \left(\frac{r-a}{4} \right) A \left(|a|^{-(n+1)}, |r|^{-(n+1)} \right).$$

PROOF. The assertion follows from Theorem 2.2 applied for $f(x) = x^{-n}$, $x \in R \setminus \{0\}$.

PROPOSITION 3.3. *Let $a, r, b \in R$, $0 \notin [a, b]$. If $a < r < b$, then we have, for all $p > 1$,*

$$\left| [G(a,r)]^{-2} - A^{-2}(a,r) \right| \leq (r-a) \left(\frac{4}{p+1} \right)^{1/p} A \left(|a|^{-3}, |r|^{-3} \right).$$

PROOF. The assertion follows from Theorem 2.3 applied for $f(x) = 1/x^2$, $x \in [a, b]$.

PROPOSITION 3.4. *Let $a, r, b \in R$, $0 \notin [a, b]$, $n \in Z \setminus \{-1, 0\}$. If $a < r < b$, then we have, for all $p > 1$,*

$$\left| L_{-n}^{-n}(a,r) - A^{-n}(a,r) \right| \leq |n| \left(\frac{r-a}{2} \right) \left(\frac{4}{p+1} \right)^{1/p} A \left(|a|^{-(n+1)}, |r|^{-(n+1)} \right).$$

PROOF. The assertion follows from Theorem 2.3 applied for $f(x) = x^{-n}$, $x \in R \setminus \{0\}$.

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