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Some Generic Properties in Fixed Point Theory

F. S. DE BLASI

Istituto Matematico "U. Dini," Viale Margagni 67/A, 50134 Firenze, Italy

Submitted by Ky Fan

It is shown that, in the sense of the Baire category, almost all continuous single valued α-nonexpansive mappings $T: C \to C$ do have fixed points. Here C is a nonempty closed convex and bounded subset of an infinite dimensional Banach space. A similar result holds for upper semicontinuous α-nonexpansive mappings which are compact convex valued. Corresponding results for single valued and set-valued nonexpansive mappings are reviewed.

1. INTRODUCTION AND RESULTS

Let C be a nonempty closed convex bounded subset of positive diameter contained in an infinite dimensional Banach space E. If E is uniformly convex, the classical theorem established by Browder [1], Gödhe [8] and (in a more general setting) by Kirk [11] ensures that C has the fixed point property for each mapping $T: C \to C$ which is *nonexpansive*, that is $\| Tx - Ty \| \leqslant \| x - y \|$, $(x, y \in C)$. This result is no longer true [15, p. 126] for T in the larger class $\mathscr{M}_1$ of all continuous maps which are α-*nonexpansive*, that is α-Lipschitz

$$\alpha[T(X)] \leqslant K_T \alpha[X] \qquad \text{for each } X \subseteq C \tag{1}$$

with Lipschitz constant $K_T = 1$. Here α denotes the Kuratowski measure of noncompactness. In this case, both, the subsets of maps with or without fixed points, can be nonempty and so it comes rather naturally the question of finding the topological size (in the sense of the Baire category) of each of these subsets.

In this note we are going to consider the question in its full generality without any additional assumption on the geometry of the Banach space E. We find that the subset $\mathscr{Z}_1$ of all $T \in \mathscr{M}_1$ which are fixed point free is of Baire I category in the Baire space $\mathscr{M}_1$ of all continuous α-nonexpansive mappings endowed with the metric of the uniform convergence

$$\rho(T, S) = \sup\{\| T(x) - S(x)\| \mid x \in C\} \qquad (T, S \in \mathscr{M}_1). \tag{2}$$

This result seems to be the best possible since, as we shall see in an example, for each $K > 1$, the subset $\mathscr{Z}_K$ of all functions which are fixed point free, can have

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nonempty interior, hence it is of Baire II category in the Baire space of all continuous α-Lipschitz mappings $T: C \to C$ with constant $K_T = K$. Thus we have:

$$\begin{aligned} \mathscr{X}_K &= \varnothing \text{ by Darbo's theorem [3]} && \text{if} \quad K < 1 \\ &= \text{I category} && \text{if} \quad K = 1 \\ &= \text{II category (possibly)} && \text{if} \quad K > 1. \end{aligned}$$

The fact that $\mathscr{X}_1$ is of I category is a consequence of the following:

THEOREM 1. *Let $\mathscr{M}_0$ be the subset of all $T \in \mathscr{M}_1$ such that:* (i) *T has at least one fixed point;* (ii) *the set Ω_T of the fixed points of T is compact and,* (iii) *for each $x \in C$, $\{T^n(x)\}$ is precompact. Then $\mathscr{M}_0$ is a residual set in the Baire space $\mathscr{M}_1$.*

Let $\mathscr{E}$ be the metric space of all nonexpansive mappings $T: C \to C$ equipped with the metric (2). The method of proof of Theorem 1 furnishes, in particular, another and different proof of the following:

PROPOSITION 1 [18]. *The subset $\mathscr{E}_0$ of all $T \in \mathscr{E}$ which have fixed points is residual in the Baire space $\mathscr{E}$.*

Indeed even more can be said, for we have:

PROPOSITION 2 [4]. *The subset $\tilde{\mathscr{E}}$ of all $T \in \mathscr{E}$ for which the successive approximations $\{T^n(x)\}$ converge for each starting point $x \in C$ is residual in the Baire space $\mathscr{E}$.*

Propositions 1 and 2 furnish some light on the problem of extending the theorem of Browder, Göhde and Kirk beyond the framework of the uniform convexity or, more generally, of the normal structure of the space (see [17]).

We notice, by the way, that Proposition 2 has a counterpart in the theory of ordinary differential equations in infinite dimensional Banach spaces [6].

Theorem 1 and Proposition 1 can be formulated also for set-valued functions.

Let $\Gamma(C)$ (resp., $\tilde{\Gamma}(C)$) be the complete metric space of all nonempty compact ((resp., compact convex) subsets of C endowed with the Hausdorff distance H. Let $\mathscr{P}$ be the set of all upper semicontinuous maps $T: C \to \tilde{\Gamma}(C)$ which satisfy (1), with constant $K_T = 1$. $\mathscr{P}$ is a Baire space under the metric of the uniform convergence,

$$\rho(T, S) = \sup\{H(T(x), S(x)) \mid x \in C\} \qquad (T, S \in \mathscr{P}). \tag{3}$$

THEOREM 2. *Let $\mathscr{P}_0$ be the subset of all $T \in \mathscr{P}$ which have properties* (i–iii) *of Theorem* 1. *Then $\mathscr{P}_0$ is a residual set in the Baire space $\mathscr{P}$.*

Let $\mathscr{U}$ be the metric space of all mappings $T: C \to \Gamma(C)$ which are non-expansive, that is Lipschitzian

$$H(T(x), T(y)) \leqslant K_T \| x - y \| \qquad (x, y \in C)$$

with constant $K_T = 1$, equipped with the metric (3).

PROPOSITION 3. *The subset* $\mathscr{U}_0$ *of all* $T \in \mathscr{U}$ *which have fixed points is residual in the Baire space* $\mathscr{U}$.

This statement is seen in the proper light if it is compared with some recent fixed point theorems [2, 9, 13, 14] proved under additional assumptions on the geometry of the space E, namely when E is uniformly convex. Without this hypothesis very little seems to be known.

2. PROOFS

Proof of Theorem 1. It is known that the set $\mathscr{N}$ of all $S \in \mathscr{M}_1$ with constant $K_S < 1$ is dense in $\mathscr{M}_1$. Let $S \in \mathscr{N}$. Let $\epsilon > 0$ and let $T \in \mathscr{M}_1$ satisfy $\rho(T, S) < \delta_S(\epsilon)$ where $0 < \delta_S(\epsilon) < (1 - K_S)\,\epsilon/2$. Let $X \subseteq C$ be any. We have

$$\alpha[(\overline{\text{co}}\, T)^n (X)] \leqslant K_S{}^n \alpha[X] + 2\delta_S(\epsilon) \sum_{i=0}^{n-1} K_S{}^i \tag{4}$$

in which $\overline{\text{co}}$ denotes the closed convex hull. Let U be the unit ball in E. In order to show that (4) is true we use induction. For $n = 1$, being $\rho(T, S) < \delta_S(\epsilon)$, we have

$$T(X) \subseteq S(X) + U\delta_S(\epsilon)$$

and so

$$(\overline{\text{co}}\, T)(X) \subseteq (\overline{\text{co}}\, S)(X) + U\delta_S(\epsilon)$$

from which

$$\alpha[(\overline{\text{co}}\, T)(X)] \leqslant \alpha[(\overline{\text{co}}\, S)(X)] + 2\delta_S(\epsilon) \leqslant K_S \alpha(X) + 2\delta_S(\epsilon).$$

Suppose now that (4) is true for a given n. From

$$T(\overline{\text{co}}\, T)^n (X) \subseteq S(\overline{\text{co}}\, T)^n (X) + U\delta_S(\epsilon)$$

we have

$$(\overline{\text{co}}\, T)^{n+1} (X) \subseteq (\overline{\text{co}}\, S)(\overline{\text{co}}\, T)^n (X) + U\delta_S(\epsilon),$$

hence

$$\begin{aligned} \alpha[(\overline{\text{co}}\, T)^{n+1} (X)] &\leqslant \alpha[(\overline{\text{co}}\, S)(\overline{\text{co}}\, T)^n (X)] + 2\delta_S(\epsilon) \\ &\leqslant K_S \alpha[(\overline{\text{co}}\, T)^n (X)] + 2\delta_S(\epsilon) \end{aligned}$$

and, using (4) in the last inequality, we find that (4) is true also for $n+1$. Since $K_S < 1$, we obtain from (4),

$$\limsup_{n\to\infty} \alpha[(\overline{\text{co}}\, T)^n (X)] \leqslant 2\delta_S(\epsilon)\,(1 - K_S)^{-1} < \epsilon.$$

Now let $B(S, \delta) = \{T \in \mathcal{M}_1 \mid \rho(T, S) < \delta\}$. Define

$$\mathcal{M}_* = \bigcap_{k=1}^{\infty} \bigcup_{S\in\mathcal{N}} B(S, \delta_S(1/k))$$

and observe that $\mathcal{M}_*$, as a dense (for $\mathcal{M}_*$ contains $\mathcal{N}$ which is so) G_δ-set in the Baire space $\mathcal{M}_1$, is residual. To complete the proof we shall show that $\mathcal{M}_* \subseteq \mathcal{M}_0$. Let $T \in \mathcal{M}_*$. This implies that there are $S_k \in \mathcal{N}$ such that $T \in B(S_k, \delta_{S_k}(1/k))$ for $k = 1, 2, \ldots$. Consequently

$$\limsup_{n\to\infty} \alpha[(\overline{\text{co}}\, T)^n (X)] < 1/k \qquad (k = 1, 2, \ldots),$$

hence

$$\lim_{n\to\infty} \alpha[(\overline{\text{co}}\, T)^n (X)] = 0, \qquad (X \subseteq C). \tag{5}$$

This equality is in particular true for $X = C$. In such case the sequence $\{(\overline{\text{co}}\, T)^n (C)\}$ of nonempty closed convex sets is monotone nonincreasing and, by Kuratowski's theorem [12, p. 412], has nonempty intersection, C_0. Since C_0 is compact convex and invariant under the continuous function T, by Schauder's theorem T has a fixed point.

From

$$\alpha[\Omega_T] = \alpha[T^n(\Omega_T)] \leqslant \alpha[(\overline{\text{co}}\, T)^n (C)]$$

by virtue of (5), if we let $X = C$, we obtain $\alpha[\Omega_T] = 0$ and so Ω_T (which is closed) is also compact. A similar argument shows that for each $x \in C$, $\{T^n(x)\}$ is precompact. This completes the proof.

EXAMPLE. Let c_0 be the Banach space of all real sequences $x = (x_1, x_2, \ldots)$ which converge to zero, with the supremum norm. Let $C = \{x \in c_0 \mid \|x\| \leqslant 4\}$. For a fixed $1 < K < 2$ set

$$\begin{aligned} p(r) &= Kr && \text{if} \quad |r| \leqslant 1 \\ &= Kr/|r| && \text{if} \quad 1 \leqslant |r| \leqslant 4 \end{aligned}$$

and define

$$T\colon C \to C \qquad \text{by} \qquad T(x) = (K, p(x_1), p(x_2), \ldots).$$

Since T is Lipschitzian with constant $K_T = K$ it satisfies $\alpha[T(X)] \leqslant K\alpha[X]$ for each $X \subseteq C$. Let $\Phi: C \to C$ be any continuous function such that $\rho(T, \Phi) < \epsilon = K - 1$. We shall see that any such Φ has not fixed points. Otherwise, for same $x \in C$, we would have $x_i = \Phi_i(x)$ $(i = 1, 2,...)$. Since $\rho(T, \Phi) < \epsilon$ we have

$$K - \epsilon < \Phi_1(x) < K + \epsilon$$

$$p(x_{i-1}) - \epsilon < \Phi_i(x) < p(x_{i-1}) + \epsilon \qquad \text{for } i \geqslant 2.$$

Thus

$$x_1 = \Phi_1(x) > K - \epsilon = 1$$

$$x_2 = \Phi_2(x) > p(x_1) - \epsilon \geqslant p(1) - \epsilon = 1$$

and, in general, $x_i > 1$ for each $i = 1, 2,...$, a contradiction. This shows that each continuous Φ in the open ball $B(T, \epsilon)$ is fixed point free.

Proof of Theorem 2. It runs as that of Theorem 1 with the difference that now the existence of a fixed point of T in C_0 is furnished by the theorem of Kakutani and Ky-Fan [10, 7].

Proof of Proposition 3. As in the proof of Theorem 1 we obtain the existence of a compact convex set C_0 which is nonempty and invariant under T. Let $\bar{x} \in C_0$. For $k = 1, 2,...$ set $S_k(x) = (1/k)\,\bar{x} + (1 - 1/k)\,T(x)$, $x \in C_0$, and observe that $S_k(C_0) \subseteq C_0$ and $S_k \to T$ uniformly on C_0. Since each S_k has Lipschitz constant $K_{S_k} < 1$, by a theorem of Nadler [16], it has a fixed point, x_k. By the compactness of C_0 we assume $x_k \to \tilde{x} \in C_0$. Suppose now, $\inf\{\| \tilde{x} - y \| \mid y \in T(\tilde{x})\} = \sigma > 0$. Then by the continuity of T and the fact that $x_k \to \tilde{x}$ and S_k converges (uniformly) to T we have, for k sufficiently large, $\inf\{\| x_k - y \| \mid y \in S_k(x_k)\} \geqslant \sigma/2$, a contradiction. Therefore $\sigma = 0$ and T has a fixed point.

Remark. Let C be a nonempty closed convex and bounded subset of a Banach space E. It has been shown in Theorem 1 that the set $\tilde{\mathscr{M}}$ of all $T \in \mathscr{M}_1$ which have fixed points is residual in $\mathscr{M}_1$. If we denote by $\tilde{\mathscr{X}}$ the subset $\mathscr{M}_1 \backslash \tilde{\mathscr{M}}$ of the pathological maps of $\mathscr{M}_1$, that is of the maps which are fixed point free, the following questions can be posed:

1. Characterize the Banach spaces E which admit nonempty $\tilde{\mathscr{X}}$.
2. Characterize the maps which are in $\tilde{\mathscr{X}}$.
3. Find how the set $\tilde{\mathscr{X}}$ is scattered in $\mathscr{M}_1$ (density, convexity, etc.).

Of course each of these questions can be formulated in correspondence with any other result of the paper. In some special case, question 3 has been partially answered. In fact, with reference to Proposition 2, it has been proved that, if E is a Hilbert space, then $\mathscr{E} \backslash \tilde{\mathscr{E}}$ is dense in $\mathscr{E}$ [5]. In a more general setting very little is known.

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