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J. Math. Anal. Appl. 334 (2007) 534–548

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

Parabolic equations with measurable coefficients II

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Received 23 October 2006

Available online 8 January 2007

Submitted by N.S. Trudinger

Abstract

We prove the existence and uniqueness of solutions in Sobolev spaces to second-order parabolic equations in non-divergence form. The coefficients (except one of them) of second-order terms of the equations are measurable in both time and one spatial variables, and VMO (vanishing mean oscillation) in other spatial variables.

Keywords: Second-order parabolic equations; Vanishing mean oscillation; Sobolev spaces

1. Introduction

This paper continues the investigation of parabolic equations in [11] dealing with more general class of coefficients than in [11]. The parabolic equation we consider is of the form

$$u_t + a^{ij}(t,x)u_{x^i x^j} + b^i(t,x)u_{x^i} + c(t,x)u = f \tag{1}$$

in the Sobolev spaces $W_p^{1,2}$, $p > 2$.

In establishing L_p-theory for equations as above, no regularity assumptions are required for b^i and c (i.e., b^i and c are bounded measurable), whereas there could not exist a unique solution to the above equation if the coefficients a^{ij} are only measurable. In fact, there have always been some regularity assumptions on a^{ij} in the literature. For example, if $a^{ij}(t,x)$ are uniformly continuous uniformly in t (see [16]), then for $f \in L_p$, the above equation has a unique solution in $W_p^{1,2}$. In [1] they assumed $a^{ij}(t,x)$ to be in the space of VMO as functions of $(t,x) \in \mathbb{R}^{d+1}$.

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2006.12.077

This result was generalized using a different approach in [14], so that $a^{ij}(t,x)$ can be measurable in $t \in \mathbb{R}$ and VMO in $x \in \mathbb{R}^d$. Piecewise continuous a^{ij} were also dealt with in [9,23]. As a predecessor of this paper, we considered in [11] coefficients $a^{ij}(t,x)$ which are measurable in $x^1 \in \mathbb{R}$ and VMO in (t,x'), $t \in \mathbb{R}$, $x' \in \mathbb{R}^{d-1}$. Here we denote $x = (x^1, x') \in \mathbb{R}^d$, $x^1 \in \mathbb{R}$, $x' \in \mathbb{R}^{d-1}$.

In this paper, we remove the regularity assumption with respect to $t \in \mathbb{R}$ on $a^{ij}(t,x)$ (in [11] a^{ij} were assumed to be VMO in $t \in \mathbb{R}$). More precisely, we assume that $a^{ij}(t,x)$ are measurable in $(t,x^1) \in \mathbb{R}^2$ and VMO in $x' \in \mathbb{R}^{d-1}$ except that $a^{11}(t,x)$ is measurable only in $x^1 \in \mathbb{R}$ and VMO in $(t,x') \in \mathbb{R}^d$. This class of coefficients a^{ij} clearly covers the class considered in [11]. By no means, however, does this paper make the results in [11] obsolete because here we have to make use of results from [11] in our main steps. In addition, the paper [11] contains a very important result about the case $p = 2$, where a^{ij} (including a^{11}) are only measurable as functions of (t,x^1). Note that the class of a^{ij} in this paper does not include the coefficients in [14] since a^{11} is not allowed to be measurable in $t \in \mathbb{R}$. However, the class of coefficients a^{ij} we consider here is considerable general, so that, for example, we do not require any regularity assumptions on a^{ij} as long as they are functions of (t,x^1) (a^{11} is a function of x^1) satisfying the ellipticity condition.

It is worth mentioning that, as noted in [10], in the elliptic case an example of an equation in $\mathbb{R}^d$, $d \geqslant 3$, was constructed in [15] (the original paper [27]) having no unique solvability in W_p^2, where the coefficients of the equation are functions of only the first *two* coordinates. Non-uniqueness of elliptic equations in a very generalized sense is proved in [20,22]. Note that due to the unique solvability in $W_p^{1,2}$ of Eq. (1), by following the arguments in [9,26], we have the weak uniqueness of stochastic processes associated with the parabolic equation. This would be another motivation to investigate parabolic equations as in (1). Besides those papers referred to earlier, we refer to papers [2–6,8,17–19,21,24,25] for more information about L_p-theory for both elliptic and parabolic equations with rough coefficients—coefficients which are not uniformly continuous. For weak uniqueness of stochastic processes, in addition to those mentioned in the above, see [13] and references therein.

This paper is organized as follow. We present the main result in Section 2. To prepare for a proof of the main result, we state and prove some auxiliary results in Section 3. Finally, we prove Theorem 2.4 in Section 4. For the results in Section 3, we needed some properties of traces of functions in parabolic Sobolev spaces [12] and parabolic equations with mixed norms [7].

2. Main results

We consider the parabolic equation (1) in the Sobolev space

$$W_p^{1,2}\big((S,T) \times \mathbb{R}^d\big) = \big\{u\colon\ u, u_t, u_x, u_{xx} \in L_p\big((S,T) \times \mathbb{R}^d\big)\big\},$$

$-\infty \leqslant S < T \leqslant \infty$. The coefficients a^{ij}, b^i, and c satisfy the following assumptions.

Assumption 2.1. The coefficients a^{ij}, b^i, and c are measurable functions defined on $\mathbb{R}^{d+1}$, $a^{ij} = a^{ji}$. There exist positive constants $\delta \in (0,1)$ and K such that

$$\big|b^i(t,x)\big| \leqslant K, \qquad \big|c(t,x)\big| \leqslant K,$$

$$\delta|\vartheta|^2 \leqslant \sum_{i,j=1}^d a^{ij}(t,x)\vartheta^i\vartheta^j \leqslant \delta^{-1}|\vartheta|^2$$

for any $(t,x) \in \mathbb{R}^{d+1}$ and $\vartheta \in \mathbb{R}^d$.

To state another assumption on the coefficients a^{ij}, we introduce some notation. Let

$$B_r(x) = \{y \in \mathbb{R}^d \colon |x-y| < r\},$$
$$B'_r(x') = \{y' \in \mathbb{R}^{d-1} \colon |x'-y'| < r\},$$
$$Q_r(t,x) = (t, t+r^2) \times B_r(x), \qquad \Gamma_r(t,x') = (t, t+r^2) \times B'_r(x'),$$
$$\Lambda_r(t,x) = (t, t+r^2) \times (x^1 - r, x^1 + r) \times B'_r(x').$$

Set $B_r = B_r(0)$, $B'_r = B'_r(0)$, $Q_r = Q_r(0)$ and so on. By $|B'_r|$ we mean the $(d-1)$-dimensional volume of $B'_r(0)$. Denote

$$\operatorname{osc}_{x'}\big(a^{ij}, \Lambda_r(t,x)\big) = r^{-3}\big|B'_r\big|^{-2} \int_t^{t+r^2} \int_{x^1-r}^{x^1+r} A^{ij}_{x'}(s,\tau)\, d\tau\, ds,$$

$$\operatorname{osc}_{(t,x')}\big(a^{ij}, \Lambda_r(t,x)\big) = r^{-5}\big|B'_r\big|^{-2} \int_{x^1-r}^{x^1+r} A^{ij}_{(t,x')}(\tau)\, d\tau,$$

where

$$A^{ij}_{x'}(s,\tau) = \int_{y',z' \in B'_r(x')} \big|a^{ij}(s,\tau,y') - a^{ij}(s,\tau,z')\big|\, dy'\, dz',$$

$$A^{ij}_{(t,x')}(\tau) = \int_{(\sigma,y'),(\varrho,z') \in \Gamma_r(t,x')} \big|a^{ij}(\sigma,\tau,y') - a^{ij}(\varrho,\tau,z')\big|\, dy'\, dz'\, d\sigma\, d\varrho.$$

Also denote

$$\mathcal{O}^{x'}_R\big(a^{ij}\big) = \sup_{(t,x)\in\mathbb{R}^{d+1}} \sup_{r \leqslant R} \operatorname{osc}_{x'}\big(a^{ij}, \Lambda_r(t,x)\big),$$

$$\mathcal{O}^{(t,x')}_R\big(a^{ij}\big) = \sup_{(t,x)\in\mathbb{R}^{d+1}} \sup_{r \leqslant R} \operatorname{osc}_{(t,x')}\big(a^{ij}, \Lambda_r(t,x)\big).$$

Finally set

$$a^{\#}_R = \mathcal{O}^{(t,x')}_R\big(a^{11}\big) + \sum_{i\neq 1 \text{ or } j \neq 1} \mathcal{O}^{x'}_R\big(a^{ij}\big).$$

Assumption 2.2. There is a continuous function $\omega(t)$ defined on $[0,\infty)$ such that $w(0) = 0$ and $a^{\#}_R \leqslant \omega(R)$ for all $R \in [0,\infty)$.

Remark 2.3. It can be seen from our proofs that we use only the fact that $R \in (0,\infty)$ can be chosen so that $a^{\#}_R$ is smaller than a constant which depends only on constants, especially, N, ν, and α in (17), appearing in the proof of Corollary 4.2.

In this paper we mean by $\mathring{W}^{1,2}_p((0,T)\times\mathbb{R}^d)$ the collection of all functions in $W^{1,2}_p((0,T)\times\mathbb{R}^d)$ vanishing at $t = T$. The differential operator is denoted by L, i.e.,

$$Lu = u_t + a^{ij} u_{x^i x^j} + b^i u_{x^i} + cu. \tag{2}$$

Here we present our main result. Throughout the paper, we write $N = N(d, \dots)$ if N is a constant depending only on $d, \dots$.

Theorem 2.4. *Let $p \in (2, \infty)$. Under Assumptions 2.1 and 2.2, for any $f \in L_p((0,T) \times \mathbb{R}^d)$, there exists a unique $u \in \mathring{W}_p^{1,2}((0,T) \times \mathbb{R}^d)$ such that $Lu = f$ in $(0,T) \times \mathbb{R}^d$. Furthermore, there is a constant $N = N(d, \delta, K, p, \omega, T)$ such that, for any $u \in \mathring{W}_p^{1,2}((0,T) \times \mathbb{R}^d)$,*

$$\|u\|_{W_p^{1,2}((0,T)\times\mathbb{R}^d)} \leqslant N \|Lu\|_{L_p((0,T)\times\mathbb{R}^d)}.$$

Remark 2.5. In case $p = 2$, by Theorem 2.2 in [11], the above theorem holds true under the assumption that a^{ij} are functions of only $(t, x^1) \in \mathbb{R}^2$ satisfying Assumption 2.1. That is, we do not need any regularity assumptions on the coefficient $a^{ij}(t, x^1)$.

3. Auxiliary results

In this section the coefficients a^{ij} are assumed to be measurable functions of only $(t, x^1) \in \mathbb{R}^2$. In addition, we assume that a^{11} is a function of only $x^1 \in \mathbb{R}$. Throughout this section we set

$$\mathcal{L}u(t,x) = u_t(t,x) + a^{ij}\big(t, x^1\big) u_{x^i x^j}(t,x).$$

We denote by $\partial' Q_r(t,x)$ the parabolic boundary of $Q_r(t,x)$ defined as

$$\partial' Q_r(t,x) = \big(\big[t, t+r^2\big] \times \partial B_r(x)\big) \cup \big\{\big(t+r^2, y\big):\ y \in B_r(x)\big\}.$$

Lemma 3.1. *There exists $N = N(d, \delta)$ such that, for $u \in W_2^{1,2}(Q_r)$ with $u|_{\partial' Q_r} = 0$, we have*

$$r^2 \int_{Q_r} |u_x|^2 \, dx \, dt + \int_{Q_r} |u|^2 \, dx \, dt \leqslant N r^4 \int_{Q_r} |\mathcal{L}u|^2 \, dx \, dt.$$

Proof. The proof is identical to that of Lemma 4.1 in [11]. In fact, Lemma 4.1 in [11] assumes that a^{ij} are functions of only x^1, but the proof there needs only the fact that a^{11} is independent of t. □

Lemma 3.2. *Let $0 < r < R$. There exists $N = N(d, \delta)$ such that, for $u \in W_2^{1,2}(Q_R)$,*

$$\|u\|_{W_2^{1,2}(Q_r)} \leqslant N\big(\|\mathcal{L}u - u\|_{L_2(Q_R)} + (R-r)^{-2}\|u\|_{L_2(Q_R)}\big).$$

Proof. The proof of this lemma is based on the estimate of solutions of the equation $\mathcal{L}u = f$ in $L_2(\mathbb{R}^{d+1})$. To find the L_2-estimate, see Remark 2.5 or, more precisely, Theorem 3.2 in [11] with $\lambda = 1$. Hence the proof of Lemma 4.2 in [11] can be repeated without any change. Also see the proof of Lemma 4.2 in [10]. □

Lemma 3.3. *Let $\gamma = (\gamma^1, \dots, \gamma^d)$ be a multi-index such that $\gamma^1 = 0, 1, 2$. Set $\gamma' = (0, \gamma^2, \dots, \gamma^d)$ and $0 < r < R$. If h is a sufficiently smooth function defined on Q_R such that $\mathcal{L}h = 0$ in Q_R, then*

$$\int_{Q_r} \big|D^{\gamma'} h_t\big|^2 \, dx \, dt + \int_{Q_r} \big|D^{\gamma} h\big|^2 \, dx \, dt \leqslant N \int_{Q_R} |h|^2 \, dx \, dt,$$

where $N = N(d, \delta, \gamma, R, r)$.

Proof. Use Lemma 3.2 and the argument in the proof of Lemma 4.4 in [10]. □

We recall some function spaces which we need in the following lemmas. As is well known, we denote by $H_p^s(\mathbb{R}^d)$, $s \in \mathbb{R}$, the space of all generalized functions u such that $(1-\Delta)^{s/2}u \in L_p(\mathbb{R}^d)$. For $k = 0, 1, 2, \dots$, $W_p^k(\Omega)$ is the usual Sobolev space and $C^{k+\nu}(\Omega)$, $0 < \nu < 1$, is the Hölder space. By $C^k(\Omega)$ we mean the space of all functions u whose derivatives $D^\alpha u$, $|\alpha| \leqslant k$, are continuous and bounded in Ω. As we recall,

$$\|u\|_{C^{k+\nu}(\Omega)} = \|u\|_{C^k(\Omega)} + \sum_{|\alpha|=k} \sup_{\substack{x,y\in\Omega\\ x\neq y}} \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x-y|^\nu},$$

where

$$\|u\|_{C^k(\Omega)} = \sum_{|\alpha|\leqslant k} \sup_{x\in\Omega} \left|D^\alpha u(x)\right|.$$

Lemma 3.4. *Let $u \in W_2^{1,2}((0,\infty)\times\mathbb{R}^d) \cap C^2([0,\infty)\times\mathbb{R}^d)$. Then we have*

$$\sup_{0\leqslant s<\infty} \left\|u(s,\cdot)\right\|_{W_2^1(\mathbb{R}^d)} \leqslant N(d)\|u\|_{W_2^{1,2}((0,\infty)\times\mathbb{R}^d)}.$$

Proof. Observe that

$$\begin{aligned}\int_{\mathbb{R}^d} \left|u_{x^i}(s,x)\right|^2 dx &= -2\int_{\mathbb{R}^d}\int_s^\infty u_{x^i t}(t,x)u_{x^i}(t,x)\,dx\,dt \\ &= 2\int_s^\infty\int_{\mathbb{R}^d} u_{x^i x^i}(t,x)u_t(t,x)\,dx\,dt \leqslant \int_{\mathbb{R}^d}\int_s^\infty |u_{x^i x^i}|^2 + |u_t|^2\,dt\,dx.\end{aligned}$$

Similarly, we obtain

$$\int_{\mathbb{R}^d} \left|u(s,x)\right|^2 dx \leqslant \int_{\mathbb{R}^d}\int_s^\infty |u_t|^2 + |u|^2\,dt\,dx.$$

The lemma is proved. □

In the rest of this paper, $h_{x'}$ represents, depending on the context, one of h_{x^i}, $i = 2, \dots, d$, or the whole collection $\{h_{x^2}, \dots, h_{x^d}\}$. By h_x we mean one of h_{x^i}, $i = 1, \dots, d$, or the gradient of h with respect to x. Thus $h_{xx'}$ is one of $h_{x^i x^j}$, where $i \in \{1, \dots, d\}$ and $j \in \{2, \dots, d\}$, or the collection of them. Norms of these collections are defined arbitrarily.

Lemma 3.5. *Let h be a sufficiently smooth function h defined on Q_4 such that $\mathcal{L}h = 0$ in Q_4. Then, for each $x' \in B_1'$,*

$$\int_0^4 \left\|h(t,\cdot,x')\right\|_{L_2(-2,2)}^p dt + \int_0^4 \left\|h_x(t,\cdot,x')\right\|_{L_2(-2,2)}^p dt$$

$$+\int_0^4 \|h_{xx'}(t,\cdot,x')\|^p_{L_2(-2,2)}\,dt \leqslant N\|h\|^p_{L_2(Q_R)},$$

where $1 \leqslant p < \infty$, $2 < R \leqslant 4$, *and* $N = N(d,\delta,p,R)$.

Proof. We prove that, for each $x' \in B_1'$,

$$\int_0^4 \|h(t,\cdot,x')\|^p_{L_2(-2,2)}\,dt + \int_0^4 \|h_{x^1}(t,\cdot,x')\|^p_{L_2(-2,2)}\,dt \leqslant N\|h\|^p_{L_2(Q_\tau)}, \tag{3}$$

where $2 < \tau < R$. If this turns out to be true, then using this and the fact that $\mathcal{L}h_{x'} = 0$ we obtain the inequality (3) with $h_{x'}$ in place of h. Furthermore, using $\mathcal{L}h_{x'x'} = 0$, we also obtain the inequality (3) with $h_{x'x'}$ in place of h. Hence the left side of the inequality in the lemma is not greater than a constant times

$$\|h\|^p_{L_2(Q_\tau)} + \|h_{x'}\|^p_{L_2(Q_\tau)} + \|h_{x'x'}\|^p_{L_2(Q_\tau)}.$$

This and Lemma 3.3 finish the proof.

To prove (3), we introduce an infinitely differentiable function η defined on $\mathbb{R}^2$ such that

$$\eta(t,x^1) = \begin{cases} 1 & \text{on } [0,4]\times[-2,2], \\ 0 & \text{on } \mathbb{R}^2 \setminus [(-r^2,r^2)\times(-r,r)], \end{cases}$$

where $2 < r < \tau$. For each $x' \in B_1'$, view ηh as a function of $(t,x^1) \in (0,\infty)\times\mathbb{R}$. Then by Lemma 3.4

$$\sup_{0\leqslant s<\infty} \|(\eta h)(s,\cdot,x')\|_{W_2^1(\mathbb{R})} \leqslant 2\|(\eta h)(\cdot,x')\|_{W_2^{1,2}((0,\infty)\times\mathbb{R})}$$

for each $x' \in B_1'$. Note that the pth power of the left side of the above inequality is greater than or equal to a constant times the left side of the inequality (3). Also note that the right side of the above inequality is no greater than a constant times

$$\|h(\cdot,x')\|_{W_2^{1,2}((0,r^2)\times(-r,r))}. \tag{4}$$

Now notice that there exists an integer k such that, for each $t \in (0,r^2)$ and $x^1 \in (-r,r)$,

$$\sup_{x'\in B_1'} |h(t,x^1,x')| \leqslant N\|h(t,x^1,\cdot)\|_{W_2^k(B_1')}.$$

This inequality remains true if we replace h with h_t, h_{x^1}, or $h_{x^1x^1}$. Hence, for all $x' \in B_1'$, the square of (4) is not greater than a constant times

$$\int_0^{r^2}\int_{-r}^{r} \big(\|h(t,x^1,\cdot)\|^2_{W_2^k(B_1')} + \|h_t(t,x^1,\cdot)\|^2_{W_2^k(B_1')}$$
$$+ \|h_{x^1}(t,x^1,\cdot)\|^2_{W_2^k(B_1')} + \|h_{x^1x^1}(t,x^1,\cdot)\|^2_{W_2^k(B_1')}\big)\,dx^1\,dt,$$

which is, by Lemma 3.3, less than or equal to a constant times $\|h\|^2_{L_2(Q_\tau)}$. The lemma is proved. □

Let $L_p\_L_q((S,T)\times\Omega)$, $\Omega\subseteq\mathbb{R}^d$, be the space of functions $u(t,x)$ on $(S,T)\times\Omega$ such that

$$\|u\|_{L_p\_L_q((S,T)\times\Omega)}:=\left(\int_S^T\left(\int_\Omega\big|u(t,x)\big|^q\,dx\right)^{p/q}dt\right)^{1/p}<\infty.$$

By $W^{1,2}_{p,q}((S,T)\times\mathbb{R}^d)$ we mean the collection of all functions defined on $(S,T)\times\mathbb{R}^d$ such that

$$\begin{aligned}\|u\|_{W^{1,2}_{p,q}((S,T)\times\mathbb{R}^d)}&:=\|u\|_{L_p\_L_q((S,T)\times\mathbb{R}^d)}+\|u_x\|_{L_p\_L_q((S,T)\times\mathbb{R}^d)}\\&\quad+\|u_{xx}\|_{L_p\_L_q((S,T)\times\mathbb{R}^d)}+\|u_t\|_{L_p\_L_q((S,T)\times\mathbb{R}^d)}<\infty.\end{aligned}$$

We say $u\in\mathring{W}^{1,2}_{p,q}((S,T)\times\mathbb{R}^d)$ if $u\in W^{1,2}_{p,q}((S,T)\times\mathbb{R}^d)$ and $u(T,x)=0$.

The following theorem is taken from [7]. Note that in the theorem coefficients a^{ij} are independent of t.

Theorem 3.6. *Let $p>q\geqslant 2$, $0<T<\infty$, and L be the operator defined in* (2). *That is, coefficients a^{ij}, b^i, and c satisfy Assumptions* 2.1 *and* 2.2. *In addition, we assume that a^{ij} are independent of t and, in case $q=2$, a^{ij} are functions of only $x^1\in\mathbb{R}^d$. Then for any $f\in L_p\_L_q((0,T)\times\mathbb{R}^d)$, there exists a unique $u\in\mathring{W}^{1,2}_{p,q}((0,T)\times\mathbb{R}^d)$ such that $Lu=f$ in $(0,T)\times\mathbb{R}^d$. Furthermore, there is a constant N, depending only on d, δ, K, p, q, T, and the function ω, such that, for any $u\in\mathring{W}^{1,2}_{p,q}((0,T)\times\mathbb{R}^d)$,*

$$\|u\|_{W^{1,2}_{p,q}((0,T)\times\mathbb{R}^d)}\leqslant N\|Lu\|_{L_p\_L_q((0,T)\times\mathbb{R}^d)}.$$

In the lemmas below we use the following notation:

$$[f]_{\mu,\nu;Q_r}:=\sup_{\substack{(t,x),(s,y)\in Q_r\\(t,x)\neq(s,y)}}\frac{|f(t,x)-f(s,y)|}{|t-s|^\mu+|x-y|^\nu}.$$

Lemma 3.7. *Let $p>4$ and $2/p<\beta<1/2$. Assume that h is a sufficiently smooth function defined on Q_4 such that $\mathcal{L}h=0$ in Q_4. Then*

$$[h_{xx'}]_{\mu,\nu,Q_1}\leqslant N\|h\|_{L_2(Q_3)},$$

where $\mu=\beta/2-1/p$, $\nu=1/2-\beta$, and $N=N(d,\delta,p,\beta)$.

Proof. First we note that $0<\mu<1$ and $0<\nu<1$. We prove

$$[h]_{\mu,\nu,Q_1}+[h_{x^1}]_{\mu,\nu,Q_1}\leqslant N\|h\|_{L_2(Q_\tau)},\tag{5}$$

where $2<\tau<3$. Once this is proved, the lemma follows from the argument in the proof of Lemma 3.5 (i.e., use $\mathcal{L}h_{x'}=0$, $\mathcal{L}h_{x'x'}=0$, and Lemma 3.3).

For the proof of the inequality (5) it is enough to prove the following: for all $s,t\in(0,1)$ and $x'\in B'_1$,

$$\big\|h(t,\cdot,x')-h(s,\cdot,x')\big\|_{C^1(-1,1)}\leqslant N|t-s|^\mu\|h\|_{L_2(Q_\tau)},\tag{6}$$

$$\big\|h(t,\cdot,x')\big\|_{C^{1+\nu}(-1,1)}+\big\|h_{x'}(t,\cdot,x')\big\|_{C^1(-1,1)}\leqslant N\|h\|_{L_2(Q_\tau)},\tag{7}$$

where $h(t, x^1, x')$ is considered as a function of only $x^1 \in (-1, 1)$. Indeed, observe that, for $(t, x), (s, y) \in Q_1$,

$$\begin{aligned}\bigl|h(t,x)-h(s,y)\bigr| &\leqslant \bigl|h\bigl(t,x^1,x'\bigr)-h\bigl(t,y^1,x'\bigr)\bigr| \\ &\quad+\bigl|h\bigl(t,y^1,x'\bigr)-h\bigl(t,y^1,y'\bigr)\bigr|+\bigl|h\bigl(t,y^1,y'\bigr)-h\bigl(s,y^1,y'\bigr)\bigr| \\ &\leqslant \sup_{-1<z^1<1}\bigl|h_{x^1}\bigl(t,z^1,x'\bigr)\bigr|\bigl|x^1-y^1\bigr|+\sup_{\substack{|z'|<1\\ -1<z^1<1}}\bigl|h_{x'}\bigl(t,z^1,z'\bigr)\bigr||x'-y'| \\ &\quad+\sup_{-1<z^1<1}\bigl|h\bigl(t,z^1,y'\bigr)-h\bigl(s,z^1,y'\bigr)\bigr| \\ &\leqslant N\bigl(|x-y|+|t-s|^{\mu}\bigr)\|h\|_{L_2(Q_\tau)},\end{aligned}$$

where the last inequality is due to (6) and (7). This proves

$$[h]_{\mu,\nu,Q_1} \leqslant N\|h\|_{L_2(Q_\tau)}.$$

Similarly, (6) and (7) imply

$$[h_{x^1}]_{\mu,\nu,Q_1} \leqslant N\|h\|_{L_2(Q_\tau)}.$$

We now prove (6) and (7), but instead of (7), we prove

$$\bigl\|h(t,\cdot,x')\bigr\|_{C^{1+\nu}(-1,1)} \leqslant N\|h\|_{L_2(Q_r)}, \tag{8}$$

where $2 < r < \tau$. If this holds true, then, as in the proof of Lemma 3.5, by the fact that $\mathcal{L}h_{x'} = 0$ it follows that

$$\bigl\|h_{x'}(t,\cdot,x')\bigr\|_{C^{1+\nu}(-1,1)} \leqslant N\|h_{x'}\|_{L_2(Q_r)}.$$

This and (8) along with Lemma 3.3 prove (7).

Let η be an infinitely differentiable function defined on $\mathbb{R}^2$ such that

$$\eta\bigl(t,x^1\bigr)=\begin{cases}1 & \text{on } [0,1]\times[-1,1],\\ 0 & \text{on } \mathbb{R}^2\setminus(-4,4)\times(-2,2).\end{cases}$$

Also let

$$g\bigl(t,x^1,x'\bigr)=-\sum_{i\neq 1 \text{ or } j\neq 1} a^{ij}\bigl(t,x^1\bigr)h_{x^ix^j}\bigl(t,x^1,x'\bigr),$$

so that

$$h_t + a^{11}\bigl(x^1\bigr)h_{x^1x^1} = g.$$

Then

$$(\eta h)_t + a^{11}\bigl(x^1\bigr)(\eta h)_{x^1x^1} = \eta g + 2a^{11}\eta_{x^1}h_{x^1} + \bigl(\eta_t + a^{11}\eta_{x^1x^1}\bigr)h.$$

For each $x' \in B_1'$, consider ηh as a function of $(t, x^1) \in (0, \infty) \times \mathbb{R}$. Then by Theorem 3.6 (note that $\eta h = 0$ for $t \geqslant 4$), we have

$$\|\eta h\|_{W^{1,2}_{p,2}((0,\infty)\times\mathbb{R})} \leqslant N\bigl\|\eta g + 2a^{11}\eta_{x^1}h_{x^1} + \bigl(\eta_t + a^{11}\eta_{x^1x^1}\bigr)h\bigr\|_{L_p\_L_2((0,\infty)\times\mathbb{R})},$$

where $N = N(\delta, p)$. We see that, for each $x' \in B_1'$, the right-hand side of the above inequality is not greater than a constant times

$$\|h\|_{L_p\_L_2((0,4)\times(-2,2))} + \|h_x\|_{L_p\_L_2((0,4)\times(-2,2))} + \|h_{xx'}\|_{L_p\_L_2((0,4)\times(-2,2))},$$

which is, by Lemma 3.5, less than or equal to a constant times $\|h\|_{L_2(Q_r)}$. Hence it follows that

$$\big\|(\eta h)(\cdot, x')\big\|_{W^{1,2}_{p,2}((0,\infty)\times\mathbb{R})} \leqslant N\|h\|_{L_2(Q_r)} \tag{9}$$

for all $x' \in B_1'$. Again we view ηh as a function of $(t, x^1) \in (0,\infty)\times\mathbb{R}$. Then by Theorem 7.3 in [12]

$$\big\|(\eta h)(t, \cdot, x') - (\eta h)(s, \cdot, x')\big\|_{H_2^{2-\beta}(\mathbb{R})} = N|t-s|^{\mu}\big\|(\eta h)(\cdot, x')\big\|_{W^{1,2}_{p,2}((0,\infty)\times\mathbb{R})} \tag{10}$$

for each $x' \in B_1'$, where N is independent of s, t, and ηh. Using an embedding theorem, we have

$$\big\|(\eta h)(t, \cdot, x') - (\eta h)(s, \cdot, x')\big\|_{C^{1+\nu}(\mathbb{R})} \leqslant N\big\|(\eta h)(t, \cdot, x') - (\eta h)(s, \cdot, x')\big\|_{H_2^{2-\beta}(\mathbb{R})},$$

where, as noted earlier, $\nu = 1/2 - \beta$. From this, (9) and (10), we finally have

$$\big\|(\eta h)(t, \cdot, x') - (\eta h)(s, \cdot, x')\big\|_{C^{1+\nu}(\mathbb{R})} \leqslant N|t-s|^{\mu}\|h\|_{L_2(Q_r)}$$

for all $x' \in B_1'$. This proves (6). To prove (8), we set $s = 4$ in the above inequality. The lemma is proved. □

Let $u \in C_0^\infty(\mathbb{R}^{d+1})$ and $f := \mathcal{L}u$, where, as we recall, $\mathcal{L}u = u_t + a^{ij}u_{x^ix^j}$. Assume that $a^{11}(x^1)$, $a^{ij}(t, x^1)$, $i \neq 1$ or $j \neq 1$, are infinitely differentiable. Then there exists a sufficiently smooth function h defined on Q_4 such that

$$\begin{cases} \mathcal{L}h = 0 & \text{in } Q_4, \\ h = u & \text{on } \partial' Q_4. \end{cases}$$

In the following lemma we establish an inequality for the functions u, f and h in the above.

Lemma 3.8. *Let $p > 4$ and $2/p < \beta < 1/2$. There exists a constant $N = N(d, \delta, p, \beta)$ such that*

$$[h_{xx'}]_{\mu,\nu,Q_1} \leqslant N\|f\|_{L_2(Q_4)} + N\|u_{xx}\|_{L_2(Q_4)},$$

where $\mu = \beta/2 - 1/p$ and $\nu = 1/2 - \beta$.

Proof. We need only follow the argument in Lemma 4.6 in [10] along with Lemmas 3.1 and 3.7. □

Denote by $(u)_{Q_r(t_0,x_0)}$ the average value of a function u over $Q_r(t_0, x_0)$, that is,

$$(u)_{Q_r(t_0,x_0)} = \fint_{Q_r(t_0,x_0)} u(t,x)\,dx\,dt.$$

Lemma 3.9. *Let $\kappa \geqslant 4$ and $r > 0$. Let a^{ij} be infinitely differentiable. For $u \in C_0^\infty(\mathbb{R}^{d+1})$, we find a smooth function h defined on $Q_{\kappa r}$ such that $\mathcal{L}h = 0$ in $Q_{\kappa r}$ and $h = u$ on $\partial' Q_{\kappa r}$. Then there exists a constant $N = N(d, \delta)$ such that*

$$\fint_{Q_r} \big|h_{xx'} - (h_{xx'})_{Q_r}\big|^2\,dx\,dt \leqslant N\kappa^{-1/4}\big[\big(|\mathcal{L}u|^2\big)_{Q_{\kappa r}} + \big(|u_{xx}|^2\big)_{Q_{\kappa r}}\big]. \tag{11}$$

Proof. We first prove that the inequality (11) holds under the assumption that the inequality is true for the case $r=1$. In fact, this is done by the dilation argument shown in [10,11]. However, rather than referring to these papers, we repeat here the argument. Let $r>0$, $\kappa\geqslant 4$, and h be a sufficiently smooth function such that $\mathcal{L}h=0$ in $Q_{\kappa r}$ and $h=u$ on $\partial' Q_{\kappa r}$. Set

$$\hat{h}(t,x)=r^{-2}h\big(r^2t,rx\big),\qquad \hat{\mathcal{L}}=\frac{\partial}{\partial t}+a^{ij}\big(r^2t,rx^1\big)\frac{\partial^2}{\partial x^i\partial x^j}.$$

Then $\hat{h}$ is defined on Q_κ and

$$\hat{\mathcal{L}}\hat{h}(t,x)=(\mathcal{L}h)\big(r^2t,rx\big)=0\quad\text{in } Q_\kappa,\qquad \hat{h}=\hat{u}\quad\text{on } \partial' Q_\kappa,$$

where $\hat{u}(t,x)=r^{-2}u(r^2t,rx)$. Note that $\hat{\mathcal{L}}$ satisfies the same ellipticity condition as $\mathcal{L}$ does. Thus by the assumption that the inequality (11) holds true for the case $r=1$, we have

$$\fint_{Q_1}\big|\hat{h}_{xx'}-(\hat{h}_{xx'})_{Q_1}\big|^2\,dx\,dt\leqslant N\kappa^{-1/4}\big[\big(|\hat{\mathcal{L}}\hat{u}|^2\big)_{Q_\kappa}+\big(|\hat{u}_{xx}|^2\big)_{Q_\kappa}\big].$$

Notice that

$$\fint_{Q_1}\big|\hat{h}_{xx'}-(\hat{h}_{xx'})_{Q_1}\big|^2\,dx\,dt=\fint_{Q_r}\big|h_{xx'}-(h_{xx'})_{Q_r}\big|^2\,dx\,dt,$$

$$\big(|\hat{\mathcal{L}}\hat{u}|^2\big)_{Q_\kappa}+\big(|\hat{u}_{xx}|^2\big)_{Q_\kappa}=\big(|\mathcal{L}u|^2\big)_{Q_{\kappa r}}+\big(|u_{xx}|^2\big)_{Q_{\kappa r}}.$$

Therefore, the inequality (11) is proved for $r>0$.

For the case $r=1$, set $p=8$ and $\beta=3/8$ in Lemma 3.8, so $2\mu=\nu=1/8$. Then using Lemma 3.8 and the dilation argument in the above, we obtain

$$[h_{xx'}]^2_{\mu,\nu,Q_{\kappa/4}}\leqslant N\kappa^{-1/4}\big[\big(|\mathcal{L}u|^2\big)_{Q_\kappa}+\big(|u_{xx}|^2\big)_{Q_\kappa}\big].\tag{12}$$

On the other hand, by the fact that $\kappa\geqslant 4$, we have

$$\fint_{Q_1}\big|h_{xx'}-(h_{xx'})_{Q_1}\big|^2\,dx\,dt\leqslant N[h_{xx'}]^2_{\mu,\nu,Q_1}\leqslant N[h_{xx'}]^2_{\mu,\nu,Q_{\kappa/4}}.$$

This and (12) prove the case $r=1$. The lemma is proved. □

Lemma 3.10. *There exists a constant $N=N(d,\delta)$ such that, for any $\kappa\geqslant 4$, $r>0$, and $u\in C_0^\infty(\mathbb{R}^{d+1})$, we have*

$$\fint_{Q_r}\big|u_{xx'}-(u_{xx'})_{Q_r}\big|^2\,dx\,dt\leqslant N\kappa^{d+2}\big(|\mathcal{L}u|^2\big)_{Q_{\kappa r}}+N\kappa^{-1/4}\big(|u_{xx}|^2\big)_{Q_{\kappa r}}.$$

Proof. Use Lemmas 3.9, 3.2, 3.1 and the argument in the proof of Lemma 4.8 in [10] (also see Lemma 4.7 in [11]). □

4. Proof of Theorem 2.4

We assume in this section that all assumptions in Section 2 are satisfied. Especially, by L we mean the operator defined in (2). In this section we set

$$L_0u=u_t+a^{ij}(t,x)u_{x^ix^j}.$$

Let $\mathbb{Q}$ be the collection of all $Q_r(t,x)$, $(t,x) \in \mathbb{R}^{d+1}$, $r \in (0,\infty)$. For a function g defined on $\mathbb{R}^{d+1}$, we denote its (parabolic) maximal and sharp functions, respectively, by

$$Mg(t,x) = \sup_{(t,x)\in Q} \fint_Q \big|g(s,y)\big|\,dy\,ds,$$

$$g^{\#}(t,x) = \sup_{(t,x)\in Q} \fint_Q \big|g(s,y) - (g)_Q\big|\,dy\,ds,$$

where the supremums are taken over all $Q \in \mathbb{Q}$ containing (t,x).

Theorem 4.1. *Let μ, $\nu \in (1,\infty)$, $1/\mu + 1/\nu = 1$, and $R \in (0,\infty)$. There exists a constant $N = N(d,\delta,\mu)$ such that, for any $u \in C_0^\infty(Q_R)$, we have*

$$(u_{xx'})^{\#} \leqslant N\big(a_R^{\#}\big)^{\frac{\alpha}{\nu}}\big[M\big(|u_{xx}|^{2\mu}\big)\big]^{\frac{1}{2\mu}} + N\big[M\big(|L_0u|^2\big)\big]^{\alpha}\big[M\big(|u_{xx}|^2\big)\big]^{\beta},$$

where $\alpha = 1/(8d+18)$ and $\beta = (4d+8)/(8d+18)$.

Proof. Let $\kappa \geqslant 4$, $r \in (0,\infty)$, and $(t_0,x_0) = (t_0,x_0^1,x_0') \in \mathbb{R}^{d+1}$. We introduce another coefficients $\bar{a}^{ij}$ defined as

$$\bar{a}^{11}\big(x^1\big) = \fint_{\Gamma_{\kappa r}(t_0,x_0')} a^{11}\big(s,x^1,y'\big)\,dy'\,ds \quad \text{if } \kappa r < R,$$

$$\bar{a}^{11}\big(x^1\big) = \fint_{\Gamma_R} a^{11}\big(s,x^1,y'\big)\,dy'\,ds \quad \text{if } \kappa r \geqslant R.$$

In case $i \neq 1$ or $j \neq 1$,

$$\bar{a}^{ij}\big(t,x^1\big) = \fint_{B'_{\kappa r}(x_0')} a^{ij}\big(t,x^1,y'\big)\,dy' \quad \text{if } \kappa r < R,$$

$$\bar{a}^{ij}\big(x^1\big) = \fint_{B'_R} a^{ij}\big(t,x^1,y'\big)\,dy' \quad \text{if } \kappa r \geqslant R.$$

Set $\bar{L}_0u = u_t + \bar{a}^{ij}u_{x^ix^j}$. Then by Lemma 3.10, we have

$$\begin{aligned}&\big(\big|u_{xx'} - (u_{xx'})_{Q_r(t_0,x_0)}\big|^2\big)_{Q_r(t_0,x_0)}\\ &\quad\leqslant N\kappa^{d+2}\big(|\bar{L}_0u|^2\big)_{Q_{\kappa r}(t_0,x_0)} + N\kappa^{-1/4}\big(|u_{xx}|^2\big)_{Q_{\kappa r}(t_0,x_0)}.\end{aligned} \tag{13}$$

Note that

$$\int_{Q_{\kappa r}(t_0,x_0)} |\bar{L}_0u|^2\,dx\,dt \leqslant 2\int_{Q_{\kappa r}(t_0,x_0)} |L_0u|^2\,dx\,dt + N(d)\sum_{i,j=1}\chi_{ij}, \tag{14}$$

where

$$\chi_{ij} = \int_{Q_{\kappa r}(t_0,x_0)} \left|\left(\bar{a}^{ij} - a^{ij}\right)u_{x^i x^j}\right|^2 dx\,dt = \int_{Q_{\kappa r}(t_0,x_0)\cap Q_R} \cdots \leqslant I_{ij}^{1/\nu} J_{ij}^{1/\mu},$$
$$I_{ij} = \int_{Q_{\kappa r}(t_0,x_0)\cap Q_R} \left|\bar{a}^{ij} - a^{ij}\right|^{2\nu} dx\,dt,$$
$$J_{ij} = \int_{Q_{\kappa r}(t_0,x_0)\cap Q_R} |u_{x^i x^j}|^{2\mu}\,dx\,dt.$$

Using the definitions of $\bar{a}^{ij}$ and assumptions on a^{ij}, we obtain the following estimates for I_{ij}. If $\kappa r < R$,

$$I_{11} \leqslant N \int_{x_0^1-\kappa r}^{x_0^1+\kappa r} \int_{\Gamma_{\kappa r}(t_0,x_0')} \left|\bar{a}^{11} - a^{11}\right| dx\,dt \leqslant N(\kappa r)^{d+2}\mathcal{O}_{\kappa r}^{(t,x')}\left(a^{11}\right) \leqslant N(\kappa r)^{d+2}a_R^{\#}.$$

In case $\kappa r \geqslant R$,

$$I_{11} \leqslant N \int_{-R}^{R}\int_{\Gamma_R} \left|\bar{a}^{11} - a^{11}\right| dx\,dt \leqslant N R^{d+2}\mathcal{O}_R^{(t,x')}\left(a^{11}\right) \leqslant N(\kappa r)^{d+2}a_R^{\#}.$$

Now let $j \neq 1$ or $k \neq 1$. If $\kappa r < R$,

$$I_{ij} \leqslant N \int_{\Lambda_{\kappa r}(t_0,x_0)} \left|\bar{a}^{ij} - a^{ij}\right| dx'\,dx^1\,dt \leqslant N(\kappa r)^{d+2}\mathcal{O}_{\kappa r}^{x'}\left(a^{ij}\right) \leqslant N(\kappa r)^{d+2}a_R^{\#}.$$

In case $\kappa r \geqslant R$,

$$I_{ij} \leqslant N \int_{\Lambda_R} \left|\bar{a}^{ij} - a^{ij}\right| dx'\,dx^1\,dt \leqslant N R^{d+2}\mathcal{O}_R^{x'}\left(a^{ij}\right) \leqslant N(\kappa r)^{d+2}a_R^{\#}.$$

From the inequality (14) and the estimates for I_{ij}, it follows that

$$\left(|\bar{L}_0 u|^2\right)_{Q_{\kappa r}(t_0,x_0)} \leqslant N\left(a_R^{\#}\right)^{1/\nu}\left(|u_{xx}|^{2\mu}\right)_{Q_{\kappa r}(t_0,x_0)}^{1/\mu} + N\left(|L_0 u|^2\right)_{Q_{\kappa r}(t_0,x_0)}.$$

This, together with (13), gives us

$$\begin{aligned}
&\left(\left|u_{xx'} - (u_{xx'})_{Q_r(t_0,x_0)}\right|^2\right)_{Q_r(t_0,x_0)} \\
&\quad \leqslant N\kappa^{d+2}\left(a_R^{\#}\right)^{1/\nu}\left(|u_{xx}|^{2\mu}\right)_{Q_{\kappa r}(t_0,x_0)}^{1/\mu} \\
&\qquad + N\kappa^{d+2}\left(|L_0 u|^2\right)_{Q_{\kappa r}(t_0,x_0)} + N\kappa^{-1/4}\left(|u_{xx}|^2\right)_{Q_{\kappa r}(t_0,x_0)}
\end{aligned} \tag{15}$$

for any $r > 0$ and $\kappa \geqslant 4$. Let

$$\mathcal{A}(t,x) = M\left(|L_0 u|^2\right)(t,x), \qquad \mathcal{B}(t,x) = M\left(|u_{xx}|^2\right)(t,x),$$
$$\mathcal{C}(t,x) = \left(M\left(|u_{xx}|^{2\mu}\right)(t,x)\right)^{1/\mu}.$$

Then we observe that $(|L_0 u|^2)_{Q_{\kappa r}(t_0,x_0)} \leqslant \mathcal{A}(t,x)$ for all $(t,x) \in Q_r(t_0,x_0)$. Similar inequalities are obtained for $\mathcal{B}$ and $\mathcal{C}$. From this and (15) it follows that, for any $(t,x) \in \mathbb{R}^{d+1}$ and $Q \in \mathbb{Q}$

such that $(t, x) \in Q$,

$$\left(\left|u_{xx'} - (u_{xx'})_Q\right|^2\right)_Q \leqslant N\kappa^{d+2}\left(a_R^{\#}\right)^{1/\nu}\mathcal{C}(t,x) + N\kappa^{d+2}\mathcal{A}(t,x) + N\kappa^{-1/4}\mathcal{B}(t,x)$$

for $\kappa \geqslant 4$. Moreover, the above inequality also holds true for $0 < \kappa < 4$ because

$$\fint_Q \left|u_{xx'} - (u_{xx'})_Q\right|^2 dx\,dt \leqslant \left(|u_{xx'}|^2\right)_Q \leqslant \sqrt{2}\kappa^{-1/4}\mathcal{B}(t,x)$$

for any $(t, x) \in Q \in \mathbb{Q}$. Therefore, we finally have

$$\left(\left|u_{xx'} - (u_{xx'})_Q\right|^2\right)_Q \leqslant N\kappa^{d+2}\left(a_R^{\#}\right)^{1/\nu}\mathcal{C}(t,x) + N\kappa^{d+2}\mathcal{A}(t,x) + N\kappa^{-1/4}\mathcal{B}(t,x)$$

for all $\kappa > 0$, $(t, x) \in \mathbb{R}^{d+1}$, and $Q \in \mathbb{Q}$ such that $(t, x) \in Q$. Take the supremum of the left-hand side of the above inequality over all $Q \in \mathbb{Q}$ containing (t, x), and then minimize the right-hand side with respect to $\kappa > 0$. Also observe that

$$\left(\left|u_{xx'} - (u_{xx'})_Q\right|\right)_Q^2 \leqslant \left(\left|u_{xx'} - (u_{xx'})_Q\right|^2\right)_Q.$$

Then we obtain

$$\left[u_{xx'}^{\#}(t,x)\right]^2 \leqslant N\left[\left(a_R^{\#}\right)^{1/\nu}\mathcal{C}(t,x) + \mathcal{A}(t,x)\right]^{\frac{1}{4d+9}}\left[\mathcal{B}(t,x)\right]^{\frac{4d+8}{4d+9}},$$

where $N = N(d, \delta, \mu)$. Upon noticing $\mathcal{B}(t, x) \leqslant \mathcal{C}(t, x)$, we arrive at the inequality in the theorem. This finishes the proof. □

Corollary 4.2. *For $p > 2$, there exist constants $R = R(d, \delta, p, \omega)$ and $N = N(d, \delta, p)$ such that, for any $u \in C_0^\infty(Q_R)$, we have*

$$\|u_{xx}\|_{L_p(\mathbb{R}^{d+1})} \leqslant N\|L_0 u\|_{L_p(\mathbb{R}^{d+1})}.$$

Proof. In this proof we set $L_p := L_p(\mathbb{R}^{d+1})$. Let μ be a real number such that $p > 2\mu > 1$. Then by applying the Fefferman–Stein theorem on sharp functions, Hölder's inequality, and the Hardy–Littlewood maximal function theorem on the inequality in Theorem 4.1, we obtain

$$\|u_{xx'}\|_{L_p} \leqslant N\left(a_R^{\#}\right)^{\frac{\alpha}{\nu}}\|u_{xx}\|_{L_p} + N\|L_0 u\|_{L_p}^{2\alpha}\|u_{xx}\|_{L_p}^{2\beta}, \tag{16}$$

where, as noted in Theorem 4.1, $1/\mu + 1/\nu = 1$ and $2\alpha + 2\beta = 1$. On the other hand, let

$$g = L_0 u + \Delta_{d-1} u - \sum_{j\neq 1, k\neq 1} a^{ij} u_{x^i x^j},$$

where $\Delta_{d-1} u = u_{x^2x^2} + \cdots + u_{x^dx^d}$. Then

$$u_t + a^{11}u_{x^1x^1} + \Delta_{d-1}u = g.$$

Note that the coefficients of the operator $a^{11}u_{x^1x^1} + \Delta_{d-1}u$ satisfy the assumptions in [11]. Thus by Corollary 5.2 in [11] there exists $R = R(d, \delta, p, \omega)$ and $N = N(d, \delta, p)$ such that

$$\|u_{x^1x^1}\|_{L_p} \leqslant N\|g\|_{L_p}, \quad u \in C_0^\infty(Q_R).$$

This leads us to

$$\|u_{x^1x^1}\|_{L_p} \leqslant N\left(\|L_0 u\|_{L_p} + \|u_{xx'}\|_{L_p}\right), \quad u \in C_0^\infty(Q_R).$$

This and (16) allow us to have

$$\|u_{xx}\|_{L_p} \leqslant N\|L_0 u\|_{L_p} + N\big(a_R^{\#}\big)^{\frac{\alpha}{\nu}}\|u_{xx}\|_{L_p} + N\|L_0 u\|_{L_p}^{2\alpha}\|u_{xx}\|_{L_p}^{2\beta}.$$

Take another sufficiently small R (we call it R again), which is equal to or smaller than the R in the above, so that it satisfies

$$N\big(a_R^{\#}\big)^{\frac{\alpha}{\nu}} \leqslant 1/2. \tag{17}$$

Then we obtain

$$\frac{1}{2}\|u_{xx}\|_{L_p} \leqslant N\|L_0 u\|_{L_p} + N\|L_0 u\|_{L_p}^{2\alpha}\|u_{xx}\|_{L_p}^{2\beta}.$$

This finishes the proof. □

Proof of Theorem 2.4. We have an L_p-estimate for functions with small compact support. Thus the rest of the proof can be done by following the argument in [14]. □

Acknowledgment

I would like to thank Nicolai V. Krylov for giving me many great ideas during the preparation of this paper.

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