Existence and uniqueness of analytic solution for Rayleigh–Taylor problem

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Abstract

We study the initial value problem for the classical Rayleigh–Taylor problem. If the initial data are analytic in disk \( R_r \) containing the unit disk, it is proved that unique solution, which is analytic in \( R_s \) for \( s \in (1, r) \), exists locally in time. The analysis is based on a Nirenberg theorem on abstract Cauchy–Kowalewski problem in properly chosen Banach spaces.

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1. Introduction

The motion of the interface of a heavy fluid resting above a lighter fluid in the presence of gravity (Rayleigh–Taylor flow) is a very basic but important problem. When the fluids are immiscible, the sharp interface deforms into a pattern containing rising bubbles of lighter fluid and falling spikes of heavier fluid.

Model equations for the location of the interface have been derived (see Baker et al. [1,2], Caflisch et al. [5], Moore [10], Sharp [15] and references therein). These studies are numerical and asymptotic, but important to furthering physical understanding of the flow dynamics. Numerical calculation ran into the traditional difficulties associated with singularity formation.

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Singularity formation is one of the most interesting phenomenon for many free surface flows such as Kelvin–Helmholtz vortex sheets, Rayleigh–Taylor flows and Saffman–Taylor flows. It has received a lot of attention in the literature over the past 25 years. These singularities, if they exist, are important to both mathematical theory and numerical calculation, while their physical significance depends on the particular problem.

There have been a lot of literature with regard to singularity formation of Rayleigh–Taylor instabilities. Numerical evidence for singularity was performed by Pugh [13] for Boussinesq limit and by Siegel [16]. Inspired by studies of the evolution of vortex sheets in homogeneous fluid, Baker et al. [1] developed a simple approximation for Rayleigh–Taylor flow as a generalization of Moore’s approximation for Kelvin–Helmholtz instability. They considered the full range of Atwood ratios and showed that singularities do exist in the complex plane and move towards the real axis. Tanveer [18,19] explored the dynamics of singularity formation in the classical Rayleigh–Taylor problem without resort to any localized approximation. Under some assumptions, Tanveer showed that the only possible singularity is of a “fold” type i.e. one-half, one-third or one-fourth singularity and so on. He also addressed the question of how does a singularity form in the unphysical region when there is none initially. He gave analytical evidence to suggest that singularities can form instantaneously at a point where the derivative of the initial conformal map is zero. Tanveer also employed numerical procedure to track the motion of a one-half singularity that is created at the initial instant of time and such singularity can actually impinge the physical domain for a very special initial condition. Goldstein et al. [6] derived nonlinear PDEs for the Rayleigh–Taylor instability of stratified fluid layers from lubrication approximation. Their numerical and analytical analysis showed that “pinching” singularities occur generically when the system is unstable. These type of pinching singularities were analyzed in works on thin films and spreading of drops [3,4].

Sulem and Sulem [17] studied the more general two fluid Rayleigh–Taylor problem for two- and three-dimensional flow. They obtained finite analyticity of the interface when the initial interface has sufficiently small gradients and is flat at infinity. Recently Kamotski and Lebeau [8] have shown that the two-dimensional two phase Rayleigh–Taylor problem is strongly ill posed in the sense that analyticity of the initial data is a necessary condition to get a local solution with interface being Hölder continuous.

In this paper, we rigorously study the initial value problem formulated by Tanveer [18]. We prove that if the initial data has no singularity in a disk $R_r$ and satisfies some conditions, then there exists unique solution to the classical Rayleigh–Taylor problem locally in time and the solution has no singularity in $R_s$ for any $s \in (1, r)$. Unlike [17], we do not require small initial data. Our analysis is based on a Nirenberg’s theorem [11,12] on nonlinear abstract Cauchy–Kowalewski equation in properly chosen Banach spaces.

Another interesting similar problem is the motion of the interface of water wave with air above water, this problem usually referred to as the water wave problem has been subject of study for several decades. For water wave problem with infinite depth, Wu [20] showed the existence and uniqueness of solutions of the full problem in Sobolev space, locally in time, for any initial interface which is nonself-intersect; Wu [21] obtained the same results for 3-D full water wave problem. For water wave problem with finite depth, Lannes [9] recently obtained the existence and uniqueness of solutions in Sobolev space locally in time. In Section 5 of this paper, based on the same conformal map formulation of the two-dimensional water wave problem as in Tanveer [18], we obtain the existence and uniqueness of analytic solution locally in time of the water wave problem; the proof is strictly parallel to that for classical Rayleigh–Taylor problem.
It is to be noted that Nirenberg’s theorem has been successfully employed to Hele-Shaw flow [14], vortex sheets problem [7] and unsteady dendritic crystal growth [22].

2. The classical Rayleigh–Taylor flow

We are going to use the same formulation of the classical Rayleigh–Taylor flow as in [18,19]. At time \( t \), consider the conformal map \( z(\xi,t) \) that maps the interior of a cut unit circle in \( \xi \) plane into a periodic strip in the physical domain such that the origin coincides with \( z = -i\infty \). The gravity is assumed to act in the positive \( y \) direction. We assume that there is a vacuum on top of the fluid and that there is no net motion at \( y = -\infty \). The unit circle boundary then corresponds to the free boundary.

2.1. Mathematical equations

The conformal map from the cut unit circle in the \( \xi \) plane into the physical domain in \( z = x + iy \) plane can be decomposed into

\[
z(\xi, t) = 2\pi + i \ln \xi + if(\xi, t), \tag{2.1}
\]

where \( f(\xi) \) is analytic in \( |\xi| < 1 \). Under this map, the free interface in the physical plane is mapped onto the unit circle \( |\xi| = 1 \) in complex \( \xi \) plane.

The kinematic condition on the free boundary can be expressed as

\[
\frac{\partial}{\partial t} \ln \rho(x, y, t) = 0, \tag{2.2}
\]
on \( \rho(x, y, t) = 1 \), where \( \xi = \rho e^{i\theta} \), with \( 0 \leq \theta \leq 2\pi \). Let \( W(\xi, t) \) be complex velocity potential, which is analytic in the unit circle \( |\xi| < 1 \). Switching the role of dependent and independent variables, the kinematic condition (2.2) becomes that

\[
\text{Re}\left[\xi W_\xi - \xi^* z_\xi^* z_t\right] = 0, \tag{2.3}
\]

where * means complex conjugate.

Using Eq. (2.1) for \( z \), we find the above is equivalent to

\[
\text{Re}\left[\frac{f_t}{1 + \xi f_\xi}\right] = \frac{\text{Re} \xi W_\xi}{|1 + \xi f_\xi|^2}, \tag{2.4}
\]
on \( \xi = e^{i\theta} \), for \( 0 \leq \theta \leq 2\pi \).

The Bernoulli’s equation on the free surface for the unsteady problem is

\[
\text{Re}\left[W_t - \frac{\xi W_\xi f_t}{1 + \xi f_\xi} - f\right] = -\frac{1}{2} \frac{|W_\xi|^2}{|1 + \xi f_\xi|^2}, \tag{2.5}
\]
on \( \xi = e^{i\theta} \), for \( 0 \leq \theta \leq 2\pi \).

The analytic continuations of (2.4) and (2.5) to \( |\xi| < 1 \) are

\[
f_t = (1 + \xi f_\xi)I_1^-[f, W](\xi, t), \tag{2.6}
\]
where $I_1^-$ is defined in $|\xi| < 1$ by

$$I_1^-[f, W](\xi, t) = \frac{1}{4\pi i} \int_{|\xi'|=1} d\xi' \frac{[\xi' + \xi'][\xi' W_\xi(\xi', t) + \xi'^{-1} \bar{W}_\xi(\xi', t)]}{[1 + \xi f_\xi(\xi', t)][1 + \xi'^{-1} \bar{f}_\xi(\xi', t)]}. \quad (2.7)$$

where $\bar{f}$ is defined by

$$\bar{f}(\xi) = \left[ f\left(\frac{1}{\xi^*}\right)\right]^*, \quad (2.8)$$

where $^*$ denotes complex conjugate, and

$$W_t - \frac{\xi W_\xi f_t}{1 + \xi f_\xi} - f = -I_2^- [f, W](\xi, t), \quad (2.9)$$

where $I_2^-$ is defined in $|\xi| < 1$ by

$$I_2^-[f, W](\xi, t) = \frac{1}{4\pi i} \int_{|\xi'|=1} d\xi' \frac{[\xi' + \xi'][\xi' W_\xi(\xi', t) + \xi'^{-1} \bar{W}_\xi(\xi', t)]}{[1 + \xi f_\xi(\xi', t)][1 + \xi'^{-1} \bar{f}_\xi(\xi', t)]}. \quad (2.10)$$

The initial conditions are

$$f(\xi, 0) = f_0(\xi), \quad W(\xi, 0) = W_0(\xi). \quad (2.11)$$

Therefore, the unsteady Rayleigh–Taylor problem is to find functions $f(\xi, t)$ and $W(\xi, t)$ which are analytic in $|\xi| < 1$ and satisfy (2.6), (2.9) and (2.11).

### 2.2. An equivalent formulation

The analytic extensions of (2.6) and (2.9) to $|\xi| > 1$ are

$$f_t = \frac{\frac{\xi W_\xi(\xi, t) + \xi^{-1} \bar{W}_\xi(\xi, t)}{(1 + \xi^{-1} f_\xi(\xi, t))} + (1 + \xi f_\xi) I_1^+[f, W](\xi, t)}{I_1^+[f, W](\xi, t)} \quad (2.12)$$

where

$$I_1^+[f, W](\xi, t) = \frac{1}{4\pi i} \int_{|\xi'|=1} d\xi' \frac{[\xi' + \xi'][\xi' W_\xi(\xi', t) + \xi'^{-1} \bar{W}_\xi(\xi', t)]}{[1 + \xi f_\xi(\xi', t)][1 + \xi'^{-1} \bar{f}_\xi(\xi', t)]}. \quad (2.13)$$

and

$$W_t - \frac{\xi W_\xi f_t}{1 + \xi f_\xi} - f = -I_2^+[f, W](\xi, t) - \frac{W_\xi(\xi, t) \bar{W}_\xi(\xi, t)}{[1 + \xi f_\xi(\xi, t)][1 + \xi^{-1} \bar{f}_\xi(\xi, t)]}. \quad (2.14)$$
where

\[ I_2^+[f, W](\xi, t) = \frac{1}{4\pi i} \int_{|\xi'|=1} d\xi' \left[ \frac{\xi' + \xi}{\xi' - \xi} \right] \frac{W_\xi(\xi', t) \tilde{W}_\xi(\xi', t)}{[1 + \xi f_\xi(\xi', t)][1 + \xi'^{-1} \tilde{f}_\xi(\xi', t)]}. \quad (2.15) \]

Plugging (2.12) into (2.14) gives

\[ W_t = f - I_2^+[f, W](\xi, t) - \frac{\xi^2 W_\xi^2(\xi, t)}{[1 + \xi f_\xi(\xi, t)][1 + \xi'^{-1} \tilde{f}_\xi(\xi, t)]} - \xi W_\xi I_1^+[f, W](\xi, t). \quad (2.16) \]

The right-hand sides of Eqs. (2.12) and (2.16) are not suitable to the application of Nirenberg theorem, so we need to change system of (2.12) and (2.16) into an equivalent one. To this end, we introduce the change of variable:

\[ g(\xi, t) = \frac{\xi W_\xi(\xi, t)}{1 + \xi f_\xi(\xi, t)}, \quad (2.17) \]

\[ h(\xi, t) = \frac{1}{1 + \xi f_\xi(\xi, t)}, \quad (2.18) \]

then (2.16) becomes [18,19]

\[ g_t = (R_3[g, h] + R_2[h]g) g_\xi + \xi g h (R_1[g])_\xi - (1 + \xi (I_2^+)_\xi) h + 1, \quad (2.19) \]

(2.12) becomes

\[ h_t = (R_3[g, h] + R_2[h]g) h_\xi - R_2[h] h g_\xi + \frac{R_3[g, h]}{\xi} h - \frac{R_3[g, h]}{\xi} h^2 - \xi R_4[g, h] h^2 \]

\[ - \xi \left( \frac{R_2[h]}{\xi} \right) g h - (R_3[g, h])_\xi h + (R_3[g, h])_\xi h^2, \quad (2.20) \]

where

\[ R_1[g](\xi, t) = -\tilde{g}(\xi, t), \quad (2.21) \]

\[ R_2[h](\xi, t) = \xi \tilde{h}(\xi, t), \quad (2.22) \]

and

\[ R_3[g, h](\xi, t) = \xi I_1^+[g, h](\xi, t), \quad (2.23) \]

\[ R_4[g, h] = I_1^+[g, h] + \tilde{g}(\xi, t). \quad (2.24) \]

\( I_1^+ \) and \( I_2^+ \) defined in \(|\xi| > 1\) can be written as
\[ I^+_1 [g, h](\xi, t) = \frac{1}{4\pi i} \int_{|\xi'|=1} \frac{d\xi'}{\xi'} \left[ \frac{\xi + \xi'}{\xi' - \xi} \right] \left[ g(\xi', t)h(\xi', t) + g(\xi', t)\tilde{h}(\xi', t) \right], \quad (2.25) \]

\[ I^+_2 [g](\xi, t) = \frac{1}{4\pi i} \int_{|\xi'|=1} \frac{d\xi'}{\xi'} \left[ \frac{\xi + \xi'}{\xi' - \xi} \right] \left[ g(\xi', t)\tilde{g}(\xi', t) \right]. \quad (2.26) \]

In \(|\xi| < 1\), \((f, g)\) satisfies

\[ h_t = \xi h + I^-_1 [f, g] - \xi h \left( I^-_1 [g, h] \right)_\xi, \quad (2.27) \]

\[ g_t = \xi g + I^-_2 [g](\xi, t) + 1 - h - \xi h \left( I^-_2 [g] \right)_\xi, \quad (2.28) \]

where \(I^-_1\) and \(I^-_2\) defined in \(|\xi| < 1\) can be written as

\[ I^-_1 [g, h](\xi, t) = \frac{1}{4\pi i} \int_{|\xi'|=1} \frac{d\xi'}{\xi'} \left[ \frac{\xi + \xi'}{\xi' - \xi} \right] \left[ g(\xi', t)h(\xi', t) + g(\xi', t)\tilde{h}(\xi', t) \right], \quad (2.29) \]

\[ I^-_2 [g](\xi, t) = \frac{1}{4\pi i} \int_{|\xi'|=1} \frac{d\xi'}{\xi'} \left[ \frac{\xi + \xi'}{\xi' - \xi} \right] \left[ g(\xi', t)g(\xi', t) \right]. \quad (2.30) \]

**Remark 2.1.** Equations (2.19) and (2.20) are analytic continuations of Eqs. (2.27) and (2.28) to \(|\xi| > 1\).

The initial conditions are

\[ h|_{t=0} = h_0(\xi) = \frac{1}{1 + \xi(f_0)_{\xi}(\xi)}, \quad g|_{t=0} = g_0(\xi) = \frac{\xi(W_0)_{\xi}}{1 + \xi(f_0)_{\xi}(\xi)}. \quad (2.31) \]

Therefore, the unsteady classical Rayleigh–Taylor problem is equivalent to finding \((g, h)\) which satisfies (2.27) and (2.28) and initial condition (2.31).

### 2.3. Notations and main result

**Definition 2.2.** Let \(R_s\) be the disk in complex \(\xi\) plane with radius \(s\), i.e. \(R_s = \{ \xi : |\xi| < s \}\); we define function space \(B_s\) so that \(B_s = \{ f(\xi) : f(\xi) \) is analytic in \(R_s\) and continuous on \(\overline{R_s} \}\) with norm \(\| f \|_s = \sup_{R_s} |f(\xi)|\).

**Remark 2.3.** Let \(r_1\) be a number such that \(1 < r_1 < r\). \(B_s\) is a Banach space and \(B_s \subset B_{s'}\) for \(r_1 < s' \leq s \leq r\). Furthermore the norm of the canonical embedding operator \(I_{s\rightarrow s'} \leq 1\).

We define \(B_s\) as

\[ B_s = B_s \times B_s \quad (2.32) \]

with norm \(\|(g, h)\|_s = \|g\|_s + \|h\|_s\). \(B_s\) is a Banach space.
We assume that the initial data \( f_0 \) and \( W_0 \) satisfy \( f_0 \in B_r \), \( W_0 \in B_r \) for some number \( r > 1 \) and
\[
1 + \xi(f_0)(\xi) \neq 0 \quad \text{for all } \xi \in \mathbb{R}_r.
\] (2.33)

Under the above assumption, from (2.31), we have \( h_0 \in B_r \), \( g_0 \in B_r \). Let \( M \) be a positive number defined by
\[
M = \|g_0\|_r + \|h_0\|_r.
\] (2.34)

In this paper, we are going to prove the following theorem:

**Theorem 2.4.** If \( g_0 \in B_r \), \( h_0 \in B_r \), \( M \) is defined as in (2.34), then there exists one and only one solution \( (g, h) \in C^1([0, T), B_s) \), \( 1 < s < r \), \( \|g, h\|_s \leq 2M \) to the problem (2.27), (2.28) and (2.31), where \( T = a_0(r - s) \), \( a_0 \) is a suitable positive constant independent of \( s \).

The proof of above theorem will be based on Nirenberg theorem [11,12]:

**Theorem 2.5 (Nirenberg).** Let \( \{B_s\}_{r_1 \leq s \leq r} \) be a scale of Banach spaces satisfying that \( B_s \subset B_{s'} \), \( \|\cdot\|_{s'} \leq \|\cdot\|_s \) for any \( r_1 < s' < s < r \). Consider the abstract Cauchy–Kowalewski problem
\[
\frac{du}{dt} = \mathcal{L}(u(t), t), \quad u(0) = 0.
\] (2.35)

Assume the following conditions on \( \mathcal{L} \):

(i) For some constants \( M > 0 \), \( \delta > 0 \) and every pair of numbers \( s, s' \) such that \( r_1 < s' < s < r \), \( (u, t) \rightarrow \mathcal{L}(u, t) \) is a continuous mapping of
\[
\left\{ u \in B_s : \|u\|_s < M \right\} \times \left\{ t : |t| < \delta \right\} \text{ into } B_{s'}.
\] (2.36)

(ii) For any \( r_1 \leq s' < s \leq r \) and all \( u, v \in B_s \) with \( \|u\|_s < M \), \( \|v\|_s < M \) and for any \( t, |t| < \delta \), \( \mathcal{L} \) satisfies
\[
\|\mathcal{L}(u, t) - \mathcal{L}(v, t)\|_{s'} \leq C \frac{\|u - v\|_s}{s - s'},
\] (2.37)

where \( C \) is some positive constant independent of \( t, u, v, s, s' \).

(iii) \( \mathcal{L}(0, t) \) is a continuous function of \( t \), \( |t| < \delta \), with values in \( B_s \) for every \( r_1 < s < r \) and satisfies, with some positive constant \( K \),
\[
\|\mathcal{L}(0, t)\|_s \leq \frac{K}{(r - s)}.
\] (2.38)

Under the preceding assumptions there is a positive constant \( a_0 \) such that there exists a unique function \( u(t) \) which, for every \( r_1 < s < r \) and \( |t| < a_0(r - s) \), is a continuously differentiable function of \( t \) with values in \( B_s \), \( \|u\|_s < M \), and satisfies (2.35).
3. Properties of Banach space $B_s$

Let $r_0$ and $r_1$ be two fixed numbers so that $r > r_1 > r_0 > 1$, then $\mathcal{R}_{r_0} \subset \mathcal{R}_{r_1} \subset \mathcal{R}_r$ and $B_r \subset B_{r_1} \subset B_{r_0}$.

In this and the following sections, $C > 0$ represents a generic constant, it may vary from line to line. $C$ may depend on $r_0, r_1$ and $r$; but it is always independent of $s$ and $s'$.

**Definition 3.1.** We define $\bar{f}$ by

$$
\bar{f}(\xi) = \left[ f\left( \frac{1}{\xi^*} \right) \right]^*,
$$

(3.1)

where $^*$ denotes complex conjugate.

**Remark 3.2.** For $s > 0$, if $f \in B_s$, then $\bar{f}$ is analytic in $|\xi| > \frac{1}{s}$ and $|\bar{f}| \leq \|f\|_s$ for $|\xi| > \frac{1}{s}$.

**Lemma 3.3.** If $f \in B_s$, $r_1 < s < r$, then $|\bar{f}(\xi)| \leq K_1 \|f\|_s$, for $r \geq |\xi| \geq \frac{1}{r_0}$, where $K_1$ is a positive constant independent of $s$ and $f$.

**Proof.** Due to Remark 3.2, $\bar{f}$ is analytic in $|\xi| \geq \frac{1}{r_0}$, by Cauchy Integral Formula, we have for $1 \leq |\xi| \leq s$

$$
\bar{f}_\xi(\xi, t) = \frac{1}{2\pi i} \int_{|\xi'| = 2r} \bar{f}(\xi', t) \frac{d\xi'}{(|\xi'| - \xi)^2} - \frac{1}{2\pi i} \int_{|\xi'| = \frac{1}{r_1}} \bar{f}(\xi', t) \frac{d\xi'}{(|\xi'| - \xi)^2}.
$$

For $\frac{1}{r_0} \leq |\xi| \leq r$, $\xi' \in \{|\xi'| = \frac{1}{r_1}\} \cup \{|\xi'| = 2r\}$, we have $|\xi' - \xi| \geq C$, where $C$ depends only on $r_0, r_1$ and $r$; hence each integral in the above equation can be bounded by $C\|f\|_s$. We obtained the lemma. □

The following lemma is essential to application of Nirenberg theorem.

**Lemma 3.4.** If $f \in B_s$, $r_0 \leq s' < s \leq r$, then $f_\xi \in B_{s'}$ and

$$
\|f_\xi\|_{s'} \leq \frac{K_2}{s - s'} \|f\|_s,
$$

(3.2)

where $K_2 > 0$ is independent of $s, s'$ and $f$.

**Proof.** Since dist($\partial B_{s'}, \partial B_s$) = $s - s'$, for $\xi \in B_{s'}$, we are able to find a disk $D(\xi)$ centered at $\xi$ with radius $s - s'$ such that $D(\xi)$ is contained in $\mathcal{R}_s$. Using Cauchy Integral Formula, we have

$$
f_\xi(\xi) = \frac{1}{2\pi i} \int_{|t - \xi| = s - s'} \frac{f(t)}{(t - \xi)^2} dt,
$$

so
\[ |f(\xi)| \leq \frac{1}{2\pi} \int \frac{|f(t)|}{|t - \xi|^2} |dt| \leq \frac{1}{2\pi} \int_0^{2\pi} \frac{|f(\xi + |s - s'|\text{e}^{i\theta})|}{|s - s'|} d\theta \leq K_2 \|f\|_s \frac{s}{s - s'}, \]

which gives the lemma. \( \square \)

**Corollary 3.5.** If \( f \in \mathcal{B}_s \), \( r_1 < s < r \), then \( \|f\|_{r_0} \leq K_3 \|f\|_s \), where \( K_3 \) is a positive constant independent of \( s \) and \( f \).

**Proof.** In Lemma 3.4, letting \( s' = r_0 \), \( K_3 = \frac{K_2}{r_1 - r_0} \), we obtain the corollary. \( \square \)

**Definition 3.6.** We define \( G_1[g, h] \) and \( G_2[g] \) by

\[ G_1[g, h] = [\bar{g}h + g\bar{h}], \quad (3.3) \]

\[ G_2[g](\xi, t) = g(\xi, t)\bar{g}(\xi, t). \quad (3.4) \]

**Remark 3.7.** If \( h \in \mathcal{B}_s \), \( g \in \mathcal{B}_s \), \( s > 1 \), then \( G_1[g, h] \) and \( G_2[g] \) are analytic in \( \frac{1}{s} \leq |\xi| \leq s \).

**Lemma 3.8.** If \( h \in \mathcal{B}_s \), \( g \in \mathcal{B}_s \) and \( r_1 < s < r \), then \( |G_1[g, h]\xi, t)| \leq 2\|h\|_s\|g\|_s \) and \( |G_2[g]\xi, t)| \leq \|g\|_s^2 \) for \( \frac{1}{s} \leq |\xi| \leq s \).

**Proof.** The lemma follows from Eqs. (3.3) and (3.4). \( \square \)

**Lemma 3.9.** If \( h \in \mathcal{B}_s \), \( g \in \mathcal{B}_s \) and \( r_1 < s < r \), then \( |I_1^+[g, h]\xi, t)| \leq C\|h\|_s\|g\|_s \), \( I_2^+[g]\xi, t)| \leq C\|g\|_s^2 \) for \( |\xi| \geq 1 \), where \( C > 0 \) is a constant independent of \( s, g \) and \( h \).

**Proof.** Due to Remark 3.7, the integrands of \( I_1^+ \), \( I_2^+ \) are analytic in \( \frac{1}{r_0} \leq |\xi| \leq 1 \). Changing the contour of integration in the definitions of \( I_1^+ \) and \( I_2^+ \) from \( |\xi| = 1 \) to \( |\xi| = \frac{1}{r_0} \) gives

\[ I_1^+[f, g]\xi, t) = \frac{1}{4\pi i} \int_{|\xi'|=\frac{1}{r_0}} \frac{d\xi'}{\xi'} \left[ \frac{\xi + \xi'}{\xi' - \xi} \right] G_1[f, g]\xi, t), \quad (3.5) \]

\[ I_2^+[f, g]\xi, t) = \frac{1}{4\pi i} \int_{|\xi'|=\frac{1}{r_0}} \frac{d\xi'}{\xi'} \left[ \frac{\xi + \xi'}{\xi' - \xi} \right] G_2[g]\xi, t). \quad (3.6) \]

For \( |\xi| > 1 \) and \( |\xi'| = \frac{1}{r_0} \), from simple geometry, we have

\[ \frac{\left| \frac{\xi + \xi'}{\xi' - \xi} \right|}{C}, \quad (3.7) \]

where \( C \) depends only on \( r_0 \).

The lemma now follows from (3.5)–(3.7) and Lemma 3.8. \( \square \)
Similarly we have

**Lemma 3.10.** If \( h \in B_s, \ g \in B_s \) and \( 1 < s < r \), then 
\[- \langle (I_1^+ \xi)[g, h], (I_1^+ \xi)[g, h] \rangle = C \|h\|_s \|g\|_s, \]
\( (I_2^+ \xi)[g, h], (I_2^+ \xi)[g, h] \leq C \|g\|^2 \) for \( |\xi| > 1 \).

**Proof.** Taking derivative in (3.5) and (3.6) gives
\[
(I_1^+[f, g])_\xi = \frac{1}{4\pi i} \int_{|\xi'|=1/0} d\xi' \left[ \frac{\xi \xi' - (\xi - \xi')^2}{(\xi' - \xi)^2} \right] G_1[f, g](\xi, t), \tag{3.8}
\]
\[
(I_2^+[f, g])_\xi = \frac{1}{4\pi i} \int_{|\xi'|=1/0} d\xi' \left[ \frac{\xi \xi' - (\xi - \xi')^2}{(\xi' - \xi)^2} \right] G_2[g](\xi, t). \tag{3.9}
\]

For \( |\xi| > 1 \) and \( |\xi'| = 1/0 \), from simple geometry, we have
\[
\left| \frac{\xi + \xi'}{\xi' - \xi} \right| \leq C, \tag{3.10}
\]
where \( C \) depends only on \( r_0 \).

The lemma now follows from (3.8)–(3.10) and Lemma 3.8.

**Lemma 3.11.** If \( h \in B_s, \ g \in B_s \) and \( 1 < s < r \), then \( |I_1^- [g, h, (\xi, t)]| \leq C \|h\|_s \|g\|_s, I_2^- [g, h, (\xi, t)] \leq C \|g\|^2 \) for \( |\xi| \leq 1 \).

**Proof.** Due to Remark 3.7, the integrands of \( I_1^- \), \( I_2^- \) are analytic in \( r_0 > |\xi| > 1 \). Changing the contour of integration in the definitions of \( I_1^- \) and \( I_2^- \) from \( |\xi| = 1 \) to \( |\xi| = r_0 \) gives
\[
I_1^- [g, h, (\xi, t)] = \frac{1}{4\pi i} \int_{|\xi'|=r_0} d\xi' \left[ \frac{\xi \xi' - (\xi - \xi')^2}{(\xi' - \xi)^2} \right] G_1[f, g](\xi, t), \tag{3.11}
\]
\[
I_2^- [g, h, (\xi, t)] = \frac{1}{4\pi i} \int_{|\xi'|=r_0} d\xi' \left[ \frac{\xi \xi' - (\xi - \xi')^2}{(\xi' - \xi)^2} \right] G_2[g](\xi, t). \tag{3.12}
\]

For \( |\xi| < 1 \) and \( |\xi'| = r_0 \), from simple geometry, we have
\[
\left| \frac{\xi + \xi'}{\xi' - \xi} \right| \leq C, \tag{3.13}
\]
where \( C \) depends only on \( r_0 \).

The lemma now follows from (3.11)–(3.13) and Lemma 3.8.

**Lemma 3.12.** If \( h \in B_s, \ g \in B_s \) and \( 1 < s < r \), then \( |(I_1^- \xi)[g, h, (\xi, t)]| \leq C \|h\|_s \|g\|_s, \]
\( (I_2^- \xi)[g, h, (\xi, t)] \leq C \|g\|^2 \) for \( |\xi| \leq 1 \).
Proof. Taking derivatives in (3.11) and (3.12), we obtain
\[
(I_1 - 1) \left[ g, h \right]_\xi (\xi, t) = \frac{1}{4\pi i} \int_{|\xi'|=r_0} \frac{d\xi'}{\xi'} \left[ \frac{\xi + \xi'}{(\xi' - \xi)^2} \right] G_1[f, g](\xi, t),
\]
(3.14)
\[
(I_2 - 2) [g]_\xi (\xi, t) = \frac{1}{4\pi i} \int_{|\xi'|=r_0} \frac{d\xi'}{\xi'} \left[ \frac{\xi + \xi'}{(\xi' - \xi)^2} \right] G_2[g](\xi, t).
\]
(3.15)
For $|\xi| < 1$ and $|\xi'| = r_0$, from simple geometry, we have
\[
\left| \xi + \xi' (\xi' - \xi)^2 \right| \leq C,
\]
(3.16)
where $C$ depends only on $r_0$.

The lemma now follows from (3.14)–(3.16) and Lemma 3.8. □

Lemma 3.13. If $g \in B_s$, $g \in B_s$ and $1 < s < r$, then $|R_1[g](\xi, t)| \leq \|g\|_s$, $R_2[h](\xi, t) \leq C\|h\|_s$ and $|R_3[g, h](\xi, t)| \leq C\|h\|_s\|g\|_s$, $|R_4[g, h](\xi, t)| \leq C\|h\|_s\|g\|_s + \|g\|_s$ for $1 \leq |\xi| \leq s$.

Proof. The lemma follows from (2.21)–(2.24), Remark 3.2 and Lemma 3.9. □

Lemma 3.14. If $g \in B_s$, $g \in B_s$ and $r_1 < s < r$, then $|(R_1[g])_\xi (\xi, t)| \leq C\|g\|_s$, $(R_2[h])_\xi (\xi, t) \leq C\|h\|_s$, $(R_3[g, h])_\xi (\xi, t) \leq C\|h\|_s\|g\|_s$ and $(R_4[g, h])_\xi (\xi, t) \leq C\|h\|_s\|g\|_s + \|g\|_s$ for $1 \leq |\xi| \leq s$.

Proof. Applying Lemma 3.3 to $f = g$, we obtain the first inequality. The second one can be proved similarly; the third and the fourth one can be obtained from Lemma 3.10 and (2.23) and (2.24). □

4. Proof of the main theorem

In this section, in order to prove the main Theorem 2.4, we apply Nirenberg’s theorem to initial problem (2.27), (2.28) and (2.31). To this end, we need more estimates of type as in (2.37).

Lemma 4.1. If $u \in B_s$, $v \in B_s$, $r_1 < s' < s < r$, then $\|u_\xi - v_\xi\|_{s'} \leq \frac{C}{s-s'} \|u - v\|_s$, where $C > 0$ is independent of $s$, $s'$ and $u$ and $v$.

Proof. Applying Lemma 3.4 with $f = u - v$, we obtain the lemma. □

Lemma 4.2. If $u \in B_s$, $v \in B_s$, $r_1 < s < r$, then $|u_\xi - v_\xi| \leq C\|u - v\|_s$ for $\frac{1}{r_1} \leq |\xi|$, where $C > 0$ is independent of $s$, $u$ and $v$.

Proof. Applying Lemma 3.3 with $f = u - v$, we obtain the lemma. □
Lemma 4.3. If $u \in B_s$, $v \in B_s$, $\|u\|_s \leq M$, $\|v\|_s \leq M$ and $h \in B_s$, $g \in B_s$, $\|g\|_s \leq M$, $\|h\|_s \leq M$, $r_1 < s < r$, then for $|\xi| > \frac{1}{r_0}$,

$$\left| G_1[g, h](\xi, t) - G_1[u, v](\xi, t) \right| \leq C\left(\|g - u\|_s + \|h - v\|_s\right),$$

$$\left| G_2[g](\xi, t) - G_2[u](\xi, t) \right| \leq C\|g - u\|_s.$$ 

Proof. From (3.3) and (3.4), we have

$$G_1[g, h] = G_1[u, v] = (\tilde{g} - \tilde{u})h + \tilde{u}(h - v) + (g - u)\tilde{h} + u(\tilde{h} - \tilde{v})$$

and

$$G_2[g] - G_2[u] = (g - u)\tilde{g} + u(\tilde{g} - \tilde{u})$$

which gives the lemma by using Remark 3.2, Lemmas 4.1 and 4.2. □

Lemma 4.4. If $u \in B_s$, $v \in B_s$, $\|u\|_s \leq M$, $\|v\|_s \leq M$ and $h \in B_s$, $g \in B_s$, $\|g\|_s \leq M$, $\|h\|_s \leq M$, $r_1 < s < r$, then for $|\xi| \leq 1$,

$$\left| I_1^-[g, h](\xi, t) - I_1^-[u, v](\xi, t) \right| \leq C\left(\|g - u\|_s + \|h - v\|_s\right),$$

$$\left| (I_1^-[g, h])_\xi(\xi, t) - (I_1^-[u, v])_\xi(\xi, t) \right| \leq C\left(\|g - u\|_s + \|h - v\|_s\right),$$

and

$$\left| I_2^-[g](\xi, t) - I_2^-[u](\xi, t) \right| \leq C\|g - u\|_s,$$

$$\left| (I_2^-[g])_\xi(\xi, t) - (I_2^-[u])_\xi(\xi, t) \right| \leq C\|g - u\|_s.$$ 

Proof. From (3.11) and (3.12), we have

$$I_1^-[g, h](\xi, t) - I_1^-[u, v](\xi, t) = \frac{1}{4\pi i} \int_{|\xi'|=r_0} \frac{d\xi'}{\xi'} \left[ \frac{\xi + \xi'}{\xi' - \xi} \right] (G_1[f, g](\xi', t) - G_1[u, v](\xi', t)),$$

$$I_2^-[g](\xi, t) - I_2^-[u](\xi, t) = \frac{1}{4\pi i} \int_{|\xi'|=r_0} \frac{d\xi'}{\xi'} \left[ \frac{\xi + \xi'}{\xi' - \xi} \right] (G_2[g](\xi', t) - G_2[v](\xi', t)).$$

Now the lemma can be proved in the same fashion as Lemmas 3.11 and 3.12 in light of Lemma 4.3. □

Lemma 4.5. If $u \in B_s$, $v \in B_s$, $\|u\|_s \leq M$, $\|v\|_s \leq M$ and $h \in B_s$, $g \in B_s$, $\|g\|_s \leq M$, $\|h\|_s \leq M$, then for $|\xi| \geq 1$,

$$\left| I_1^+[g, h](\xi, t) - I_1^+[u, v](\xi, t) \right| \leq C\left(\|g - u\|_s + \|h - v\|_s\right),$$

$$\left| (I_1^+[g, h])_\xi(\xi, t) - (I_1^+[u, v])_\xi(\xi, t) \right| \leq C\left(\|g - u\|_s + \|h - v\|_s\right),$$

$$\left| I_2^+[g](\xi, t) - I_2^+[u](\xi, t) \right| \leq C\|g - u\|_s,$$

$$\left| (I_2^+[g])_\xi(\xi, t) - (I_2^+[u])_\xi(\xi, t) \right| \leq C\|g - u\|_s.$$
and
\[
\left| (I_2^+[g])_\xi (\xi, t) - (I_2^+[u])_\xi (\xi, t) \right| \leq C \| g - u \|_s.
\]

**Proof.** The lemma can be proved in the same fashion as Lemmas 3.9 and 3.10 in light of Lemma 4.3. \(\Box\)

**Lemma 4.6.** If \(u \in B_s, \ v \in B_s, \|u\|_s \leq M, \|v\|_s \leq M\) and \(g \in B_s, \ g \|_s \leq M, \ h \|_s \leq M, \) then for \(r \geq |\xi| \geq 1,
\[
\left| (R_1[g])_\xi (\xi, t) - (R_1[u])_\xi (\xi, t) \right| \leq C \| g - u \|_s, \\
\left| R_2[h](\xi, t) - R_2[v](\xi, t) \right| \leq C \| h - v \|_s, \\
\left| (R_2[h])_\xi (\xi, t) - (R_2[v])_\xi (\xi, t) \right| \leq C \| h - v \|_s, \\
\left| R_3[g, h](\xi, t) - R_3[u, v](\xi, t) \right| \leq C (\| g - u \|_s + \| h - v \|_s), \\
\left| (R_3[g, h])_\xi (\xi, t) - (R_3[u, v])_\xi (\xi, t) \right| \leq C (\| h - v \|_s + \| g - u \|_s), \\
\left| (R_4[g, h])_\xi (\xi, t) - (R_4[u, v])_\xi (\xi, t) \right| \leq C (\| h - v \|_s + \| g - u \|_s),
\]
where \(C > 0\) is a constant independent of \(s.\)

**Proof.** The proof follows easily from (2.21)–(2.24), Lemmas 4.1, 4.2, 3.13 and 3.14. \(\Box\)

**Definition 4.7.** Let \((g, h) \in B_s.\) We define the following operators: for \(|\xi| < 1, L_1[g, h]\) and \(L_2[g, h]\) are defined by
\[
L_1[g, h](\xi, t) = \xi g\xi I_1^-[g, h](\xi, t) + 1 - h - \xi h(I_2^-[g](\xi, t))_\xi, \quad (4.1) \\
L_2[g, h](\xi, t) = \xi h\xi I_1^-[f, g](\xi, t) - \xi h(I_1^-[g, h](\xi, t))_\xi. \quad (4.2)
\]

Analytic continuations of \(L_1[g, h]\) and \(L_2[g, h]\) to \(|\xi| > 1\) are
\[
L_1[g, h] = (R_3 + R_2 g)g\xi + \xi g h(R_1)\xi - \left( 1 + \xi (I_2^-[g])_\xi \right) h + 1, \quad (4.3) \\
L_2[g, h] = (R_3 + R_2 g)h\xi - R_2 h g\xi + \frac{R_3}{\xi} h - \frac{R_3}{\xi} h^2 - \xi R_4 h^2 \\
- \xi \left( \frac{R_2}{\xi} \right) h - (R_3)\xi h + (R_3)\xi h^2. \quad (4.4)
\]

**Lemma 4.8.** If \(u \in B_s, \ v \in B_s, \|u\|_s \leq M, \ v \|_s \leq M\) and \(g \in B_s, \ g \|_s \leq M, \|h\|_s \leq M, \) then for \(|\xi| \geq 1, r_1 < s' < s < r,
\[
\| L_1[g, h] - L_1[u, v] \|_{s'} \leq \frac{C}{s - s'} (\| g - u \|_s + \| h - v \|_s),
\]
where \(C > 0\) is independent of \(s\) and \(s'.\)
Lemma 4.9. If $|\xi| < 1$,

\[
L_1[g, h](\xi, t) - L_1[u, v](\xi, t) = \xi(g_\xi - u_\xi)I_1^- [g, h] + \xi u_\xi \{ I_1^- [g, h] - I_1^- [u, v] \} + (v - h) - \xi(h - v) (I_2^- [g])_\xi + \xi v \{ (I_2^- [u])_\xi - (I_2^- [g])_\xi \}.
\]

and for $|\xi| > 1$,

\[
L_1[g, h](\xi, t) - L_1[u, v](\xi, t) = (R_3[g, h] - R_3[u, v]) g_\xi + R_3[u, v] (g_\xi - u_\xi) + (R_2[h] - R_2[v]) g h_\xi + R_2[v] g_\xi (g - u) + R_2[v] u (g_\xi - u_\xi) + \xi g h \{ (R_2[g])_\xi - (R_2[u])_\xi \} (g - u) (R - 1[u])_\xi + \xi u (R_1[u])_\xi (h - v) + (v - h) - \xi(h - v) (I_2^- [g])_\xi + \xi v \{ (I_2^- [u])_\xi - (I_2^- [g])_\xi \}.
\]

By Lemmas 3.3, 3.14 and 4.1–4.6, each term in Eqs. (4.5) and (4.6) can be bounded by \( \frac{C}{s-s'}(\|g - u\|_s + \|h - v\|_s) \); hence we obtain the lemma. \( \square \)

Lemma 4.9. If \( u \in B_s, \ v \in B_s, \|u\|_s \leq M, \|v\|_s \leq M \) and \( h \in B_s, \ g \in B_s, \|g\|_s \leq M, \|h\|_s \leq M \), then for \( |\xi| \geq 1, \ r_1 < s' < s < r, \)

\[
\|L_2[g, h](\xi, t) - L_2[u, v](\xi, t)\|_{s'} \leq \frac{C}{s-s'}(\|g - u\|_s + \|h - v\|_s),
\]

where \( C > 0 \) is independent of \( s \) and \( s' \).

Proof. By (4.1) and (4.4), for \( |\xi| < 1, \)

\[
L_2[g, h] - L_2[u, v] = \xi(h_\xi - v_\xi)I_1^- [g, h] + \xi u_\xi \{ I_1^- [g, h] - I_1^- [u, v] \} - \xi(h - v) (I_2^- [g, h])_\xi - \xi v \{ (I_2^- [u])_\xi - (I_2^- [g])_\xi \},
\]

and for \( |\xi| > 1, \)

\[
L_2[g, h](\xi, t) - L_2[u, v](\xi, t) = (R_3[g, h] - R_3[u, v]) h_\xi + R_3[u, v] (h_\xi - v_\xi) + (R_2[h] - R_2[v]) g h_\xi + R_2[v] h_\xi (g - u) + (R_2[h] - R_2[v]) h g_\xi - R_2[h] h (g_\xi - u_\xi) - R_2[h] u_\xi (h - v) + \xi^{-1} (R_3[g, h] - R_3[u, v]) h^2 + \xi^{-1} R_3[u, v] (h - v) (v + h) - \xi \{ (\xi^{-1} R_2[h])_\xi - (\xi^{-1} R_2[v])_\xi \} g h - \xi (\xi^{-1} R_2[v])_\xi g (h - v) + \xi (\xi^{-1} R_2[v])_\xi (g - u) v
\].
\[-\{(R_3[g, h])_\xi - (R[u, v])_\xi\}h - (R_3[u, v])_\xi(h - v)
+ \{(R_3[g, h])_\xi - (R[u, v])_\xi\}h^2 - (R_3[u, v])_\xi(h - v)(h + v).\tag{4.8}\]

By Lemmas 3.3–3.14 and 4.1–4.6, each term in Eqs. (4.7) and (4.8) can be bounded by $C_{s-s'}(\|g - u\|_s + \|h - v\|_s)$; hence we obtain the lemma.

Let $p = g - g_0$, $q = h - h_0$, then $(g, h)$ is a solution of initial problem (2.27), (2.28) and (2.31) if and only if $(p, q)$ solves the following initial problem
\[(p_t, q_t) = \mathcal{L}(p, q),\quad (p, q)|_{t=0} = (0, 0),\tag{4.9}\]

where the operator $\mathcal{L}$ is defined by
\[\mathcal{L}(p, q) = (L_1[p + g_0, q + h_0], L_2[p + g_0, q + h_0]).\tag{4.10}\]

**Lemma 4.10.** If $(p, q) \in B_s$, $(u, v) \in B_s$, $\|(p, q)\|_s \leq M$ and $\|(u, v)\|_s \leq M$, then
\[\|\mathcal{L}(p, q) - \mathcal{L}(u, v)\|_{s'} \leq \frac{C}{s-s'}\|(p, q) - (u, v)\|_s.\]

**Proof.** The proof follows from Lemmas 4.8, 4.9 and (4.10).

**Lemma 4.11.** If $r_1 < s' < r$, then $\|\mathcal{L}(0, 0)\|_{s'} \leq \frac{K}{r-r'}$.

**Proof.** From (4.10), we have
\[\mathcal{L}(0, 0) = (L_1[g_0, h_0], L_2[g_0, h_0]) = (L_1[g_0, h_0] - L_1[0, 0], L_2[g_0, h_0] - L_2[0, 0]).\]

for any $s$ such that $s' < s < r$. Using Lemmas 4.8 and 4.9 with $g = g_0, h = h_0, u = 0, v = 0$, we obtain
\[\|\mathcal{L}(0, 0)\|_{s'} \leq \frac{C}{s-s'}\|(g_0, h_0)\|_s.\]

Letting $s \to r$ in the above equation, we have the lemma.

**Proof of the main theorem.** We first apply Nirenberg theorem to system (4.9). For $(p, q) \in B_s$, by Lemmas 4.8 and 4.9 with $(g, h) = (p, q)$, $(u, v) = (0, 0)$, we have $(L_1[p, q], L_2[p, q]) \in B_{s'}$, hence $\mathcal{L}(p, q) \in B_{s'}$ from (4.10). Since the system (4.9) is autonomous, the continuity of the operator $\mathcal{L}$ is implied by Lemma 4.10; hence (2.36) holds. (2.37) and (2.38) are given by Lemma 4.10 and Lemma 4.11, respectively. Therefore, there exists unique solution $(p, q) \in B_s$, $\|(p, q)\|_s \leq M$, so $g = p + g_0, h = q + h_0$ is the unique solution of Rayleigh–Taylor problem (2.27), (2.28) and (2.31).
5. The unsteady deep water wave problem

We consider a two-dimensional periodic deep water wave. Without loss of generality, the period is taken to be $2\pi$ and acceleration due to gravity $g$, acting in the negative $y$ direction is taken to be unity. We use the same mathematical formulation of the problem as in [18]. The conformal map from the cut unit circle in the $\xi$ plane into the physical domain in $z = x + iy$ plane can be decomposed into

$$z(\xi, t) = 2\pi + i \ln \xi + if(\xi, t),$$

(5.1)

where $f(\xi, t)$ is analytic in $|\xi| < 1$. Under this map, the free interface in the physical plane is mapped onto the unit circle $|\xi| = 1$ in complex $\xi$ plane.

However, unlike the classical Rayleigh–Taylor problem, we also decompose the complex velocity $W(\xi, t)$ into

$$W(\xi, t) = ic \ln \xi + 2\pi c + icw(\xi, t),$$

(5.2)

where the log singularity at $\xi = 0$ is due to the uniformly translating flow at $y = -\infty$ and $c$ is the speed of the uniform flow. $w(xi, t)$ is analytic in $|\xi| < 1$.

The kinematic equation (2.3) is valid and in this case becomes

$$\text{Re} \left[ \frac{f_t}{1 + \xi f_\xi} \right] = \frac{\text{Re} ic\xi w_\xi}{|1 + \xi f_\xi|^2},$$

(5.3)

on $\xi = e^{i\theta}$, for $0 \leq \theta \leq 2\pi$.

The Bernoulli’s equation on the free surface for the unsteady problem is

$$\text{Re} \left[ icw_t - ic \frac{(1 + \xi w_\xi)f_t}{1 + \xi f_\xi} + f \right] = -\frac{1}{2} c^2 \frac{|1 + \xi w_\xi|^2}{|1 + \xi f_\xi|^2},$$

(5.4)

on $\xi = e^{i\theta}$, for $0 \leq \theta \leq 2\pi$.

From this point on, the discussion will be strictly parallel to that of the classical Rayleigh–Taylor problem, except that every occurrence of $W_t$ is to be replaced by $icw_t$ and $\xi W_\xi$ by $ic(1 + \xi w_\xi)$.

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