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AN ELEMENTARY GEOMETRIC PROOF OF TWO THEOREMS OF THOM

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§1. STIEFEL WHITNEY NUMBERS

Notation. All manifolds will be smooth, unoriented; $H_*(-)$, $H^*(-)$ $\mathbb{Z}_2$-homology and cohomology; $[M]$ either bordism class of M or homology fundamental class, depending on the context; τ_M classifying map of the stable tangent bundle of M. $\cong$ isomorphism or diffeomorphism depending on the context. $\amalg$ disjoint union.

THEOREM A (Thom). *If $(\tau_M)_*[M]=0$ then M bounds.*

Proof. The theorem is trivially true for $n=1$. So let $n>1$ and assume the theorem for dimensions $<n$. Let M be a closed n-manifold such that $(\tau_M)_*[M]=0$. Let X be the Grassman manifold of K-planes in $\mathbf{R}^{K+L}$ (K,L large) and X^k the k-skeleton consisting of all Schubert cells of dimension $\leqslant k$. ($X^{-1}=$ empty set.)

Step 1. After a homotopy we may assume $\tau_M: M\to X^n$. After a further homotopy in X^n, we may assume τ_M is smoothly t-regular to the center $\hat{e}$ of each n-cell e of X^n. Now $H_n(X^n)\cong H_n(X)$ by inclusion and $(\tau_M)_*[M]=0$ in $H_n(X^n)$ since it is zero in $H_n(X)$.

Thus $\tau_M{}^{-1}(\hat{e})$ consists of an even number of points for each e. An obvious surgery now produces a bordism between τ_M and f_1: $M_1\to X^{n-1}$.

Step 2. Suppose that $F: Q\to X$ is a bordism between $f: N\to X$ and $\tau_M: M\to X$ with $f(N)\subseteq X^k$ and $0<k<n$. After a homotopy in X^k, we may assume f is t-regular to $\hat{e}$ for each k-cell e of X.

Claim. $\mathrm{V}=\mathrm{f}^{-1}(\hat{\mathrm{e}})$ bounds.

Notice that V is framed in N so the claim implies that f (and hence τ_M) is bordant to f': $N'\to X$ with $f'(N')\subseteq X^{k-1}$, by a surgery which replaces the projection $D^k\times V\to D^k$ by the projection $\partial D^k\times W\to\partial D^k$ (D^k is a small k-disk in e, center $\hat{e}$; $\partial W=V$).

Proof of claim. dim $V=n-k<n$. By Theorem A for dimension $n-k$, it suffices to show that $\langle\tau_V^*w,[V]\rangle=0$ for all $w\in H^{n-k}(X)$. Now observe that

(1) $\tau_V=\tau_N\circ j$, where j: $V\subseteq N$, since V is framed in N.

(2) $j_*[V]=f^*\Delta\cap[N]$, where $\Delta\in H^k(X^k)$ takes the value 1 on e and zero on all other k-cells of X.

(3) $(i_M)_*[M]=(i_N)_*[N]$ (because $\partial Q=M\amalg N$) where $i_M: M\subseteq Q$, $i_N: N\subseteq Q$. Then

$$\langle\tau_V^*w,[V]\rangle\underset{(1)}{=}\langle\tau_N^*w,j_*[V]\rangle\underset{(2)}{=}\langle\tau_N^*w,f^*\Delta\cap[N]\rangle=\langle\tau_N^*w\cup f^*\Delta,[N]\rangle$$

$$=\langle\tau_Q^*w\cup F^*\Delta,(i_N)_*[N]\rangle\underset{(3)}{=}\langle\tau_Q^*w\cup F^*\Delta,(i_M)_*[M]\rangle=\langle\tau_M^*w\cup\tau_M^*\Delta,[M]\rangle$$

$$=\langle w\cup\Delta,(\tau_M)_*[M]\rangle=0.$$

Step 3. By iterating Step 2 we obtain a bordism $F: Q \to X$ between τ_M and $f: N \to X$ where $f(N) = \text{point}$. Since the pair $BO(K)$, $BO(n+1)$ is $(n+1)$-connected $(K \geq n+1)$ we may assume that $F: Q \to BO(n+1)$ and $F|M = \tau_M$, $F(N) = \text{point}$.

Step 4. F pulls back an $(n+1)$-disk bundle ξ which restricts to (the disk bundle of) τ_M over M and to a trivial bundle ϵ_N over N. Let $S(\xi)$, $S(\tau_M)$, $S(\epsilon_N)$ be the associated sphere bundles, and $A(\xi)$, etc. the fibrewise antipodal involution on ξ etc. Consider $U = M \times M \times [-1, 1]$ with involution $T(x, y, t) = (y, x, -t)$. This has fixed point set $F = \{(x, x, 0)|x \in M\}$. We may identify T in a neighbourhood, R, of F in W with the antipodal involution on τ_M. Now remove the interior of R from U and attach $S(\xi)$, by identifying ∂R and $S(\tau_M)$, to obtain a smooth manifold W with $\partial W = \partial U \amalg S(\epsilon_N)$ and involution T, which, on ∂W, restricts to $T(x, y, \pm 1) = (y, x, \mp 1)$ and $A(\epsilon_N)$. The double cover $W \to W/T$ is classified by a map $\varphi: W/T \to \mathbb{P}^q$ ($\mathbb{P}^q$ a projective space, q large). Its restriction to $\partial U/T$ is trivial and its restriction to $S(\epsilon_N)/T$ may be identified with $Id_N \times g$ (where $g: S^n \to \mathbb{P}^n$ is the usual double cover). Now g is classified by the inclusion $\mathbb{P}^n \subseteq \mathbb{P}^q$ so the restriction of φ to $S(\epsilon_N)/T$ can be identified with projection $N \times \mathbb{P}^n \to \mathbb{P}^n \subseteq \mathbb{P}^q$. Let $\mathbb{P}^{q-n}$ be a projective space in $\mathbb{P}^q$ meeting $\mathbb{P}^n$ transversely in one point. After a homotopy of φ we may assume that $\varphi\backslash\partial(W/T)$ is transverse to $\mathbb{P}^{q-n}$ with inverse image N (in particular $\varphi(\partial U/T)$ does not meet $\mathbb{P}^{q-n}$). Now make φ t-regular to $\mathbb{P}^{q-n}$ keeping φ fixed on $\partial(W/T)$. Then $\partial\varphi^{-1}(\mathbb{P}^{q-n}) \cong N$, so N (and hence M) bounds, as required. This completes the proof of Theorem A.

§2. STEENROD REPRESENTATION

THEOREM B (Thom). *Any $\mathbb{Z}_2$-homology class is representable by a smooth manifold.*

Proof. Let X be a finite simplicial complex and X^k its k-skeleton. If $f: N \to X^k$ is smoothly t-regular to the center $\hat{A}$ of each k-simplex A of X, define $C(f) = \sum_A A \otimes [V]$, where the coefficient $[V]$ is the bordism class of $V = f^{-1}(\hat{A})$. If f bounds in X^k then $C(f) = 0$, by relative transversality.

LEMMA 1. *If $C = \sum_B B \otimes [W_B]$ is a simplicial $(k+1)$-chain then there exists a manifold Q and a map $G: Q \to X^{k+1}$ with $\partial C = C(G|\partial Q)$.*

Proof. Let D_B be a $(k+1)$-disk in B, center $\hat{B}$, of small radius. Let $p_B: D_B, \partial D_B \to B, \partial B$ be the obvious radial map. Let $G_B: D_B \times W_B \to X^{k+1}$ be projection followed by p_B and take $G: Q \to X^{k+1}$ to be $\amalg G_B$.

LEMMA 2. *$C(f)$ is a cycle.*

Proof. Let $W_B = f^{-1}(\hat{B})$ for each k-simplex B. After a homotopy, we may identify f near $f^{-1}(\amalg D_B)$ with G (notation as in Lemma 1, with $k+1$ replaced by k), and assume that $G\backslash\partial Q$ bounds in X^{k-1} so that $C(G\backslash\partial Q) = 0$. But $g^{-1}(\hat{A}) \cong \coprod_{B>A} W_B$. Thus $\partial C(f) = C(G|\partial Q) = 0$.

COROLLARY. *If $C(f)$ is homologous to zero then f is bordant in X^{k+1} to g where $g(N) \subseteq X^{k-1}$.*

Suppose now that z is a simplicial $(n+1)$-cycle in X. Lemma 2 may be applied to produce a map $F\colon Q, \partial Q \to X^{n+1}, X^n$ with $C(F|\partial Q) = \partial z = 0$. (Here $W_B = \text{point}$.) To complete the proof of the theorem we need

LEMMA. *If $F\colon Q, \partial Q \to X^{n+1}, X^k$ $(k < n)$ then $C(F|\partial Q)$ is homologous to zero.*

Proof. Write $N = \partial Q$ and $f = F|N$. By Theorem A choose a $\mathbb{Z}_2$-basis $\{[V_\lambda]\}$ for the bordism group of $(n-k)$-manifolds and let $\{w_\lambda\}$ be dual to these in $H^{n-k}(BO)$ (i.e. $\langle \tau^*_{V_\lambda} w_\mu, [V_\lambda]\rangle = \text{Kronecker } \delta_{\lambda\mu}$). We have $C(f) = \sum_A A \otimes [V] = \Sigma_\lambda C_\lambda \otimes [V_\lambda]$ where $C_\lambda = \sum_A \langle \tau^*_V w_\lambda, [V]\rangle A$, a cycle in X^k.

Claim 1. $f_*(\tau^*_N w_\lambda \cap [N]) = \text{the class of } C_\lambda \text{ in } H_k(X)$.

Proof. f factors through a map $\tilde f\colon N \to X^k$. Therefore it suffices to prove $\tilde f_*(\tau^*_N w_\lambda \cap [N]) = \text{class of } C_\lambda$ in $H_k(X^k)$. To do this, define, for any A, $\Delta \in H^k)$ by the simplicial cocycle which takes the value 1 on A and 0 elsewhere. Then, by trans versality of f, $j_*[V] = \tilde f^*\Delta \cap [N]$ where $j\colon V \subseteq N$. Thus

$$\begin{aligned}
\langle \Delta, \tilde f_*(\tau^*_N w_\lambda \cap [N])\rangle &= \langle \tilde f^*\Delta \cup \tau^*_N w_\lambda, [N]\rangle \\
= \langle \tau^*_N w_\lambda, \tilde f^*\Delta \cap [N]\rangle &= \langle \tau^*_N w_\lambda, j_*[V]\rangle \\
&= \langle \tau^*_V w_\lambda, [V]\rangle \text{ as required.}
\end{aligned}$$

Claim 2. $f_*(\tau^*_N w_\lambda \cap [N]) = 0$ in $H_k(X)$.

Proof. Let $\alpha \in H^k(X)$. Then $\langle \alpha, f_*(\tau^*_N w_\lambda \cap [N])\rangle$

$$\begin{aligned}
&= f^*\alpha \cup \tau^*_N w_\lambda, [N]\rangle \\
&= \langle F^*\alpha \cup \tau^*_Q w_\lambda, (i_N)_*[N]\rangle = 0 \text{ since } (i_N)_*[N] = 0
\end{aligned}$$

(where $i_N\colon N \subseteq Q$). Thus $f_*(\tau^*_N w_\lambda \cap [N]) = 0$. This completes the proof of the lemma.

Therefore by Lemma 2, f is bordant, in X^{k+1}, to $f'\colon N' \to X^{k-1}$. Starting from $F\colon Q, \partial Q \to X^{n+1}, X^n$ representing z, as above, we iterate the lemma to obtain $F\colon Q, \partial Q \to X^{n+1}, X^{-1}$, i.e. a closed manifold representing z. This proves Theorem B.

Remark. In the important special case when M is stably parallelizable, the argument of Step 4 in Theorem A gives a geometric proof that M bounds.

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