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ANALYSIS OF FUZZY QUEUES

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Abstract—A general approach for queuing systems in a fuzzy environment is proposed based on Zadeh's extension principle, the possibility concept and fuzzy Markov chains. To illustrate the approach, analytical results for M/F/1 and FM/FM/1 systems are presented. Fuzzy queues are much more realistic than the commonly used crisp queues in many practical situations. A simple numerical example is also presented.

INTRODUCTION

Although Poisson arrival in a queuing system is a fairly reasonable assumption, the service rate is really more possibilistic than probabilistic. Furthermore, in many practical situations, the parameters λ and μ in the M/M/1 or M/D/1 system are frequently fuzzy and cannot be expressed in exact terms. Thus, linguistic expressions for these parameters, such as "the mean arrival rate is approx. 5" and "the mean service rate is approx. 10" are much more realistic under these circumstances.

This paper outlines a general approach for a queuing system in a fuzzy environment. To illustrate the approach, two typical fuzzy queues denoted by M/F/1 and FM/FM/1 are investigated. The former represents a queue with exponential interarrival time and fuzzy service rate while the latter represents a queue with fuzzified exponential arrivals and service rates. Zadeh's extension principle forms the basic approach for this investigation of these fuzzified stochastic processes.

Although only M/F/1 and FM/FM/1 systems are investigated, the general approach can be used and extended easily to other fuzzy queuing systems.

M/F/1 QUEUES

Consider a queuing system with one server and with Poisson arrival. The mean of the arrival rate is λ and the service discipline is first-come-first-serve. Suppose that the service time $\tilde{S}$ is just approximately known and is represented by a possibility distribution $\pi(t) = \mu_{\tilde{S}}(t)$ which restricts its more or less possible values. This possibility distribution on the possible non-fuzzy values of t is induced into all the system measures such that the originally queuing system becomes fuzzified. In a broad sense, a fuzzy queuing system may be considered as a perception of a usual queuing system which will be called the original of the fuzzy queue. It should be emphasized that the location of the original of the fuzzy queuing system is unknown. We only know that it is located in a set Q of the queuing systems. This fuzzy queuing system will be denoted by M/F/1 and its original by M/D/1. The set Q of all possible originals of M/F/1 can be written as $Q = \{(\text{M/D/1}) | t \in \text{SUP}\,\tilde{S}\}$ where t is the service time of the M/D/1 system and $\sup \tilde{S} = \{t \in R^+, \mu_{\tilde{S}}(t) > 0\}$ is the support of $\tilde{S}$. The statement "M/D/1 is an origianl of M/F/1" is fuzzy. By fuzzy logic we know that it has the truth value $\mu_{\tilde{S}}(t)$.

The possibility distributions of the system performance measures of M/F/1 can be obtained using Zadeh's expansion principle [4] from the solutions of the original problem M/D/1 with t known precisely.

Symbolically, we have

$$\mu_{f(\tilde{S})}(x) = \mathrm{SUP}\{\mu_{\tilde{S}}(t) \mid t \in F^{-1}(x)\}$$

where f stands for any entity parametered by $\tilde{S}$. A similar idea to solve fuzzy queuing problems has been hinted by Prade in Ref. [2].

FUZZY MARKOV CHAIN

On the other hand, we can also define an imbedded fuzzy Markov chain for the M/F/1 queuing system. Let $X(\tilde{S})$ denote the imbedded fuzzy stochastic process which can be shown to be fuzzy Markovian by looking at the system immediately after a customer's service is completed and service is about to begin on the next customer in the queue. Let $\tilde{P}_i$ be the probability of i arrivals during service time $\tilde{S}$. Notice that $\tilde{P}_i$ is a fuzzy function such that there exists a possibility distribution, induced by $\mu_{\tilde{S}}$ on each of its points. Since the arrival process is assumed to be Poisson with parameter λ, the fuzzy probability function can be defined by

$$\mu_{\tilde{P}_i}(x) = \sup_{t \in R^+} \left\{ \mu_{\tilde{S}}(t) \mid x = \frac{\exp(-\lambda t)(\lambda t)^i}{i!} \right\}. \tag{1}$$

The one-step transition matrix for a fuzzy Markov chain can now be constructed in a straightforward manner. Note that a transition from state zero to state j or a transition from state i to state j both require the arrival of j customers during the service interval. Moreover, moving from state i to state j where $j \geqslant i - 1 > 0$ requires that $(j - i + 1)$ customers arrived during a service interval, the extra arrival being necessary to account for the customer known to be departing at the transition point. Moving from i to j where $j < i - 1$ is clearly impossible as long as service occurs only one at a time.

Denote the transition probability matrix of the imbedded fuzzy Markov chain by $\tilde{P} = [\tilde{P}_{ij}]$. We can then write for all $j \geqslant i - 1$, $i \geqslant 1$ as

$$\tilde{P} = [\tilde{P}_{ij}] = \begin{bmatrix} \tilde{P}_0, & \tilde{P}_1, & \tilde{P}_2, & \dots \\ \tilde{P}_0, & \tilde{P}_1, & \tilde{P}_2, & \dots \\ 0, & \tilde{P}_0, & \tilde{P}_1, & \dots \\ 0, & 0, & \tilde{P}_0, & \dots \\ \vdots & \vdots & \vdots & \end{bmatrix}$$

where $\tilde{P}_{ij}$ are defined by

$$\forall j \geqslant i - 1, \quad i \geqslant 1; \quad \mu_{\tilde{P}_{ij}}(x_{ij}) = \sup_{t \in R^+} \left\{ \mu_{\tilde{S}}(t) \mid x_{ij} = \frac{\exp(-\lambda t)(\lambda t)^{j-i+1}}{(j - i + 1)!} \right\}. \tag{2}$$

Similarly, a fuzzy Markov chain can be viewed as a perception of a usual Markov chain which is called the original of the fuzzy Markov chain. This original is unknown but located in a set U of Markov chains. Based on queuing theory we can solve the stationary equations for each possible original of the fuzzy Markov chain. The steady-state solutions are [1, 3]

$$\forall t, \lambda \in R^+, \quad t < \frac{1}{\lambda}: \pi_0 = 1 - \lambda t, \quad \pi_1 = (1 - \lambda t)[\exp(\lambda t) - 1],$$

$$\pi_n = (1 - \lambda t) \sum_{k=1}^{n} (-1)^{n-k} \exp(k\lambda t) \left[\frac{(k\lambda t)^{n-k}}{(n-k)!} + \frac{(k\lambda t)^{n-k-1}}{(n-k-1)!} \right] \quad \text{for } n \geqslant 2, \tag{3}$$

and furthermore, the system performance measures can be shown to be

$$L = \frac{\lambda t(2 - \lambda t)}{2(1 - \lambda t)}, \quad W = \frac{L}{\lambda} = \frac{t(2 - \lambda t)}{2(1 - \lambda t)}, \tag{4}$$

where π_n denotes the steady-state probability of n customers in the system at a departure point, L denotes the expected queue length, and W denotes the expected sojourn time. By means of the extension principle the corresponding results in the fuzzy case can be defined by their membership functions for the steady-state solutions:

$$\forall \lambda \in R^+ : \mu_{\tilde{\pi}_0}(x) = \operatorname*{SUP}_{t \in R^+, t < 1/\lambda} \{\mu_{\tilde{S}}(t) \mid x = 1 - \lambda t\},$$

$$\mu_{\tilde{\pi}_1}(x) = \operatorname*{SUP}_{t \in R^+, t < 1/\lambda} \{u_{\tilde{S}}(t) \mid x = (1 - \lambda t)[\mathrm{e}(\lambda t) - 1]\}. \tag{5}$$

$$\forall n \geq 2, \quad \mu_{\tilde{\pi}_n}(x) = \operatorname*{SUP}_{t \in R^+, t 1/\lambda} \left\{ \mu_{\tilde{S}}(t) \mid x = (1 - \lambda t) \sum_{k=1}^{n} (-1)^{n-k} \times \exp(k\lambda t)\left[\frac{(k\lambda t)^{n-k}}{(n-k)!} + \frac{(k\lambda t)^{n-k-1}}{(n-k-1)!}\right]\right\}. \tag{6}$$

The system performance measure are

$$\forall \lambda \in R^+, \quad \mu_{\tilde{L}}(x) = \operatorname*{SUP}_{t \in R^+, t < 1/\lambda} \left\{\mu_{\tilde{S}}(t) \mid x = \frac{\lambda t(2 - \lambda t)}{2(1 - \lambda t)}\right\},$$

$$\mu_{\tilde{W}}(x) = \operatorname*{SUP}_{t \in R^+, t < 1/\lambda} \left\{\mu_{\tilde{S}}(t) \mid x = \frac{t(2 - \lambda t)}{2(1 - \lambda t)}\right\}. \tag{7}$$

Furthermore, let $\tilde{L}_k$ denote the k-th factorial moment of the system size and $\tilde{W}_k$ the regular k-th moment of the system waiting time. Then the important generalization of Little's formula in fuzzy case can be given as

$$\forall \lambda \in R^+, \quad \mu_{\tilde{W}_k}(y) = \operatorname*{SUP}_{x \in R^+} \left\{\mu_{\tilde{L}}(x) \mid y = \frac{x}{\lambda^k}\right\}$$

$$= \operatorname*{SUP}_{\substack{x \in R^+ \\ y = x/\lambda^k}} \operatorname*{SUP}_{\substack{t \in R^+ \\ t < 1/\lambda}} \left\{\mu_{\tilde{S}}(t) \,\middle|\, x = \left.\frac{\mathrm{d}^k\left(\dfrac{(1 - \lambda t)(1 - z)}{1 - z\exp[\lambda t(1 - z)]}\right)}{\mathrm{d}z^k}\right|_{z=1}\right\} \tag{8}$$

(FM/FM/1) FUZZY QUEUES

Consider a one-server queuing system, denoted by FM/FM/1, in which arrivals and departures are both Poisson processes with fuzzy parameters. The density functions for the interarrival times and service times are given, respectively, as

$$\tilde{a}(t) = \tilde{\lambda}\Theta \exp(-\tilde{\lambda}\Theta t), \quad \tilde{b}(t) = \tilde{\mu}\Theta \exp(-\tilde{\mu}\Theta t) \tag{9}$$

where $\tilde{\lambda}$ and $\tilde{\mu}$ are both linguistic. Note that there exists a possibility distribution associated with the two fuzzy parameters, $\tilde{\lambda}$ and $\tilde{\mu}$, in an FM/FM/1 queuing system. The original of a fuzzy queue FM/FM/1 is a usual queue M/M/1, with membership function $\mu_{(\mathrm{FM/FM/1})}(\mathrm{M/M/1}) = \min\{\mu_{\tilde{\lambda}}(\lambda), \mu_{\tilde{\mu}}(\mu)\}$. In general, all the fuzzy functions parametered by $\tilde{\lambda}$ and $\tilde{\mu}$ can be defined by $\mu_{f(\tilde{\lambda},\tilde{\mu})}(z) = \mathrm{SUP}_{x,y \in R} \min\{\mu_{\tilde{\lambda}}(x), \mu_{\tilde{\mu}}(y) \mid z = f(x,y)\}$.

To define the imbedded fuzzy Markov chains in a FM/FM/1 system, we are again concerned with changes in state between successive departure epochs. However, the time between every set of transitions is now a fuzzy random variable whose possibility is the possibility distribution of the service time, a fuzzy function $\tilde{b}(t)$. Since the arrival process is Poisson with fuzzy parameter $\tilde{\lambda}$, we can consider a conditional possibility distribution for i arrivals, given a service time t, denoted by $\mu_{\tilde{P}(A=i|s=t)}$, as

$$\forall t \in R^+; \quad \mu_{\tilde{P}(A=i|s=t)}(y) = \operatorname*{SUP}_{x \in R^+} \left\{\mu_{\tilde{\lambda}}(x) \mid y = \frac{\exp(-xt)(xt)^i}{i!}\right\}. \tag{10}$$

To obtain the marginal possibility distribution for i arrivals in an arbitrary service interval t, denoted by $\mu_{\tilde{P}_i}$, we must weight this function by $\tilde{b}(t)$ and integrate or sum over all t. That is

$$\mu_{\tilde{P}_i}(z) = \operatorname*{SUP}_{\substack{x,y\in R^+\\ x/y<1}} \min\left\{\mu_{\tilde{\lambda}}(x), \mu_{\tilde{\mu}}(y) \,\middle|\, \forall t \in R^+,\right.$$

$$\left. z = \int_t P\{A = n \mid S = t\} b(t)\,\mathrm{d}t = \int_0^\infty \frac{y(xt)^i \exp[-t(x+y)]}{i!}\,\mathrm{d}t \right\}. \qquad (11)$$

Denote the square matrix of the imbedded fuzzy Markov chain in FM/FM/1 system by $\tilde{P} = [\tilde{P}_{ij}]$. Note that $\tilde{P}_{ij} = \tilde{P}\{x_{n+1} = j \mid x_n = i\} = \tilde{P}\{A = j - i + 1\}$. Thus the one-step transition probabilities can be defined as $\forall j \geqslant i - 1, i \geqslant 1, i, j \in I$;

$$\mu_{\tilde{P}_{ij}}(z) = \operatorname*{SUP}_{x,y\in R_+, x/y<1} \min\left\{\mu_{\tilde{\lambda}}(x), \mu_{\tilde{\mu}}(y) \,\middle|\, \forall t \in R^+, z = \int_0^\infty \frac{y(xt)^{j-i+1}\exp[-t(x+y)]}{(j-i-1)!}\,\mathrm{d}t\right\}. \qquad (12)$$

Solving each possible original of the imbedded fuzzy Markov chain, we know the steady-state solutions and system performance measures to be

$$\forall \lambda, \mu \in R^+, \frac{\lambda}{\mu} < 1; \quad \pi_n = \left(\frac{\lambda}{\mu}\right)^n\left(1 - \frac{\lambda}{\mu}\right), \quad L = \frac{\lambda}{\mu - \lambda}, \quad W = \frac{1}{\mu - \lambda}. \qquad (13)$$

The corresponding results in the fuzzy case for FM/FM/1 are given by

$$\forall n \in I: \mu_{\tilde{\pi}_n}(\sigma) = \operatorname*{SUP}_{x,y\in R^+, x/y<1} \min\left\{\mu_{\tilde{\lambda}}(x), \mu_{\tilde{\mu}}(y) \,\middle|\, \sigma = \left(\frac{x}{y}\right)^n\left(1 - \frac{x}{y}\right)\right\},$$

$$\mu_{\tilde{L}}(\sigma) = \operatorname*{SUP}_{x,y\in R^+, x/y<1} \min\left\{\mu_{\tilde{\lambda}}(x), \mu_{\tilde{\mu}}(y) \,\middle|\, \sigma = \frac{1}{y - x}\right\}$$

and

$$\mu_{\tilde{W}}(\sigma) = \operatorname*{SUP}_{x,y\in R^+, x/y<1} \min\left\{\mu_{\tilde{\lambda}}(x), \mu_{\tilde{\mu}}(y) \,\middle|\, \sigma = \frac{1}{y - x}\right\}.$$

Furthermore, Little's formula in higher moments can be obtained from $\forall k \in I$:

$$\mu_{\tilde{W}_k}(\sigma) = \operatorname*{SUP}_{t\in R^+}\left\{\mu_{\tilde{L}}(\tau) \,\middle|\, \sigma = \frac{\tau}{\lambda^k}\right\} = \operatorname*{SUP}_{\substack{t\in R^+\\ \sigma = \tau/\lambda^k}} \operatorname*{SUP}_{\substack{x,y\in R^+\\ x/y<1}} \min\left\{\mu_{\tilde{\lambda}}(x), \mu_{\tilde{\mu}}(y) \,\middle|\, \tau = \left.\frac{\mathrm{d}^k\left(\dfrac{y-k}{y-zx}\right)}{\mathrm{d}z^k}\right|_{z=1}\right\}.$$

A NUMERICAL EXAMPLE

A special tune-up station has been established at thhe end of an automotive assembly line to make adjustment on those vehicles which can not meet federal exhaust gas emission standards. Failures appear to be completely random and hence justify the Poisson arrival assumption with $\lambda = 0.1$ vehicle per minute. Each arrival is serviced by an adjustment requiring approx. 5 min which can be expressed by a trapezoidal fuzzy number with membership function:

$$\begin{aligned}
\forall t \in R^+: \quad & \mu_{\tilde{S}}(t) = 0, \quad t \leqslant 2,\\
& \mu_{\tilde{S}}(t) = 1/2t - 1, \quad 2 \leqslant t \leqslant 4,\\
& \mu_{\tilde{S}}(t) = 1, \quad 4 \leqslant t \leqslant 6,\\
& \mu_{\tilde{S}}(t) = -1/2t + 4, \quad 6 \leqslant t \leqslant 8,\\
& \mu_{\tilde{S}}(t) = 0, \quad t \geqslant 8.
\end{aligned}$$

In attempting to evaluate storage space requirements, management needs to know (a) the mean number of vehicles in the station, (b) the expected sojourn time per vehicle and (c) the probability that there will be more than two vehicles present.

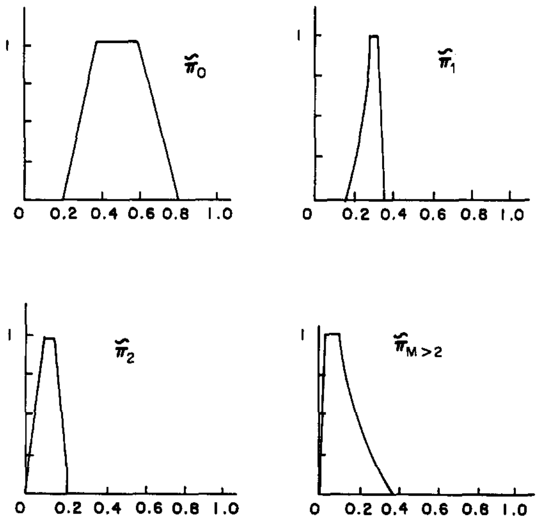

Fig. 1. Stationary fuzzy probabilities.

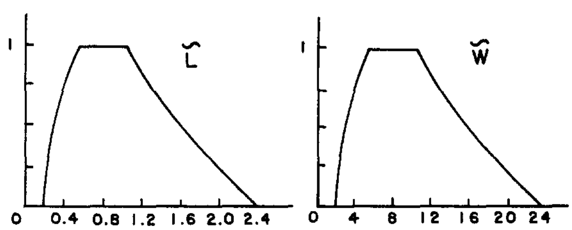

Fig. 2. Fuzzy measures of system performance.

Figures 1 and 2 show the solutions of the problem, in which Fig. 1 gives the stationary fuzzy probabilities and Fig. 2 gives the fuzzy measures of the system performances. It is noted that the membership functions of the results are no longer trapezoidal fuzzy numbers except for $\tilde{\pi}_0$.

DISCUSSION

Although only two simple queuing systems are investigated, the approaches presented in this paper can be extended easily to other more complicated fuzzy queues. For example, systems like M/F/C, M/F/C/k etc., can be easily treated by essentially the same basic approach. The important idea is to view a fuzzy queue as a perception of a usual crisp queue. The corresponding fuzzy measures of the system performances can then be defined by the membership functions which are obtained from the solutions of each of the possible original queues in terms of the extension principle.

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