SEMI-DISCRETE FOURIER SPECTRAL APPROXIMATIONS
OF INFINITE DIMENSIONAL HAMILTONIAN SYSTEMS AND
CONSERVATION LAWS

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Abstract—In this paper, we discuss the semi-discrete Fourier spectral approximation for infinite dimensional Hamiltonian systems and pay more attention to the Hamiltonian structure. In this manner we consider further its preservation of conservation laws of the original Hamiltonian systems.

INTRODUCTION

Symplectic numerical method for finite dimensional Hamiltonian systems has been widely developed by Feng's research group [1-5]. There are also some works on symplectic numerical method for infinite dimensional Hamiltonian systems [6-8]. Reference [6] is a finite difference method. It discretizes simultaneously the time and space variables such that the transition from one time step to the next is a symplectic mapping. Reference [7] is a semi-discretization in time direction such that in the reduced system the transition from the one time step to the next is a symplectic mapping in the sense of the infinite dimensional symplectic manifold. Reference [8] emphasizes the consistency of the model in semi-discretization with the original Hamiltonian system. It chooses the translation invariance of Hamiltonian systems as the starting point. In the Hamiltonian systems which consist of a single equation with one space variable, the translation invariance implies many important properties, such as the total mass and energy conservation laws. But in the Hamiltonian system with more than one equation, the translation invariance has no such properties. In fact, for Hamiltonian systems, the most important is its Hamiltonian structure. In this paper we consider the semi-discrete Fourier spectral approximation of infinite dimensional Hamiltonian systems and discuss the preservation of the conservation laws of the original Hamiltonian system. For the Hamiltonian system whose Hamiltonian operator is a constant differential operator, the Fourier semi-discrete system is also a Hamiltonian system. In this case, the semi-discrete system preserves all linear and quadratic conservation laws of the original infinite dimensional Hamiltonian system and the Hamiltonian function, i.e., it preserves the total mass, momenta and quadratic energy and the Hamiltonian energy. Moreover, when the original Hamiltonian system is linear, the semi-discrete system preserves all conservation laws of the original system. But if the original Hamiltonian operator is non-linear, the reduced operator, generally speaking, is not Hamiltonian. However, the corresponding semi-discrete system still has some conservation laws.

INFINITE DIMENSIONAL HAMILTONIAN SYSTEMS

In this section, we shall review briefly infinite dimensional Hamiltonian systems (see [10]) and list some examples. Firstly, we introduce some notation.

$\mathcal{C}^\infty(\mathbb{R}^k)$: the set of all infinitely differentiable functions on $\mathbb{R}^k$;
$\mathcal{C}_p^\infty(\mathbb{R}^k)$: the set of all periodic functions in $\mathcal{C}^\infty(\mathbb{R}^k)$ with period 2$\pi$;

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\[ U = \bigotimes_{i=1}^{l} C_0^\infty(\mathbb{R}^k) \] is the direct product of \( C_0^\infty(\mathbb{R}^k) \) \( l \) times; 

\[ u = (u_1, \ldots, u_l)^T \in U: \text{ an element of } U; \]

\[ u_i^{|i|} \text{ the } |i|\text{-th derivative of } u_i, \quad i = (i_1, \ldots, i_k), \quad |i| = i_1 + \cdots + i_k; \]

\[ u_i^{(n)} = (u_i^T, u_i^{nT})^T; \]

\[ H[u] \equiv H(x, u^{(n)}) \quad \text{ when } (x, u^{(n)}) \text{ is viewed as independent variables, } H \text{ is infinitely differentiable}; \]

\[ A \equiv \{ H[u] \equiv H(x, u^{(n)}) \mid H \text{ being periodic in } x \text{ with period } 2\pi \}; \]

\[ A' = \bigotimes_{i=1}^{l} A. \]

Evidently, \( C_0^\infty(\mathbb{R}^k) \) is a linear space and \( C_0^\infty(\mathbb{R}^k) \) is a subspace of it. To every \( H[u] \in A \) there corresponds a functional \( \mathcal{H} = \int H[u] \, dx \). The set of all functionals is denoted by \( \mathcal{F} \) (\( \mathcal{F} = \{ \mathcal{H} = \int H[u] \, dx \mid H[u] \in A \} \)). \( \delta \mathcal{H} = (\delta_{u_1} \mathcal{H}, \ldots, \delta_{u_l} \mathcal{H})^T \) is the variational derivative of the functional \( \mathcal{H} \) with respect to \( u = (u_1, \ldots, u_l)^T \). Obviously, \( \delta \mathcal{H} \in A' \). Let \( D : A' \rightarrow A' \) be a linear operator. It is an \( l \times l \) matrix differential operator. With the aid of the differential operator \( D \), we can define a binary operator on \( \mathcal{F} \):

\[ \{ \mathcal{H}, \mathcal{G} \} = \int \delta \mathcal{H}^T D \delta \mathcal{G} \, dz, \quad \forall \mathcal{H}, \mathcal{G} \in \mathcal{F}. \]  

(1)

If it is anti-symmetric, i.e.,

\[ \{ \mathcal{H}, \mathcal{G} \} = - \{ \mathcal{G}, \mathcal{H} \} \]

(2)

and satisfies the Jacobi identity

\[ \{ \{ \mathcal{H}, \mathcal{G} \}, \mathcal{K} \} + \{ \{ \mathcal{G}, \mathcal{K} \}, \mathcal{H} \} + \{ \{ \mathcal{K}, \mathcal{H} \}, \mathcal{G} \} = 0, \quad \forall \mathcal{H}, \mathcal{G}, \mathcal{K} \in \mathcal{F}, \]

(3)

then it is called a Poisson bracket. In this case, the differential operator \( D \) is called a Hamiltonian operator. Condition (2) holds if and only if \( D \) is skew-adjoint: \( D^* = -D \). Verification of the Jacobi identity is very complicated. There have been many methods to do it, such as I. Gelfand and I. Dorfman's algebraic method [9], P. J. Olver's functional multi-vector method [10], etc. For very simple case, we have

**Theorem 1** [10]. *Let \( D \) be a skew-adjoint \( l \times l \) matrix differential operator. If its elements do not depend on \( u \) and the derivatives of \( u \), then it is a Hamiltonian operator.*

*In particular, if \( D \) is skew-adjoint and the coefficients of its elements are all constant, then it is a Hamiltonian operator.*

Let \( H[u] \in \mathcal{F} \) be a given functional and \( D \) a Hamiltonian operator. Then the evolution equation

\[ u_t = D \delta H[u] \]

(4)

is called an infinite dimensional Hamiltonian system, \( H[u] \) is called a Hamiltonian functional, \( \delta H[u] = (\delta_{u_1} H[u], \ldots, \delta_{u_l} H[u])^T \) is the variational derivative of \( H[u] \).

We now give some examples.

1. The simplest infinite dimensional Hamiltonian system is the first order wave equation

\[ \frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}. \]  

(5)

The Hamiltonian operator is \( D = \partial_x \), the Hamiltonian functional is \( H = \frac{1}{2} \int u^2 \, dx \).

2. The 1-dim wave equation

\[ \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}. \]  

(6)
It can be rewritten as two forms of the first order equations

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \frac{\partial v}{\partial x} \\
\frac{\partial v}{\partial t} &= \frac{\partial u}{\partial x} \\
\frac{\partial u}{\partial x} &= v \\
\frac{\partial v}{\partial x} &= \frac{\partial^2 u}{\partial x^2}
\end{align*}
\]

They correspond two Hamiltonian systems

\[
\frac{\partial \tilde{u}}{\partial t} = D_1 \delta H_1, \quad \frac{\partial \tilde{u}}{\partial t} = D_2 \delta H_2,
\]

where \(\tilde{u} = (u, v)^T\), \(D_1 = \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix}\) and \(D_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\) are the Hamiltonian operators and \(H_1 = \frac{1}{2} \int (u^2 + v^2) \, dx\) and \(H_2 = \frac{1}{2} \int (v^2 + u_x^2) \, dx\) are the Hamiltonian functionals respectively.

3. The 3-dim wave equation

\[
\frac{\partial^2 p}{\partial t^2} = \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2}.
\]

It can be rewritten as

\[
\begin{align*}
\frac{\partial u_1}{\partial t} &= \frac{\partial u_4}{\partial x} \\
\frac{\partial u_2}{\partial t} &= \frac{\partial u_4}{\partial y} \\
\frac{\partial u_3}{\partial t} &= \frac{\partial u_4}{\partial z} \\
\frac{\partial u_4}{\partial t} &= \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z},
\end{align*}
\]

where \(u_4 = P\). Its matrix form is

\[
u_t = D \cdot u,
\]

where \(u = (u_1, u_2, u_3, u_4)^T\),

\[
D = \begin{pmatrix}
0 & 0 & 0 & \frac{\partial}{\partial x} \\
0 & 0 & 0 & \frac{\partial}{\partial y} \\
0 & 0 & 0 & \frac{\partial}{\partial z} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} & 0
\end{pmatrix}
\]

Obviously, \(D\) is a Hamiltonian operator. The corresponding Hamiltonian functional is

\[
H = \frac{1}{2} \int (u_1^2 + u_2^2 + u_3^2 + u_4^2) \, dx \, dy \, dz.
\]

4. Generally speaking, for the first order linear differential equation

\[
\frac{\partial u}{\partial t} = A_1 \frac{\partial u}{\partial x_1} + \cdots + A_k \frac{\partial u}{\partial x_k},
\]

if \(A_i, i = 1, \cdots, k\), all are \(l \times l\) constant symmetric matrices, then (13) is a Hamiltonian system with the Hamiltonian operator

\[
D = A_1 \frac{\partial}{\partial x_1} + \cdots + A_k \frac{\partial}{\partial x_k}
\]
and the Hamiltonian functional
\[ \mathcal{H}[u] = \frac{1}{2} \int (u_x^2 + \cdots + u_n^2) \, dx. \]

5. KdV equation
\[ u_t = uu_x + u_{xxx}. \]

It has two Hamiltonian structures
\[ u_t = D\delta \mathcal{H}_1[u] = E\delta \mathcal{H}_0[u] \]
where
\[ D = \frac{\partial}{\partial u}, \quad \mathcal{H}_1 = \int \left( \frac{1}{2} u^3 - \frac{1}{2} u_x^2 \right) \, dx, \]
\[ E = \frac{1}{3} \left( u \frac{\partial}{\partial u} + \frac{\partial}{\partial x} u \right) + \frac{\partial^3}{\partial x^3}, \quad \mathcal{H}_0 = \frac{1}{2} \int u^2 \, dx. \]

Evidently, \( D \) is a Hamiltonian operator. \( E \) is also a Hamiltonian operator, see [9] [10]. The lower order conservation laws of KdV equation are \( \int u \, dx, \mathcal{H}_0 \) and \( \mathcal{H}_1 \).

6. Long water wave equation
\[ u_t = \frac{1}{2} \partial \left( u^2 + 2h - u_x \right), \quad \partial = \frac{\partial}{\partial x}. \]
\[ h_t = \frac{1}{2} \partial \left( 2uh + h_x \right), \]

Kuperschmidt has proved in [11] that (20) has three Hamiltonian structures.
\[ \tilde{u}_t = D_1 \delta \mathcal{H}_1 = D_2 \delta \mathcal{H}_2 = D_3 \delta \mathcal{H}_3, \]
where \( \tilde{u} = (u, h)^T \),
\[ D_1 = \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix}, \]
\[ D_2 = \begin{pmatrix} 2\partial & \partial u - \partial^2 \\ u\partial + \partial^2 & h\partial + \partial h \end{pmatrix}, \]
\[ D_3 = \begin{pmatrix} 2(u\partial + \partial u) & 2(h\partial + \partial h) + \partial(u - \partial)^2 \\ 2(h\partial + \partial h) + (u + \partial)^2\partial & (u + \partial)(h\partial + \partial h) + (h\partial + \partial h)(u - \partial) \end{pmatrix}. \]

The corresponding Hamiltonian functionals are
\[ \mathcal{H}_1 = \frac{1}{2} \int \left( u_x^2 + h^2 - u_xh \right) \, dx, \]
\[ \mathcal{H}_2 = \frac{1}{2} \int uh \, dx, \]
\[ \mathcal{H}_3 = \frac{1}{2} \int h \, dx. \]

Naturally, \( \mathcal{H}_1, \mathcal{H}_2 \) and \( \mathcal{H}_3 \) all are the conservation laws of the long water wave equation (20). Another first order conservation law is \( \int u \, dx \). Moreover, (20) has a series of conservation laws of high order.

We have seen in the examples 5 and 6 that an evolution equation may have more than one Hamiltonian structure. If there is a coupled relation between two different Hamiltonian operators, i.e., if \( D \) and \( E \) are Hamiltonian operators, so is their linear combination \( aD + bE, \forall a, b \in \mathbb{R} \), then they are called a Hamiltonian pair. In fact, two Hamiltonian operators \( D \) and \( E \) form a
Hamiltonian pair if and only if $D + E$ is also a Hamiltonian operator. If an evolution equation has two different Hamiltonian structures which form a Hamiltonian pair, then it is called a bi-
Hamiltonian system. It is well known that KdV equation is a bi-Hamiltonian system. For long
water wave equation, Kuperschmidt has proved that any two of three Hamiltonian operators (21),
(22) and (23) form a Hamiltonian pair [11]. For a bi-Hamiltonian system there exists a series
of conservation laws $\mathcal{H}_0, \mathcal{H}_1, \cdots$. They are in involution mutually, i.e.,
\begin{equation}
\{\mathcal{H}_m, \mathcal{H}_n\} = 0, \quad \forall m, n.
\end{equation}

**SEMI-DISCRETE FOURIER SPECTRAL APPROXIMATION**

Case I. We first consider the case of a single equation with one space variable, i.e., $k = l = 1$.
In this case, $U = C^\infty_p(\mathbb{R})$, i.e., $U$ is the set of all infinitely differentiable periodic functions on $\mathbb{R}$
with period $2\pi$. The inner product on $U$ is defined by
\begin{equation}
(u, v) = \int_0^{2\pi} u \cdot v \, dz, \quad \forall u, v \in U.
\end{equation}
Take the orthogonal basis
\begin{equation}
\frac{1}{\sqrt{2}} \cos z, \sin z, \cdots, \cos nz, \sin nz, \cdots.
\end{equation}
They are not normal. But the module is a constant $\sqrt{\pi}$. Set
\begin{equation}
B = \text{span} \left\{ \frac{1}{\sqrt{2}}, \cos z, \sin z, \cdots, \cos Nz, \sin Nz \right\} \subset U.
\end{equation}
It is a $2N + 1$ dimensional linear subspace of $U$. Set $P : U \to B$,
\begin{equation}
u \to \tilde{u} = Pu = a_0/\sqrt{2} + \sum_{n=1}^N (a_n \cos nz + b_n \sin nz),
\end{equation}
where
\begin{align}
a_0 &= \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} u(z) \, dz, \\
n_n &= \frac{1}{\pi} \int_0^{2\pi} u(z) \cos nz \, dz, \quad n = 1, \cdots, \\
b_n &= \frac{1}{\pi} \int_0^{2\pi} u(z) \sin nz \, dz, \quad n = 1, \cdots,
\end{align}
are the Fourier coefficients of $u$. $P$ is an orthogonal projection from $U$ to $B$. Denote $(\cdot, \cdot) \equiv ((\cdot, \cdot))_B$ being the reduced inner product in $B$ of the inner product $(\cdot, \cdot)$ in $U$, i.e.,
\begin{equation}
((\tilde{u}, \tilde{v})) = ((\tilde{u}, \tilde{v}))_B = (\tilde{u}, \tilde{v}), \quad \forall \tilde{u}, \tilde{v} \in B.
\end{equation}
Denote $\tilde{B} = \mathbb{R}^{2N+1}$ \{ $\tilde{u} = (a_0, a_1, b_1, \cdots, a_N, b_N)^T \in \mathbb{R}^{2N+1}$ \}. The inner product of $\tilde{B}$, denoted by $(\cdot, \cdot)$, is the usual Euclidean inner product, i.e.,
\begin{equation}
\langle \tilde{u}, \tilde{v} \rangle = a_0 \tilde{a}_0 + \sum_{n=1}^N (a_n \tilde{a}_n + b_n \tilde{b}_n), \quad \forall \tilde{u}, \tilde{v} \in \tilde{B},
\end{equation}
where $\tilde{u} = (\tilde{a}_0, \tilde{a}_1, \tilde{b}_1, \cdots, \tilde{a}_N, \tilde{b}_N)^T, I : \tilde{B} \to B$,
\begin{equation}
u \to \tilde{u} = I\tilde{u} = a_0/\sqrt{2} + \sum_{n=1}^N (a_n \cos nz + b_n \sin nz),
\end{equation}
is an isomorphic mapping. Denote $\hat{P} = I^{-1} \circ P : U \to \hat{B}$. Then $P$, $\hat{P}$ and $I$ have the following properties:

\begin{align}
((\hat{u}, \hat{v})) &= (P\hat{u}, P\hat{v}) = (P\hat{u}, \hat{v}) = (u, P\hat{v}), \quad \forall u, v \in U, \quad (35) \\
(I\hat{u}, I\hat{v}) &= ((I\hat{u}, I\hat{v})) = \pi(\hat{u}, \hat{v}), \quad \forall \hat{u}, \hat{v} \in \hat{B}, \quad (36) \\
(Pu, Pv) &= (P(Pu, P\hat{v})) = \pi(Pu, P\hat{v}), \quad \forall u, v \in U, \quad (37) \\
P \circ I = I, \quad \hat{P} \circ I = \text{id}_{\hat{B}}, \quad I \circ \hat{P} u = Pu \iff u \in B. \quad (38)
\end{align}

Equations (35) and (38) are obvious. Equation (37) follows from (36). Equation (36) is because

\begin{equation}
((I\hat{u}, I\hat{v})) = (\hat{u}, \hat{v}) = \int_0^{2\pi} \left( \frac{a_0}{\sqrt{2}} + \sum_{n=1}^N (a_n \cos nx + b_n \sin nx) \right) \left( \frac{\hat{a}_0}{\sqrt{2}} + \sum_{n=1}^N (\hat{a}_n \cos nx + \hat{b}_n \sin nx) \right) dx = \pi(a_0\hat{a}_0 + \sum_{n=1}^N (a_n\hat{a}_n + b_n\hat{b}_n)) = \pi(\hat{u}, \hat{v}).
\end{equation}

From (36) we know that the isomorphic mapping $I$ does not preserve the inner product, but only different from a constant factor. If the orthogonal basis in $U$ is also normal, then $I$ preserves the inner product.

The discretization of the Hamiltonian operator $\mathcal{D}$ is

\begin{equation}
D = \hat{P} \circ \mathcal{D} \circ I : \hat{B} \to \hat{B}, \quad (D(\hat{u})\hat{v}) = \hat{P} \circ \mathcal{D}(I\hat{u}) \cdot I\hat{v}, \quad \forall \hat{u}, \hat{v} \in \hat{B}. \quad (39)
\end{equation}

**Lemma 2.** Let $\mathcal{D}$ be skew-adjoint. Then $D$ defined above is skew-symmetric.

**Proof.** $\forall \hat{u}_1, \hat{v}_1, \hat{v}_2 \in T_{\hat{u}}\hat{B},$

\begin{align}
(D(\hat{u})\hat{v}_1, \hat{v}_2) &= (\hat{P} \circ \mathcal{D}(I\hat{u})I\hat{v}_1, \hat{v}_2) \\
&= \pi^{-1}(I \circ \hat{P} \circ \mathcal{D}(I\hat{u})I\hat{v}_1, I\hat{v}_2) \quad \text{(by (36))} \\
&= \pi^{-1}(P \circ \mathcal{D}(I\hat{u})\hat{v}_1, I\hat{v}_2) \quad \text{(by (38))} \\
&= \pi^{-1}(\mathcal{D}(I\hat{u})\hat{v}_1, I\hat{v}_2) \quad (I\hat{v}_2 \in B) \\
&= -\pi^{-1}(I\hat{v}_1, \mathcal{D}(I\hat{u})\hat{v}_2) \quad (I = P \circ I \text{ and the symmetry of } P) \\
&= -(\hat{v}_1, \hat{P} \circ \mathcal{D}(I\hat{u})I\hat{v}_2) \\
&= -(\hat{v}_1, D(\hat{u})\hat{v}_2). \quad \square
\end{align}

The discretization of a functional in $U$ is

\begin{equation}
H = \hat{\mathcal{H}} = \frac{1}{\pi} \mathcal{H} \circ I, \quad (40)
\end{equation}

i.e.,

\begin{equation}
H(\hat{u}) = \frac{1}{\pi} \mathcal{H}(I\hat{u}), \quad \forall \hat{u} \in \hat{B}. \quad (40')
\end{equation}

We now consider the discretization of variational derivatives. For every $\hat{u} \in \hat{B}$, $I\hat{u} \in B$. Hence, $\mathcal{H}(I\hat{u}) = \mathcal{H}(\hat{u})(I\hat{u} = \hat{u})$ can be viewed as a functional of $\hat{u}$. By (40'), $H(\hat{u}) = \frac{1}{\pi} \mathcal{H}(I\hat{u})$ is also a function of $\hat{u}$. Then there is a relation between the variational derivative as a functional and the gradient as a function.
Lemma 3.
\[ \nabla_\delta H(\hat{u}) = \hat{P} \delta H(I\hat{u}), \quad \forall \hat{u} \in \hat{B}. \]  

Proof. For any \( \hat{u} \in \hat{B}, \)
\[
\nabla_\delta H(\hat{u}) \cdot \hat{v} = \langle \nabla_\delta H(\hat{u}), \hat{v} \rangle = \frac{d}{dx} \bigg|_{x=0} H(\hat{u} + \epsilon \hat{v})
\]
\[
= \frac{1}{\pi} \frac{d}{dx} \bigg|_{x=0} \mathcal{H}(I\hat{u} + \epsilon I\hat{v})
\]
\[
= \frac{1}{\pi} \langle (\delta \mathcal{H}(I\hat{u}), I\hat{v}) \rangle \quad \text{(by (38) and the symmetry of } P) \]
\[
= \langle \hat{P} \delta \mathcal{H}(I\hat{u}), \hat{v} \rangle. \quad \square
\]

Now we define the semi-discrete approximative equation in \( \hat{B} \) of the infinite dimensional Hamiltonian system (4) as
\[ \frac{d\hat{u}}{dt} = D \nabla_\delta H(\hat{u}). \]  

If \( D \) is still a Hamiltonian operator, then (42) is exactly a finite dimensional Hamiltonian system.

As an example, we consider the first order wave equation again
\[
\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} = D \mathcal{H}[u],
\]
where \( D = \frac{\partial}{\partial x}, \mathcal{H}[u] = \frac{1}{2} \int_{-\pi}^{\pi} u^2 dx. \) By above definitions,
\[
D = \text{diag} \left( 0, \left[ \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right], \cdots, \left[ \begin{array}{cc} 0 & N \\ -N & 0 \end{array} \right] \right),
\]
\[ H(\hat{u}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( a_0/\sqrt{2} + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx) \right)^2 \ dx
\]
\[
= \frac{1}{2} \left( a_0^2 + \sum_{n=1}^{N} (a_n^2 + b_n^2) \right).
\]

Then
\[ \nabla_\delta H(\hat{u}) = (a_0, a_1, b_1, \cdots, a_N, b_N)^T. \]

The corresponding semi-discrete approximation is
\[ \frac{da_0}{dt} = 0, \]
\[ \frac{d a_n}{dt} = n b_n, \quad n = 1, 2, \cdots, N. \]  

If we expand \( u \) as a Fourier series and substitute it into the first order wave equation, we get
\[ \frac{da_0}{dt} = 0, \]
\[ \frac{d a_n}{dt} = n b_n, \quad n = 1, 2, \cdots. \]
Hence, (43) is the $N$-th order truncation of (44). If $u_0 \in B$, then the approximate solution of (43) is also the exact solution of the first order wave equation.

By the operator $D$ we can define a bracket

\[ \{ \{ H, G \} \} = (\nabla_u H)^T D \nabla_u G. \] (45)

Since by Lemma 2, $D$ is anti-symmetric, $D$ is a Hamiltonian operator if and only if (45) satisfies Jacobi identity. If $H$ and $G$ are the discretization of the functionals $\mathcal{H}$ and $\mathcal{G}$ respectively, then by Lemma 3, (35), (37) and (38), we have

\[
\{ \{ H(\bar{u}), G(\bar{u}) \} \} = (\nabla_u H(\bar{u}))^T D \nabla_u G(\bar{u})
= (\hat{P} \delta H(I\bar{u}))^T \hat{P} \circ D(I\bar{u}) \cdot I(\hat{P} \delta G(I\bar{u}))
= (\hat{P} \delta H(I\bar{u}), \hat{P} \circ D(I\bar{u}) \cdot \hat{P} \delta G(I\bar{u}))
= \frac{1}{\pi} (\delta H(I\bar{u}), \hat{P} \circ D(I\bar{u}) \cdot \hat{P} \delta G(I\bar{u}))
= \frac{1}{\pi} (P \delta H(I\bar{u}), D(I\bar{u}) \cdot \delta G(I\bar{u})).
\] (46)

On the other hand, $\{ \mathcal{H}, \mathcal{G} \}$ is also a functional on $U$. So it has its own discretization.

\[
\{ \hat{\mathcal{H}}, \hat{\mathcal{G}} \}(\bar{u}) = \frac{1}{\pi} \{ \mathcal{H}, \mathcal{G} \} \circ I(\bar{u})
= \frac{1}{\pi} \int_0^{2\pi} \delta \mathcal{H}(I\bar{u}) D(I\bar{u}) \delta \mathcal{G}(I\bar{u}) \, dx
= \frac{1}{\pi} (\delta \mathcal{H}(I\bar{u}), D(I\bar{u}) \delta \mathcal{G}(I\bar{u})).
\] (47)

Comparing (46) and (47), we have

**Theorem 4.** If $\mathcal{H}$ and $\mathcal{G}$ all are quadratic functionals, i.e., $\mathcal{H}$ and $\mathcal{G}$ have the form

\[
\int_0^{2\pi} \sum_{i,j=0}^M c_{ij} \delta^i u \delta^j u \, dx,
\] (48)

where $\delta^i u = \frac{\partial^i u}{\partial x^i}$, $\delta^0 u = u$, then

\[
\{ \{ H(\bar{u}), G(\bar{u}) \} \} = \{ \hat{\mathcal{H}}, \hat{\mathcal{G}} \}(\bar{u}) = \{ \mathcal{H}, \mathcal{G} \}(I\bar{u}).
\] (49)

If the Hamiltonian operator $D$ does not depend on $u$ and its derivatives and either $\mathcal{H}$ or $\mathcal{G}$ is a quadratic functional, then (49) is also valid.

**Proof.** When $\mathcal{H}$ is a quadratic functional, $\delta \mathcal{H}$ is linear. In this case,

\[
P \delta \mathcal{H}[u] = \delta \mathcal{H}[u], \quad u \in B.
\]

Therefore

\[
(P \delta \mathcal{H}(I\bar{u}), D(I\bar{u}) \cdot P \delta \mathcal{G}(I\bar{u})) = (\delta \mathcal{H}(I\bar{u}), D(I\bar{u}) \delta \mathcal{G}(I\bar{u})).
\]

This is just the right hand side of (47). So we know that the first part is true.

If $D$ does not depend on $u$ and its derivatives, then $D \delta \mathcal{G}(I\bar{u}) = P(D \delta \mathcal{G}(I\bar{u}))$. Hence when $\mathcal{H}$ is quadratic,

\[
(P \delta \mathcal{H}(I\bar{u}), D \delta \mathcal{G}(I\bar{u}))
\]
Therefore (49) is also valid.

Although we have Theorem 4, it does not know when \( D \) satisfies the Jacobi identity. If \( D \) does not depend on \( u \) and its derivatives, \( D \) is naturally a Hamiltonian operator. Generally speaking, we can not know whether or not \( D \) is Hamiltonian even if \( D \) is a Hamiltonian operator. For example, the operator \( D = u \frac{\partial}{\partial x} + \frac{\partial}{\partial z} \) is Hamiltonian. For \( N = 1 \),

\[
D = \begin{bmatrix}
0 & \frac{-\sqrt{2}}{2}b_1 & \frac{\sqrt{2}}{2}a_1 \\
\frac{\sqrt{2}}{2}b_1 & 0 & \sqrt{2}a_0 \\
-\frac{\sqrt{2}}{2}a_1 & -\sqrt{2}a_0 & 0
\end{bmatrix}
\]

(50)

is a Hamiltonian operator. For \( N = 2 \),

\[
D = \begin{bmatrix}
0 & \frac{-\sqrt{2}}{2}b_1 & \frac{\sqrt{2}}{2}a_1 & -\sqrt{2}b_2 & \sqrt{2}a_2 \\
\frac{\sqrt{2}}{2}b_1 & 0 & \frac{3}{2}b_1 & \frac{3}{2}a_1 & 3 \sqrt{2}a_0 \\
\frac{-\sqrt{2}}{2}a_1 & -\sqrt{2}a_0 & 0 & -\frac{3}{2}a_1 & -\frac{3}{2}b_1 \\
\sqrt{2}b_2 & \frac{3}{2}b_1 & \frac{3}{2}a_1 & 0 & 2\sqrt{2}a_0 \\
-\sqrt{2}a_2 & -\frac{3}{2}a_1 & \frac{3}{2}b_1 & -2\sqrt{2}a_0 & 0
\end{bmatrix}
\]

(51)

It can be verified that (51) is not a Hamiltonian operator. Even in these cases, we may still discuss the lower order conservation laws of the semi-discrete approximative equation (42).

**Theorem 5.**

1°. If the original Hamiltonian system is linear, i.e., the Hamiltonian operator \( D \) does not depend on \( u \) and the derivatives of \( u \) and the Hamiltonian functional is quadratic, or \( D \) depends linearly on \( u \) and the derivatives of \( u \) and the Hamiltonian functional is linear, then the semi-discrete approximative system (42) preserves all conservation laws of the original Hamiltonian system.

2°. If \( D \) does not depend on \( u \) and the derivatives of \( u \), then the semi-discrete system (42) preserves all linear and quadratic conservation laws of the original Hamiltonian system and the Hamiltonian energy.

3°. If \( \mathcal{H} \) is quadratic, then the semi-discrete system (42), in spite of whether it is Hamiltonian or not, preserves all linear and quadratic conservation laws.

4°. In any case, (42) at least preserves the Hamiltonian energy.

4° follows from the anti-symmetry of \( D \). 2° and 3° and the first part of 1° follow from Theorem 4. For the second part of 1°, noting that in this case \( \delta \mathcal{H} \) is a constant, hence

\[
(\delta \mathcal{H}(\hat{u}), DP \cdot \delta G(\hat{u})) = -(D \delta \mathcal{H}(\hat{u}), P \delta G(\hat{u}))
\]

\[
= -(P D \delta \mathcal{H}(\hat{u}), \delta G(\hat{u}))
\]

\[
= -(D \delta \mathcal{H}(\hat{u}), \delta G(\hat{u})) = (\delta \mathcal{H}(\hat{u}), D \delta G(\hat{u})).
\]

Therefore for any functional \( G \),

\[
\{\{H(\hat{u}), G(\hat{u})\}\} = \{\hat{H}, \hat{G}\}(\hat{u}) = \{\mathcal{H}, \mathcal{G}\}(\hat{u}),
\]
where \( G(\bar{u}) = \hat{G}(\bar{u}) = \frac{1}{\pi} G(I\bar{u}) \). Then we get the result. 

In the previous section we have known that KdV equation has two different Hamiltonian structures

\[
\partial_t u = uu_x + u_{xxx} = \mathcal{D}\delta\mathcal{H}_1[u] = \mathcal{E}\delta\mathcal{H}_0[u],
\]

where

\[
\mathcal{D} = \frac{\partial}{\partial x}, \quad \mathcal{H}_1[u] = \int_0^{2\pi} \left( \frac{1}{8} u^3 - \frac{1}{4} u_x^2 \right) dx,
\]

\[
\mathcal{E} = \frac{1}{8} \left( u \frac{\partial}{\partial x} + \frac{\partial}{\partial x} u \right) + \frac{\partial^3}{\partial x^3}, \quad \mathcal{H}_0[u] = \frac{1}{2} \int_0^{2\pi} u^2 dx.
\]

Hence it has two different discrete forms

\[
\frac{d\bar{u}}{dt} = D\nabla \mathcal{H}_1(\bar{u}), \quad \frac{d\bar{u}}{dt} = E\nabla \mathcal{H}_0(\bar{u}),
\]

where \( D, E, \mathcal{H}_1, \mathcal{H}_0 \) are the discretizations of \( \mathcal{D}, \mathcal{E}, \mathcal{H}_1 \) and \( \mathcal{H}_0 \) respectively. \( D \) is linear, so it is a Hamiltonian operator. Thus (52) is a finite dimensional Hamiltonian system. But \( E \) is not a Hamiltonian operator. So (53) is not a Hamiltonian system.

Because

\[
D\nabla \mathcal{H}_1(\bar{u}) = \hat{P} \circ \mathcal{D} \cdot \mathcal{I} \cdot \hat{P}\delta\mathcal{H}_1(I\bar{u})
\]

\[
= I^{-1}(P \circ \mathcal{D} \cdot \mathcal{P}\delta\mathcal{H}_1(I\bar{u}))
\]

\[
= I^{-1}(P \circ D\delta\mathcal{H}_1(I\bar{u}))
\]

and \( D\delta\mathcal{H}_1 = \mathcal{E}\delta\mathcal{H}_0 \), we have

\[
E\nabla \mathcal{H}_0(\bar{u}) = \hat{P} \circ \mathcal{E}(I\bar{u}) \cdot \mathcal{I} \cdot \hat{P}\delta\mathcal{H}_0(I\bar{u})
\]

\[
= I^{-1}(P \circ \mathcal{E}(I\bar{u}) \cdot \mathcal{P}\delta\mathcal{H}_0(I\bar{u}))
\]

\[
= I^{-1}(P \circ L(I\bar{u}) \delta\mathcal{H}_0(\bar{u})) \quad \text{(since \( \mathcal{H}_0 \) is a quadratic functional)}
\]

\[
= I^{-1}(P \circ D(I\bar{u}) \delta\mathcal{H}_1(\bar{u})) \quad \text{(by (38))}
\]

\[
= I^{-1}(P \circ D(I\bar{u}) \delta\mathcal{H}_1(\bar{u}))
\]

\[
= D\nabla \mathcal{H}_1(\bar{u}).
\]

The right hand sides of the equations (52) and (53) are the same. Viewed as ordinary differential equations, they are the same one. Of course, they have same conservation laws. By Theorem 5, the conservation laws of the Hamiltonian system (52) are \( \mathcal{H}_0, \mathcal{H}_1 \) and \( G = \hat{G} = \frac{1}{\pi} G(I\bar{u}) \), where \( G = \int_0 u dx \).

Case II. We next consider the case of \( l \geq 1 \). In this case, \( U = C_p^\infty (\mathbb{R}) \times \cdots \times C_p^\infty (\mathbb{R}) \). The element in \( U \) is denoted by \( u = (u_1, \cdots, u_l)^T, \quad u_i \in C_p^\infty (\mathbb{R}) \). The inner product is

\[
(u, v) = \sum_{i=1}^l (u_i, v_i).
\]

The basis of every component is (29). Their combination forms the basis of \( U \). For example, for \( l = 2 \), the basis of \( U \) is

\[
\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} \cos x \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \cos x \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \sin x \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \sin x \\ 0 \end{pmatrix}, \quad \cdots, \quad \begin{pmatrix} \cos nx \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \cos nx \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \sin nx \\ 0 \end{pmatrix}, \quad \cdots.
\]
The orthogonal projection is $P : U \rightarrow B$

$$\tilde{u} = Pu = \begin{pmatrix} Pu_1 \\ \vdots \\ Pu_l \end{pmatrix}, \quad (56)$$

where $B$ is a $(2N + 1)l$ dimensional subspace of $U$. Hence, $B$ is isomorphic to a real linear space $\mathbb{R}^{(2N + 1)l}$. The inner product in $B$ is also the Euclidean inner product. Similar to (34), we can define an isomorphic mapping $I : \hat{B} \rightarrow B$ and further define $\hat{P} = I^{-1} \circ P$. The inner product \((\cdot, \cdot)_B\) in $B$, the discretizations of the Hamiltonian operators and functionals are the same with the previous ones formally. Thus the semi-discrete system is still (42). And the properties (35) – (38), Lemmas 2 and 3, Theorems 4 and 5 all are valid for the present case. The different point is just the form of a quadratic functional. It is at present

$$\int_0^{2\pi} \sum_{i,j=0}^{M} \sum_{a,b=0}^{l} C_{ij}^{a,b} \partial^a_x u_\alpha \partial^b_x u_\beta \, dx.$$ 

We now consider the semi-discrete Fourier spectral approximation of the long water wave equation. From the previous section we know that it has three different forms

$$u_t = D_1 \delta \mathcal{H}_1[u] = D_2 \delta \mathcal{H}_2[u] = D_3 \delta \mathcal{H}_3[u],$$

where $u = (u_1, u_2)^T$, $D_1$, $D_2$, $D_3$ and $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{H}_3$ are given by (21 – 26) respectively. Here $u_1$ corresponds to $u$ in (20) and $u_2$ to $h$ in (20). Denote $D_1$, $D_2$, $D_3$ and $H_1$, $H_2$, $H_3$ the discretizations of $D_1$, $D_2$, $D_3$ and $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{H}_3$ respectively. Then we get three different semi-discrete systems

$$\frac{d\tilde{u}}{dt} = D_1 \nabla_x H_1(\tilde{u}), \quad (57)$$

$$\frac{d\tilde{u}}{dt} = D_2 \nabla_x H_2(\tilde{u}), \quad (58)$$

$$\frac{d\tilde{u}}{dt} = D_3 \nabla_x H_3(\tilde{u}). \quad (59)$$

$D_1$ is linear, so it is a Hamiltonian operator. Therefore (57) is a finite dimensional Hamiltonian system. Hence (57) is of the conservation laws $H_1$, $H_2$, $H_3$ and $G = \mathcal{G}$, $G = \int u_1 \, dx$.

For $N = 1$, $D_2$ is a $6 \times 6$ matrix

$$D_2(\tilde{u}) = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{\sqrt{2}} \tilde{u}_5 & -\frac{1}{\sqrt{2}} \tilde{u}_6 & \frac{1}{\sqrt{2}} \tilde{u}_3 & \frac{1}{\sqrt{2}} \tilde{u}_4 \\
0 & 1 \sqrt{2} \tilde{u}_5 & 0 & 1 & 2 & \frac{1}{\sqrt{2}} \tilde{u}_1 \\
0 & 1 \sqrt{2} \tilde{u}_6 & -1 & 0 & \frac{1}{\sqrt{2}} \tilde{u}_1 & \sqrt{2} \tilde{u}_2 \\
0 & -\frac{1}{\sqrt{2}} \tilde{u}_3 & -2 & -\frac{1}{\sqrt{2}} \tilde{u}_1 & 0 & 1 \\
0 & -\frac{1}{\sqrt{2}} \tilde{u}_4 & -\frac{1}{\sqrt{2}} \tilde{u}_1 & -\sqrt{2} \tilde{u}_2 & -1 & 0
\end{bmatrix}.$$

Direct verification shows that it is a Hamiltonian operator. So as $N = 1$, (58) is a Hamiltonian
system. For $N = 2$, $D_2$ is a $10 \times 10$ matrix

$$
D_2(\vec{u}) =
\begin{bmatrix}
\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{\dot{u}_2}{\sqrt{2}} & 1 + \frac{\dot{u}_9}{2} & 2 & \frac{\dot{u}_1 - \dot{u}_7}{\sqrt{2}} & 0 & -\frac{\dot{u}_6}{2} & 0 & \frac{\dot{u}_3}{2} & 0 & 0 \\
\frac{\dot{u}_2}{\sqrt{2}} & 0 & 1 + \frac{\dot{u}_9}{2} & 2 & \frac{\dot{u}_1 - \dot{u}_7}{\sqrt{2}} & 0 & -\frac{\dot{u}_6}{2} & 0 & \frac{\dot{u}_3}{2} & 0 & 0 \\
0 & -\frac{\dot{u}_3}{\sqrt{2}} & -2 & -\left(\frac{\dot{u}_1 + \dot{u}_7}{\sqrt{2}}\right) & 0 & 1 - \frac{\dot{u}_9}{2} & 0 & -\frac{\dot{u}_3}{2} & 0 & -\frac{\dot{u}_5}{2} \\
0 & -\frac{\dot{u}_4}{\sqrt{2}} & \frac{\dot{u}_7}{\sqrt{2}} & -\frac{\dot{u}_1}{\sqrt{2}} & -1 + \frac{\dot{u}_9}{2} & 0 & -\frac{\dot{u}_3}{2} & -\frac{\dot{u}_4}{2} & -\frac{\dot{u}_5}{2} & -\frac{\dot{u}_6}{2} \\
0 & \sqrt{2}\dot{u}_9 & 0 & \dot{u}_5 & 0 & \dot{u}_3 & 0 & 4 & 4 & \sqrt{2}\dot{u}_1 \\
0 & \sqrt{2}\dot{u}_9 & \dot{u}_5 & \frac{3}{2}\dot{u}_6 & \dot{u}_3 & \frac{3}{2}\dot{u}_4 & 0 & 4 & 0 & \sqrt{2}\dot{u}_1 & 2\sqrt{2}\dot{u}_2 \\
0 & -\sqrt{2}\dot{u}_7 & 0 & -\dot{u}_3 & 0 & \dot{u}_5 & 0 & 4 & \sqrt{2}\dot{u}_1 & 0 & 0 \\
0 & -\sqrt{2}\dot{u}_8 & -\frac{\dot{u}_3}{2} & -\frac{3}{2}\dot{u}_4 & \dot{u}_5 & \frac{3}{2}\dot{u}_6 & 0 & \sqrt{2}\dot{u}_1 & -2\sqrt{2}\dot{u}_2 & -4 & 0
\end{bmatrix}
$$

It is not a Hamiltonian operator. Hence, for $N \geq 2$, $D_2(\vec{u})$ is not a Hamiltonian operator. $D_3$ has the similar property.

However, since $\mathcal{H}_2$ is a quadratic functional and $\mathcal{H}_3$ is a linear functional, similar to the KdV equation, we have the equation

$$
D_1\nabla_{\vec{u}}H_1(\vec{u}) = D_2\nabla_{\vec{u}}H_2(\vec{u}) = D_3\nabla_{\vec{u}}H_3(\vec{u}).
$$

So (57), (58) and (59) are the same ordinary differential equation. Of course, they have same conservation laws.

This method can be generalized to the case of $k > 1$, $l > 1$. The conclusion is similar. We omit it here.

Discussion

As we mentioned above, the semi-discrete Fourier approximation can not preserve the bi-Hamiltonian structure for nonlinear bi-Hamiltonian systems. The important point of bi-Hamiltonian systems is that they have a series of (possibly infinite) conservation laws. If in semi-discretization, we can get semi-discrete bi-Hamiltonian systems of finite dimensions approximating the original bi-Hamiltonian systems, then the semi-discrete systems also have a series of conservation laws. But it is still an open problem whether there exists a discretizing method which can preserve the bi-Hamiltonian structure for bi-Hamiltonian systems.

REFERENCES


