Wronskian and Grammian solutions for the
(2 + 1)-dimensional BKP equation

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Abstract The (2 + 1)-dimensional BKP equation in the Hirota bilinear form is studied during this work. Wronskian and Grammian techniques are applied to the construction of Wronskian and Grammian solutions of this equation, respectively. It is shown that these solutions can be expressed as not only Pfaffians but also Wronskians and Grammians.

Keywords (2 + 1)-dimensional BKP equation, Wronskian solution, Grammian solution

Wronskian and Grammian determinant solutions are common features of soliton equations, and their corresponding techniques have been proved to be powerful and direct to search for exact solutions of soliton equations.1–4 More significantly, many physically important solutions such as rational solutions, soliton solutions, positon solutions, negaton solutions, and complexiton solutions can be derived based on the Wronskian or Grammian formulations.5–8 In the present letter, the (2 + 1)-dimensional BKP equation (here BKP stands for B-type Kadomtsev–Petviashvili) in the Hirota bilinear form is studied9,10

\[ (D_x D_t - D_y^3 + 3D_y^2) f \cdot f = 0, \]

where \(D_x, D_y,\) and \(D_t\) are Hirota bilinear operators.11 Up to now, few attention has been paid to the study of Eq. (1). In Ref. 11, Hirota presented the \(N\)-soliton solution and Pfaffian solution of Eq. (1) and concluded that the \(N\)-soliton solution can be expressed as a Pfaffian. However, to our knowledge, neither the Wronskian solutions nor Grammian solutions for Eq. (1) has been revealed so far by the Wronskian and Grammian techniques.

This paper aims to construct Wronskian solutions and Grammian solutions for Eq. (1) through the Wronskian and Grammian techniques. In the construction, every generating function of matrix entry satisfies systems of partial differential equations involving free parameters.

Proposition 1 Assume that continuous derivative with any order exists for \(\phi_i = \phi_i(x,y,t)\) \((i = 1, 2, \ldots, N)\) with \(-\infty < x, y < +\infty, t \geq 0\) and it satisfies the set of linear partial differential equations

\[ \phi_{i,x}(2) = \sum_{j=1}^{N} \lambda_{ij}(t) \phi_j, \quad \phi_{i,y}(1) = A \phi_{i,x}(1), \quad \phi_{i,t}(1) = 4 \phi_{i,x}(3) + B \phi_{i,x}(1), \]

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where \( \phi_{i,x}^{(m)} = \partial^m \phi_i / \partial x^m \) represents the \( m \)-th order partial derivative of \( \phi_i \) with respect to \( x \), and \( \lambda_{ij}, A, B \) are arbitrary real constants and \( A, B \) satisfy \( AB = -3 \).

Then, \( N \)-th order Wronskian determinant

\[
W = W(\phi_1, \phi_2, \ldots, \phi_N) = \begin{vmatrix}
\phi_{1,x}^{(0)} & \phi_{1,x}^{(1)} & \cdots & \phi_{1,x}^{(N-1)} \\
\phi_{2,x}^{(0)} & \phi_{2,x}^{(1)} & \cdots & \phi_{2,x}^{(N-1)} \\
\vdots & \vdots & \ddots & \vdots \\
\phi_{N,x}^{(0)} & \phi_{N,x}^{(1)} & \cdots & \phi_{N,x}^{(N-1)}
\end{vmatrix}
\]

presents a solution of Eq. (1), where \( \phi_{j,x}^{(m)} = \partial^m \phi_j / \partial x^m \).

**Proof** For simplicity, we adopt the abbreviated notation of Freeman and Nimmo \(^{12,13}\) for the Wronskian

\[
f = |0, 1, \ldots, N-1| = |N-1|.
\]

Through the conditions (2), derivatives of expression (4) can be easily obtained. With applying Lemma 1 and Lemma 3 in Ref. 7, calculations based on notation (4) can prove that the function \( f = |N-1| \) is a Wronskian determinant solution to Eq. (1), under the differential conditions (2).

It is proved that except the Wronskian determinant solutions, Eq. (1) also has Grammian solutions. \(^{14}\)

**Proposition 2** Assume that \( f_i = f_i(x,y,t) \) and \( g_j = g_j(x,y,t) \) with \( 1 \leq i \) and \( j \leq N \) satisfy a set of conditions

\[
f_{i,j}^{(1)} = Af_{i,j}^{(1)}; \quad f_{i,j}^{(1)} = 4f_{i,j}^{(3)} + Bg_{i,j}^{(1)}; \quad g_{j,j}^{(1)} = Ag_{j,j}^{(1)}; \quad g_{j,j}^{(1)} = 4g_{j,j}^{(3)} +Bg_{j,j}^{(1)}.
\]

The Grammian determinant defined by

\[
f = \det (a_{ij})_{1 \leq i,j \leq N}, \quad a_{ij} = c_{ij} + \int f_i g_j \, dx
\]

solves Eq. (1), where \( c_{ij} \) is a constant.

**Proof** Let us rewrite the determinant \( f \) as a Pfaffian (see Ref. 11 for details)

\[
f = (1, 2, \ldots, N, N^*, (N-1)^*, \ldots, 2^*, 1^*), \quad (i, j^*) = a_{ij}, \quad (i, j) = (i^*, j^*) = 0, \quad 1 \leq i, j^* \leq N.
\]

To derive the derivatives of \( a_{ij} \) and \( f \), we introduce Pfaffian entries \((m, n = 0, 1, 2, \ldots)\) (see Ref. 11 for details)

\[
(d_{n}, j^*) = \frac{\partial^n}{\partial x^m} g_j; \quad (d_{n}^*, i) = \frac{\partial^n}{\partial x^m} f_i; \quad (d_{m}, d_{n}^*) = (d_{m}, i) = (d_{m}^*, i^*) = 0.
\]

Based on these entries, derivatives of \( a_{ij} \) with respect to \( x, y, t \) are

\[
\frac{\partial}{\partial x} a_{ij} = f_i g_j = (d_0, d_0^*, i, j^*),
\]
Then it follows from formula (10) that the last equality of Eq. (9) is equivalent to

\[ \frac{\partial}{\partial y} a_{ij} = Af_jg_j = A(d_0, d_0^*, i, j^*), \]

\[ \frac{\partial}{\partial t} a_{ij} = 4[(d_2, d_0^*, i, j^*) - (d_1, d_1^*, i, j^*) + (d_0, d_1^*, i, j^*)] + B(d_0, d_0^*, i, j^*). \]

For brevity, we use the abbreviated notation \( f = (\bullet) \). Through the conditions (7) and the derivatives of \( a_{ij} \), derivatives of \( f \) can be easily obtained. After some calculations, we have

\[ (f_{y(1)} - f_{y(3)}) + 3f_{y(2)} f = 3A[(d_3, d_0^*, \bullet) - 2(d_0, d_0^*, d_1, d_1^*, \bullet) + (d_0, d_1^*, \bullet) - (d_1, d_2^*, \bullet) - (d_2, d_1^*, \bullet)]((\bullet), \]

\[ (f_{x(1)} - f_{x(3)}) f_{y(1)} + 3(f_{x(2)} y_{y(1)} - f_{x(1)} y_{y(1)}) f_{x(1)} = 12A(d_0, d_0^*, \bullet)(d_1, d_1^*, \bullet), \]

\[ 3f_{x(1)} y_{y(1)} f_{x(2)} = 3A[(d_1, d_0^*, \bullet) + (d_0, d_1^*, \bullet)]^2. \]

Substituting the above results into Eq. (1) leads to

\[ (D_1 D_2 - D_2 D_1 + 3D_1^3) f^2 f = 2((f_{y(1)} - f_{y(3)}) + 3f_{y(2)} f) f + (f_{y(1)} - f_{y(3)}) f_{y(1)} + 3(f_{x(2)} y_{y(1)} - f_{x(1)} y_{y(1)}) f_{x(1)} = 2\{3A[(d_3, d_0^*, \bullet) + 2(d_0, d_0^*, d_0, d_1, d_1^*, \bullet) + (d_0, d_1^*, \bullet) - (d_1, d_2^*, \bullet) - (d_2, d_1^*, \bullet)]((\bullet) - 12A(d_0, d_0^*, d_0, d_1, d_1^*, \bullet)((\bullet) + 12A(d_0, d_0^*, \bullet)(d_1, d_1^*, \bullet) - 3A[(d_1, d_0^*, \bullet) - (d_0, d_1^*, \bullet)]^2 - 12A(d_0, d_1^*, \bullet)(d_1, d_0^*, \bullet)). \]

In terms of the identities of determinant, we can get the following identity

\[ [(d_3, d_0^*, \bullet) + 2(d_0, d_0^*, d_1, d_1^*, \bullet) + (d_0, d_1^*, \bullet) - (d_1, d_2^*, \bullet) - (d_2, d_1^*, \bullet)]((\bullet) = \]

\[ [((d_0, d_1^*, \bullet) - (d_1, d_0^*, \bullet))]^2. \]

Then it follows from formula (10) that the last equality of Eq. (9) is equivalent to

\[ -24A[(d_0, d_0^*, d_1, d_1^*, \bullet)(d_0, d_1^*, \bullet) + (d_0, d_1^*, \bullet)(d_1, d_0^*, \bullet)](\bullet) = 0, \]

in which Jacobi identity for the determinant is utilized. Thus the Grammian determinant \( f = (\bullet) \) is proved to be able to solve Eq. (1) under the differential conditions (7).

In the present letter, we have investigated the (2+1)-dimensional BKP equation in the Hirota bilinear form using the Wronskian and Grammian approaches. As a result, both the Wronskian determinant solution and Grammian one to the BKP equation were constructed. Our results show that solutions of Eq. (1) can be expressed as not only Pfaffians but also Wronskians and Grammans. This property is different from the known property of the KP equation that it only has Wronskian and Grammian solutions. This work was supported by the National Natural Science Foundation of China (11202161 and 11172233) and the Basic Research Fund of the Northwestern Polytechnical University (GBKY1034).


