Generalized Schwarz Algorithm for Obstacle Problems

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Abstract—In this paper, we present so-called generalized additive and multiplicative Schwarz algorithms for solving the discretization problems of obstacle problems with a self-adjoint elliptic operator. We establish convergence theorems for the proposed algorithms. Numerical tests show that a faster convergence rate can be obtained by choosing suitable parameters in the algorithms.

Keywords—Variational inequalities, Obstacle problems, Generalized Schwarz algorithms, Convergence.

1. INTRODUCTION

Schwarz algorithms were proposed more than one hundred years ago for proving the solvability of PDEs on a complicated domain. Over the last two decades, numerical solution for PDEs has developed into a very active research area, see, e.g., [1,2] and the references therein.

A Schwarz algorithm was proposed to solve a variational inequality by Lions in [3]. Then various Schwarz algorithms for solving variational inequalities were constructed and analyzed, cf. [4-14]. Recently, a Generalized Schwarz Algorithm (GSA) has been introduced to solve boundary value problems of elliptic equations, see [15] and the references therein. Compared with classical Schwarz algorithms, in which the subproblems are coupled by the Dirichlet boundary condition, the generalized Schwarz algorithm replaces the inner boundary condition by a Robin condition with a parameter. Numerical experiments show that GSA is much faster than the classical Schwarz algorithm for the appropriate choice of the parameters.

The aim of this paper is to construct and analyze a generalized Schwarz algorithm for solving discrete variational inequalities. As a model problem, we discuss the most basic and important inequality—obstacle problem with a self-adjoint elliptic operator.

The paper is organized as follows. In Sections 2 and 3, we introduce the discrete obstacle problem as well as generalized Schwarz algorithms with two subdomains, and establish the convergence theorems for the algorithms. In Section 4, we discuss the convergence of generalized Schwarz algorithms with more subdomains. Finally, in Section 5, we give some numerical results.

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We thank the anonymous referee for his helpful suggestions. The result in Section 4 was motivated by the referee.
2. OBSTACLE PROBLEM AND GENERALIZED SCHWARZ ALGORITHM WITH TWO SUBDOMAINS

Consider the obstacle problems: find \( u \in K \) such that
\[
a(u, v - u) \geq (f, v - u), \quad \forall v \in K,
\]
where \( K = \{ v \in H^1_0(\Omega) : \phi \leq v \leq \psi \text{ a.e. in } \Omega \} \), \( \Omega \) is a bounded convex polygonal domain in \( \mathbb{R}^d \) (\( d = 1, 2 \)) with a boundary \( \partial \Omega \), \( \phi \) and \( \psi \in H^2(\Omega) \) satisfying \( \phi \leq 0 \leq \psi \text{ a.e. on } \partial \Omega \),
\[
(f, v) = \int_{\Omega} f v \, dx, \quad f \in L^2(\Omega), \quad a(u, v) \text{ is a coercive, continuous, symmetric bilinear form on } H^1_0(\Omega).
\]

The discrete problem of (1) is as follows: find \( u_h \in K^h \) such that
\[
a(u_h, v - u_h) \geq (f, v - u_h), \quad \forall v \in K^h,
\]
where
\[
K^h = \{ v \in V^h : \phi \leq v \leq \psi \text{ on } \Omega_h \},
\]
\( V^h \) is the conforming linear finite element space, \( \Omega_h \) is the set of all the nodes. Now, we consider generalized Schwarz algorithms with two subdomains for solving (2). By using the two-level triangulation of Dryja and Widlund, we obtain open subdomains \( \Omega_1, \Omega_2 \) such that \( \Omega = \Omega_1 \cup \Omega_2 \) and \( \Omega_1 \cap \Omega_2 \neq \emptyset \). Define
\[
\Gamma_i = \{ p \in \Omega_h : p \in \partial \Omega_i \setminus \partial \Omega \}, \quad i = 1, 2.
\]
Let \( n_i \) be the outer normal direction of \( \partial \Omega_i \) at \( p \). For \( v \in V^h \), the directional derivative \( \frac{\partial v}{\partial n_i} \) is indeed a difference quotient. In order to pose the Robin condition at \( \Gamma_i \), we define
\[
g_i(v) = \theta_i v + (1 - \theta_i) \frac{\partial v}{\partial n_i}, \quad p \in \Gamma_i,
\]
where \( 0 < \theta_i \leq 1 \). For \( d = 2 \) and \( p \in \Gamma_i \), there are two edges containing vertex \( p \) and located on \( \partial \Omega_i \setminus \partial \Omega \). Denote the two edges by \( s_1 \) and \( s_2 \). Let \( n_{ij} \) be the outer normal direction of \( s_j \) on \( \partial \Omega_i \). Define for \( v \in V^h \) and \( p \in \Gamma_i \) that
\[
\frac{\partial v(p)}{\partial n_i} = \frac{1}{2} \sum_{j=1}^{2} \frac{\partial v(p)}{\partial n_{ij}}.
\]
Then we may use \( g_i(v) \) defined in (3) also for the case of \( d = 2 \). (In this case, the outer normal direction at \( p \in \Gamma_i \) in the ordinary sense does not exist since \( v \) is piecewise linear.)

Now we may define generalized Schwarz algorithms with two subdomains.

Algorithm AGSA1.

Step 1. Given \( \omega_1, \omega_2 > 0 \), \( \omega_1 + \omega_2 = 1 \), \( u^0 \in K^h \), \( n := 0 \).
Step 2. For \( i = 1, 2 \), solve the subproblems: find \( u^{n,i} \in K^n_i \) such that
\[
a(u^{n,i}, v - u^{n,i}) \geq (f, v - u^{n,i}), \quad \forall v \in K^n_i,
\]
where
\[
K^n_i = \{ v \in K^h : v = u^n \text{ in } \Omega_h \setminus \overline{\Omega}_{ih}, \quad g_i(v) = g_i(u_n) \text{ at } p \in \Gamma_i \},
\]
and \( \overline{\Omega}_{ih} = \Omega_{ih} \cup \Gamma_i \), \( \Omega_{ih} \) is the set of all the nodes in \( \Gamma_i \).
Step 3. \( u^{n+1} = \sum_{i=1}^d \omega_i u^{n,i} \), \( n := n + 1 \), go to Step 2.

The above algorithm is additive. We may define a multiplicative algorithm as follows.
ALGORITHM MGSA1.

Step 1. Given \( u^0 \in K_h, n := 0 \).

Step 2. Solve the subproblem over \( \Omega_1 \): find \( u^{n,1} \in K_h^n \) such that

\[
a(u^{n,1}, v - u^{n,1}) \geq (f, v - u^{n,1}), \quad \forall v \in K_h^n,
\]

then \( u^n := u^{n,1} \).

Step 3. Solve the subproblem over \( \Omega_2 \): find \( u^{n,2} \in K_h^n \) such that

\[
a(u^{n,2}, v - u^{n,2}) \geq (f, v - u^{n,2}), \quad \forall v \in K_h^n.
\]

Step 4. \( u^{n+1} = u^{n,2} \), \( n := n + 1 \), go to Step 2.

REMARK 1. It is obvious that AGSA1 and MGSA1 with \( \theta_1 = \theta_2 = 1 \) are classical Schwarz algorithms in [4,5,7-11,13,14,16].

3. CONVERGENCE THEOREMS

At first we prove that Algorithms AGSA1 and MGSA1 are well-defined, i.e., (4) has a unique solution.

**LEMMA 1.** \( K_h^n \) is a nonempty, closed and convex subset of \( V_h \).

**PROOF.** \( K_h^n \) is nonempty since \( u^n \in K_h^n \). It is easy to see that \( g_i(\lambda v_1 + (1 - \lambda)v_2) = \lambda g_i(v_1) + (1 - \lambda)g_i(v_2) = g_i(u^n) \) and then \( \lambda v_1 + (1 - \lambda)v_2 \in K_h^n \) for \( v_1, v_2 \in K_h^n \) and \( 0 \leq \lambda \leq 1 \). So \( K_h^n \) is convex. Obviously, \( K_h^n \) is a closed subset of \( V_h \). The proof is completed. \( \blacksquare \)

By Lemma 1 and a well-known theorem (cf. [16]), we know that (4) has a unique solution.

In order to prove the convergence theorem, we introduce the so-called Condition A. We say nodes \( p_k \) and \( p_l \in \Omega_h \) are adjacent if they belong to the same element. Let

\[ \Gamma_i = \{ p \in \Omega_{ih} : p \text{ is adjacent to a node in } \Gamma_i \} . \]

**CONDITION A.** \( (\Gamma_1 \cap \Gamma_1) \cap (\Gamma_2 \cap \Gamma_2) = \emptyset \).

**REMARK 2.** Assume that \( \Omega_1 \) and \( \Omega_2 \) are connected subdomains, respectively. Then Condition A means that \( \Omega_1 \cap \Omega_2 \) contains at least two nodes for \( d = 1 \), and it means, roughly speaking, \( \Omega_1 \cap \Omega_2 \) contains at least two "columns" of nodes for \( d = 2 \).

**REMARK 3.** Let \( \delta = \text{dist}(\partial \Omega_1 \cap \Omega, \partial \Omega_2 \cap \Omega) \). Then \( \delta \) is the overlapping size. \( \delta > 0 \) means that it is uniform overlapping (see [3]). Let \( h \) be finite element meshsize. It is obvious that if \( h < \delta/3 \), then Condition A holds. So Condition A is natural in a uniform overlapping case.

**LEMMA 1.** Assume Condition A holds. Then (2) is equivalent to the following problem: find \( u^* \in K_h \) such that

\[
a(u^*, v - 2u^*) \geq (f, v - 2u^*), \quad \forall v \in K_h^* + K_h^* ,
\]

where

\[ K_h^* = \{ v \in K_h : v = u^* \text{ in } \Omega_h \setminus \Omega_h, g_i(v) = g_i(u^*) \text{ at } p \in \Gamma_i \} . \]

**PROOF.** Assume \( u_h \) is the solution of (2). Let \( u^* = u_h \), since \( K_h^* \subset K_h \), by (2) we have for \( i = 1, 2 \) that

\[
a(u^*, v_i - u^*) \geq (f, v_i - u^*), \quad \forall v_i \in K_h^* .
\]

Summing (6) for \( i = 1, 2 \), we obtain

\[
a(u^*, v_1 + v_2 - 2u^*) \geq (f, v_1 + v_2 - 2u^*), \quad \forall v_1 + v_2 \in K_h^* + K_h^* ,
\]

which means \( u_h = u^* \) is a solution of (5).
Now assume that $u^* \text{ is a solution of (5) and prove that } u^* \text{ is the solution of (2). For any } v \in K^h, \text{ take } v_1 \in V^h \text{ such that}

\begin{align*}
  v_1 &= u^*, & \text{at } p \in (\Omega_h \setminus \Omega_{1h}) \cup \Gamma'_1, \\
  v_1 &= v, & \text{at } p \in (\Omega_h \setminus \Omega_{2h}) \cup \Gamma'_2, \\
  v_1 &= v, & \text{elsewhere (maybe empty)}. 
\end{align*}

From Condition A, we know that $v_1$ is well defined by (8)–(10). Obviously, $v_1 \in K^h$ since $v, u^* \in K^h$. From (8) and (9), we have

\begin{align*}
  g_1(v_1) &= g_1(u^*), & \text{at } p \in \Gamma_1, \\
  g_2(v_1) &= g_2(v), & \text{at } p \in \Gamma_2. 
\end{align*}

It follows from (8) and (11) that $v_1 \in K^*_1$.

Let $v_2 = v + u^* - v_1$. Then it is easy to see that $v_2 \in K^h$ and $v_2 = u^* \in \Omega_h \setminus \Omega_{2h}$. By (12), we have for $p \in \Gamma_2$,

\[ g_2(v_2) = g_2(v + u^* - v_1) = g_2(v) + g_2(u^*) - g_2(v_1) = g_2(u^*). \]

Hence, $v_2 \in K^*_2$. So it follows from (5) that for any $v \in K^h$,

\[ a(u^*, v - u^*) = a(u^*, v_1 + v_2 - 2u^*) \geq (f, v_1 + v_2 - 2u^*) = (f, v - u^*), \]

which means $u^*$ is the solution of (2). The lemma has been proved.

Now we can prove the following theorem.

**Theorem 3.** Let sequence $\{u^n\}$ be produced by Algorithm AGSA1. Then $\{u^n\}$ converges to $u_h$ when $n \to \infty$.

**Proof.** Let $J(v) = a(v, v)/2 - (f, v)$. Since $a(v, v)$ is symmetric, we know that the solution $u^{n, i}$ of (4) satisfies

\[ J(u^{n, i}) = \min_{v \in K^*_i} J(v), \quad i = 1, 2. \]

Since $u^n \in K^*_n$, we have

\[ J(u^{n, i}) \leq J(u^n), \quad i = 1, 2. \]

It is easy to see $J(v)$ is a strictly convex, coercive functional. Hence,

\[ J(u_h) \leq J(u^{n, 1}) \leq \sum_{i=1}^2 \omega_i J(u^{n, i}) \leq J(u^n) \leq \cdots \leq J(u^0). \]

Since $a(u, v)$ is coercive, there exists a constant $\alpha > 0$ such that

\[ J(u^n) = \frac{1}{2} a(u^n, u^n) - (f, u^n) \geq \|u^n\| (\alpha \|u^n\| - \|f\|), \]

which combining with (14) yields the boundedness of $\{u^n\}$. Similarly, we can prove the boundedness of $\{u^{n, 1}\}$ and $\{u^{n, 2}\}$ by using (13) and (14). So they have convergence subsequences, denoted still by $\{u^n\}, \{u^{n, 1}\}$, and $\{u^{n, 2}\}$ converging to $u^*, u^*_1$, and $u^*_2$, respectively. Then it follows from (13), (14), and Step 3 in Algorithm AGSA1 that

\[ J(u^*) = J(u^*_i), \quad i = 1, 2, \]

\[ u^* = \sum_{i=1}^2 \omega_i u^*_i. \]
The strict convexity of \( J(v) \) and (15),(16) means that
\[
u^* = u_1^* = u_2^*.
\]
Then letting \( n \to \infty \) in Step 2, we obtain
\[
a(u^*, v - u^*) \geq (f, v - u^*), \quad \forall v \in K_i^*, \quad i = 1, 2,
\]
where
\[
K_i^* = \{v \in K^h : v = u^* \text{ in } \Omega_h \backslash \Omega_{ih}, g_i(v) = g_i(u^*) \text{ at } p \in \Gamma_i\}.
\]
Summing (17) for \( i = 1, 2 \), we have
\[
a(u^*, v - 2u^*) \geq (f, v - 2u^*), \quad \forall v \in K_1^* + K_2^*,
\]
which is (5). It follows from Lemma 2 that \( u^* = u_h \). The proof is completed.

For Algorithm MGSA1, we have results similar to Theorem 3.

**Theorem 4.** Let sequence \( \{u^n\} \) be produced by Algorithm MGSA1. Then \( \{u^n\} \) converges to \( u_h \) when \( n \to \infty \).

**Proof.** Similarly, we may prove that
\[
J(u^{n+1}) \leq J(u^{n,2}) \leq J(u^{n,1}) \leq J(u^n) \leq \cdots \leq J(u^0).
\]
Then for the limits \( u^* \), \( u_1^* \), and \( u_2^* \) of the convergent subsequences of \( \{u^n\} \), \( \{u^{n,1}\} \), and \( \{u^{n,2}\} \), respectively, we have
\[
J(u^*) \leq J(u_1^*) \leq J(u_2^*) \leq J(u^*),
\]
and then
\[
J(u^*) = J(u_1^*) = J(u_2^*),
\]
which implies that
\[
\frac{1}{2}a(u^*, u^*) = \frac{1}{2}a(u_1^*, u_1^*) - (f, u_1^* - u^*).
\]
On the other hand, it follows from Steps 2 and 3 of Algorithm MGSA1 that
\[
a(u_1^*, v - u_1^*) \geq (f, v - u_1^*), \quad \forall v \in K_1^*,
\]
\[
a(u_2^*, v - u_2^*) \geq (f, v - u_2^*), \quad \forall v \in K_2^{**},
\]
where
\[
K_2^{**} = \{v \in K^h : v = u_2^* \text{ in } \Omega_h \backslash \Omega_{2h}, g_2(v) = g_2(u_1^*) \text{ at } p \in \Gamma_2\}.
\]
Hence, by (19) and (20) with \( v = u^* \), we have
\[
\frac{1}{2}a(u^* - u_1^*, u^* - u_1^*) = \frac{1}{2}a(u^*, u^*) + \frac{1}{2}a(u_1^*, u_1^*) - a(u_1^*, u^*)
= \frac{1}{2}a(u_1^*, u_1^*) - (f, u_1^* - u^*) + \frac{1}{2}a(u_1^*, u_1^*) - a(u_1^*, u^*)
= a(u_1^*, u_1^*) - (f, u_1^* - u^*) \leq 0,
\]
which combining with the coerciveness of \( J(v) \) yields that \( u^* = u_1^* \). Similarly, we obtain \( u_1^* = u_2^* \) and \( K_2^{**} = K_2^* \). Adding (20) and (21), we know that
\[
a(u^*, v - 2u^*) \geq (f, v - 2u^*), \quad \forall v \in K_1^* + K_2^*,
\]
and \( u^* = u_h \) by Lemma 2, again. The proof is completed.
4. GENERALIZED SCHWARZ ALGORITHM WITH MANY SUBDOMAINS

In this section, we consider generalized Schwarz algorithms with many subdomains. We assume that subdomains $\Omega_1, \Omega_2, \ldots, \Omega_m$ satisfy $\bigcup_{i=1}^m \Omega_i = \Omega$. We also assume that any finite element does not cross the boundary of each $\Omega_i$. Similarly, we can define $\Gamma_i$, $\Gamma_i'$, $\Omega_{ih}$, $\Gamma_{ih}$, $g_i(v)$, $\frac{\partial v}{\partial n_i}$, $K_i^+$, $K_i^-$ and construct the corresponding algorithms as follows.

**Algorithm AGSA2.**

Step 1. Given $\omega_1, \omega_2, \ldots, \omega_m > 0$ such that $\sum_{i=1}^m \omega_i = 1$, $u_0 \in K_h$, $n := 0$.

Step 2. For $i = 1, 2, \ldots, m$, solve the subproblems: find $u_n^i \in K_i^+$ such that

$$a(u_n^i, v - u_n^i) \geq (f, v - u_n^i), \quad \forall v \in K_i^+.$$

Step 3. $u_n^i = \sum_{i=1}^m \omega_i u_n^i$, $n := n + 1$, go to Step 2.

**Algorithm MGSA2.**

Step 1. Given $u^0 \in K_h$, $n := 0$.

Step 2. For $i = 1, 2, \ldots, m$, solve the following subproblems over $\Omega_i$: find $u_n^i \in K_i^+$ such that

$$a(u_n^i, v - u_n^i) \geq (f, v - u_n^i), \quad \forall v \in K_i^+,$$

and $u_n^i = u_n^i$.

Step 3. $u_n^{i+1} = u_n^i$, $n := n + 1$, go to Step 2.

To conclude the convergence, instead of Condition A, we introduce the following condition.

**Condition B.** If $\Omega_i \cap \Omega_j \neq \emptyset$ and $i \neq j$, then $(\Gamma_i' \cup \Gamma_i) \cap (\Gamma_j' \cup \Gamma_j) = \emptyset$.

Similar to Lemma 2, we can prove the following lemma.

**Lemma 5.** Assume Condition B holds. Then (2) is equivalent to the following problem: find $u^* \in K_h$ such that

$$a(u^*, v - mu^*) \geq (f, v - mu^*), \quad \forall v \in K_1^+ + K_2^+ + \cdots + K_m^+. \quad (22)$$

Using the above lemma, it is not difficult to derive the following conclusion.

**Theorem 6.** Let sequence $\{u_n\}$ be produced by Algorithms AGSA2 or MGSA2. Then $\{u_n\}$ converges to $u_h$ when $n \to \infty$.

5. NUMERICAL EXAMPLES

In this section, we give some numerical experiments in order to confirm theoretical results and to investigate the behavior of the methods presented in this paper. In the tests, we use the Lagrange linear finite element space as $V_h$ and set the initial $u^0 = 0$, the meshsize $h = 0.01$. The stopping criterion is that the maximum norm of the difference between successive iteration solutions is less than $\epsilon = 10^{-8}$. The subproblems are solved by PSOR (cf. [2]), in which the stopping criterion is that the maximal norm of difference between the successive iterative solutions is less than $10^{-10}$.

In the first example, we consider the one-dimensional obstacle problem (1) for $\Omega = (0, 1)$, $\phi = 0$, $\psi = 1$, and $a(u, v) = \int_0^1 \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} \, dx$. Let subdomains $\Omega_1 = (0, 1/2 + \rho)$ and $\Omega_2 = (1/2 - \rho, 1)$. In the case of $h$-overlapping, we define $\rho = 0$. In this case, we choose $\Omega_1 = (0, 1/2)$, $\Omega_2 = (1/2 - h, 1)$. We solve problem (2) by our generalized Schwarz algorithms (AGSA1 and MGSA1) for different
Generalized Schwarz Algorithm

Table 1. Algorithm AGSA1 with \( f = \cos(\pi x) \) and \( \rho = 0.2 \).

<table>
<thead>
<tr>
<th>( \theta_1 ) ( \theta_2 )</th>
<th>0</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.6</th>
<th>0.8</th>
<th>0.9</th>
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Table 2. Algorithm MGSA1 with \( f = \cos(\pi x) \) and \( \rho = 0.05 \).

<table>
<thead>
<tr>
<th>( \theta_1 ) ( \theta_2 )</th>
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Table 3. Algorithm AGSA1 with \( f = 4\sin(4x) \).

<table>
<thead>
<tr>
<th>( \rho ) ( \theta_1, \theta_2 )</th>
<th>(0,0)</th>
<th>(0.2,0.2)</th>
<th>(0.4,0.4)</th>
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</table>

Table 4. Algorithm MGSA1 with \( f = (4x^2 - 1)(9x^2 - 1)/10 \).

<table>
<thead>
<tr>
<th>( \rho ) ( \theta_1, \theta_2 )</th>
<th>(0,0)</th>
<th>(0.2,0.2)</th>
<th>(0.4,0.4)</th>
<th>(0.6,0.6)</th>
<th>(0.8,0.8)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>233</td>
<td>21</td>
<td>17</td>
<td>18</td>
<td>18</td>
</tr>
<tr>
<td>2h</td>
<td>127</td>
<td>21</td>
<td>18</td>
<td>16</td>
<td>16</td>
</tr>
<tr>
<td>0.1</td>
<td>71</td>
<td>21</td>
<td>18</td>
<td>17</td>
<td>17</td>
</tr>
<tr>
<td>0.2</td>
<td>24</td>
<td>19</td>
<td>19</td>
<td>18</td>
<td>18</td>
</tr>
</tbody>
</table>

In the second example, we consider the two-dimensional obstacle problem (1) for "L" domain: \( \Omega = \{(x,y) : 0 < y < 1 \text{ for } 0 < x < 1/2, 0 < y < 1/2 \text{ for } 1/2 \leq x < 1 \} \) and \( \psi = 0 \), \( \alpha(u,v) = \int_{\Omega} \frac{\partial u \partial v}{\partial x \partial y} \, dx \, dy \). We let \( \Omega_1 = \{(x,y) : 0 < x < 1/2, 0 < y < 1 \} \) and \( \Omega_2 = \{(x,y) : 0 < x < 1, 0 < y < 1/2 \} \). We solve problem (2) by our generalized Schwarz algorithms for different \( f \) and different choices of parameters \( \theta_1 \) and \( \theta_2 \). The numbers of iteration in different cases are shown in Tables 5 and 6. In this case, Condition A does not hold, but the algorithms are still convergent.

In the last example, we consider a rectangle domain in \( \mathbb{R}^2 \) with \( \Omega = (0, 1) \times (0, 1), \alpha(\cdot, \cdot), \phi, \) and \( \psi \) are the same as that in Example 2. We let \( \Omega_1 = \{(x,y) : 0 < x < 1/2 + \rho, 0 < y < 1 \}, \) \( \Omega_1 = \{(x,y) : 1/2 - \rho < x < 1, 0 < y < 1 \}, \) and \( f = (x - 1/2)(y - 1/2) \). The iteration numbers are shown in Tables 7 and 8.
From the tests above, we may see the following.

(1) The classical algorithm (θ₁ = θ₂ = 0, corresponding to the iteration numbers with underline) is slower than the cases θ₁² + θ₂² ≠ 0.

(2) When the overlapping size becomes larger, the algorithms become less sensitive to the changes of θ₁ and θ₂.

REFERENCES