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Explicit bounds for rational points near planar curves and metric Diophantine approximation

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Abstract

The primary goal of this paper is to complete the theory of metric Diophantine approximation initially developed in Beresnevich et al. (2007) [10] for C^3 non-degenerate planar curves. With this goal in mind, here for the first time we obtain fully explicit bounds for the number of rational points near planar curves. Further, introducing a perturbational approach we bring the smoothness condition imposed on the curves down to C^1 (lowest possible). This way we broaden the notion of non-degeneracy in a natural direction and introduce a new topologically complete class of planar curves to the theory of Diophantine approximation. In summary, our findings improve and complete the main theorems of Beresnevich et al. (2007) [10] and extend the celebrated theorem of Kleinbock and Margulis (1998) [20] in dimension 2 beyond the notion of non-degeneracy.
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1. Introduction

Problems about rational points lying near curves and surfaces are widespread in number theory and include, for instance, questions regarding small values of homogeneous polynomials on the integer lattice. Within this paper we study the distribution of rational points near curves $\mathcal{C}$ embedded in $\mathbb{R}^2$. With this in mind we now introduce some basic notation. First of all, without loss of generality, let us agree that the curves we consider are given as a graph $\mathcal{C}_f = \{(x, f(x)) : x \in I\}$ of some function f defined on an interval I. Given $\delta > 0$, $Q > 1$ and a subinterval $J \subset I$, consider the following counting function

$$N_f(Q, \delta, J) := \#\left\{(p_1/q, p_2/q) \in \mathbb{Q}^2 : \begin{array}{l} p_1/q \in J, \ 0 < q \leqslant Q, \\ |f(p_1/q) - p_2/q| \leqslant \delta Q^{-1} \end{array}\right\},$$

where $\#A$ stands for the cardinality of a set A and q, p_1, p_2 denote coprime integers. Essentially, this function counts rational points $(p_1/q, p_2/q)$ with denominator $q \leqslant Q$ lying at a distance comparable to δ/Q from the arc $\{(x, f(x)) : x \in J\}$ of $\mathcal{C}_f$.

To begin with, we give a brief account of known results. Let $I \subset \mathbb{R}$ be a compact interval, $c_2 \geqslant c_1 > 0$ and let $\mathcal{F}(I; c_1, c_2)$ be the set of C^2 functions $f : I \to \mathbb{R}$ such that

$$c_1 \leqslant |f''(x)| \leqslant c_2 \quad \text{for all } x \in I. \tag{1}$$

In 1994 Huxley [18, Th. 1] proved that

$$N_f(Q, \delta, I) \ll_\varepsilon C^{10/3} \delta^{1-\varepsilon} Q^2 + C^{1/3} Q, \quad \text{where } C = \max\{c_2, c_1^{-1}\} \tag{2}$$

and the constant implicit in $\ll_\varepsilon$ depends on ε but does not depend on δ, Q, C or f. For $\delta > Q^{-2/3}$ Huxley's result was improved by Vaughan and Velani [27, Th. 1] on showing that for any $f \in \mathcal{F}(I; c_1, c_2)$, any $Q > 1$ and $0 < \delta < \frac{1}{2}$

$$N_f(Q, \delta, I) \ll \delta Q^2 + \delta^{-\frac{1}{2}} Q. \tag{3}$$

Additionally assuming that $f'' \in \mathrm{Lip}_\theta(I)$ with $0 < \theta < 1$ they proved [27, Th. 3] that for any $\varepsilon > 0$

$$N_f(Q, \delta, I) \ll \delta Q^2 + \delta^{-\frac{1}{2}} Q^{\frac{1}{2}+\varepsilon} + \delta^{\frac{\theta-1}{2}} Q^{\frac{3-\theta}{2}}. \tag{4}$$

Estimates (3) and (4) were also extended to an inhomogeneous case in [11].

When $\delta = o(Q^{-1})$ the quantity $N_f(Q, \delta, I)$ can vary from 0 to Q depending on the choice of $f \in \mathcal{F}(I; c_1, c_2)$ irrespectively of the actual value of δ – see [5, §2.2] for examples. In the case $\delta \gg Q^{-1}$ lower bounds for $N_f(Q, \delta, I)$ were obtained in [10, Th. 6] and in the inhomogeneous form in [11, Th. 5]. More precisely, it was shown that for any $f \in \mathcal{F}(I; c_1, c_2)$ additionally satisfying the condition $f \in C^3(I)$ there exist constants $k_1, k_2, c, Q_0 > 0$ such that for any $Q > Q_0$ and any δ satisfying $k_1 Q^{-1} \leqslant \delta \leqslant k_2$ one has

$$N_f(Q, \delta, I) \geqslant c\delta Q^2. \tag{5}$$

However, the above undoubtedly remarkable results fall short of providing a complete theory for the whole class $\mathcal{F}(I; c_1, c_2)$. Indeed, within (4) and (5) the additional differentiability and

Lipschitz assumptions are imposed while (2) is not optimal. In this paper we resolve this issue in full in relation to lower bounds and furthermore expand the results to a genuinely larger, topologically closed class of functions f introduced in the next paragraph. Furthermore, our results are fully explicit and uniform in f.

Throughout $\overline{\mathcal{F}}(I; c_1, c_2)$ stands for the closure of $\mathcal{F}(I; c_1, c_2)$ in the C^0 (uniform convergence) topology, *i.e.* the topology induced by the norm $\|f\|_{C^0} := \sup_{x \in I} |f(x)|$. In order to state the main counting result of this paper we now gather the definitions of various explicit constants appearing in the course of establishing this result and depending on c_1 and c_2 only. These are

$$\hat{E} := \frac{3^6 2^5 c_2}{c_1 \min\{1, \sqrt{c_1}\}}, \qquad c_0 := 2^{-13} \hat{E}^{-6} c_2^{-1}, \tag{6}$$

$$C_1 := \frac{c_2}{2c_1 c_0^2} = \frac{2^{25} \hat{E}^{12} c_2^3}{c_1}, \qquad C_2 := \left(\frac{c_2 C_1}{2c_0}\right)^{1/3}. \tag{7}$$

Theorem 1. *Let I be a compact interval, $c_2 \geqslant c_1 > 0$ and the constants $\hat{E}$, c_0, C_1 and C_2 be given by (6) and (7). Let a subinterval $J \subset I$ of length $|J| \leqslant \frac{1}{2}$, $Q > 1$ and $\delta \leqslant 1$ satisfy the conditions*

$$\delta Q^2 |J| \geqslant 8C_1 \quad \text{and} \quad Q\delta \geqslant C_2. \tag{8}$$

Assume also that either

$$Q \geqslant \frac{128}{c_0 c_1^2} |J|^{-3} \quad \text{or} \quad \frac{c_0^2}{c_2} \delta^{-2} \geqslant Q \geqslant \frac{16}{c_0 c_1^2} |J|^{-2}. \tag{9}$$

Then for any $f \in \overline{\mathcal{F}}(I; c_1, c_2)$

$$N_f(Q, \delta, J) \geqslant \frac{1}{4C_1} \delta Q^2 |J|.$$

Theorem 1 is a consequence of a more general covering result (Theorem 2 below) which will require the following notation. Let

$$\mathcal{R}_f^c(Q, \delta, J) := \left\{(q, p_1, p_2) \in \mathbb{N} \times \mathbb{Z}^2 : \begin{array}{l} p_1/q \in J, \ cQ < q \leqslant Q, \\ |f(p_1/q) - p_2/q| \leqslant \delta/Q, \\ \gcd(q, p_1, p_2) = 1 \end{array} \right\},$$

where $Q > 1$, $\delta > 0$, $c \geqslant 0$ and $J \subset I$; and let

$$\Delta_f^c(Q, \delta, J, \rho) := \bigcup_{(q, p_1, p_2) \in \mathcal{R}_f^c(Q, \delta, J)} \{x : |x - p_1/q| \leqslant \rho\}.$$

Also let $|A|$ denote the Lebesgue measure of a measurable set $A \subset \mathbb{R}^n$.

Theorem 2. *Let I be a compact interval, $c_2 \geqslant c_1 > 0$ and the constants $\hat{E}$, c_0, C_1 and C_2 be given by* (6) *and* (7). *Let a subinterval $J \subset I$ of length $|J| \leqslant \frac{1}{2}$, $Q > 1$ and $\delta \leqslant 1$ satisfy* (8) *and* (9). *Then for any $f \in \overline{\mathcal{F}}(I; c_1, c_2)$*

$$\left|\Delta_f^{c_0}(Q, \delta, J, \rho) \cap J\right| \geqslant \frac{1}{2}|J|, \quad \text{where } \rho := C_1\left(\delta Q^2\right)^{-1}. \tag{10}$$

The proof of Theorem 1 modulo Theorem 2 is easy and left to the reader, but see [10, §4.1] for a hint.

Since the constant in (2) implied by the Vinogradov symbol is independent of f, this estimate can also be extended to the class $\overline{\mathcal{F}}(I; c_1, c_2)$. We state this formally as

Theorem 3. *Let I be a compact interval and $c_2 \geqslant c_1 > 0$. Then* (2) *remains true for any $f \in \overline{\mathcal{F}}(I; c_1, c_2)$, where the implicit constant does not depend on δ, Q, C or f.*

We believe that (4) can also be extended to $f \in \overline{\mathcal{F}}(I; c_1, c_2)$, however this requires techniques of a very different nature and we plan to return to this issue in a subsequent publication.

For the rest of this section we discuss various consequences of the above results to metric Diophantine approximation. In what follows unless otherwise mentioned we follow the terminology of Bernik and Dodson [15] and Kleinbock and Margulis [20]. The foundations of a general metric theory of Diophantine approximation for planar curves were laid by Schmidt [24] in 1964 who proved that every non-degenerate planar curve is extremal (in the sense of Sprindžuk [26]). Recall that a curve $\mathcal{C}_f$ defined as a graph of function $f : I \to \mathbb{R}$ is non-degenerate if for almost every point $\mathbf{x} \in I$ we have that $f''(x) \neq 0$ and $f \in C^2$ on some neighbourhood of x (for the definition of non-degeneracy in higher dimensions see [4] or [20]). In this case we will also say that f is non-degenerate. In particular, by definition, any $f \in \mathcal{F}(I; c_1, c_2)$ is non-degenerate for any choice of $c_2 \geqslant c_1 > 0$. In the case of approximation by linear forms Baker [3] refined Schmidt's theorem with a Hausdorff dimension result and recently Badziahin [1] established the inhomogeneous version of Baker's theorem. Furthermore, non-degenerate curves have been shown to be of Groshev type [6,14] (see also [4,8,16] for higher dimensional results and the review [19]). Unlike the dual case, the progress with simultaneous approximation was rather slow. For quite a while nothing was known apart from Bernik's Khintchine type theorem for convergence for parabola [13]. However, in the last 5 years or so a general theory of simultaneous Diophantine approximation was developed in [10,27], which was subsequently generalised to multiplicative Diophantine approximation [2,12] and to the inhomogeneous case [11]. In short, the progress was based on the development of the theory of ubiquitous systems [9] and on the study of the distribution of rational points near planar curves. In particular, the various results on metric Diophantine approximation on planar curves inherited the extra smoothness and/or Lipschitz conditions imposed within (4) and (5). Theorem 2 enables us to remove these indeed unnecessary constrains within the divergence results and furthermore broaden them to a genuinely larger class, which is now introduced.

The curve $\mathcal{C}_f = \{(x, f(x)): x \in I\}$ (resp. the function f) will be called *weakly non-degenerate at $x_0 \in I$* if there exist constants $c_2 \geqslant c_1 > 0$ and a compact subinterval $J \subset I$ centred at $x_0 \in J$ such that $f|_J \in \overline{\mathcal{F}}(J; c_1, c_2)$. We will say that $\mathcal{C}_f$ (resp. the function f) is *weakly non-degenerate* if $\mathcal{C}_f$ is weakly non-degenerate at almost every point $x \in I$.

Remark 1. Recall that the non-degeneracy of $\mathcal{C}_f$ (resp. f) at $x_0 \in I$ requires $f|_J \in \mathcal{F}(J; c_1, c_2)$ for some compact subinterval $J \subset I$ centred at x_0. Therefore, every non-degenerate curve $\mathcal{C}_f$ is

weakly non-degenerate. However the converse is not always true as follows from the example of Section 2, which shows that a weakly non-degenerate curve may be degenerate everywhere. More precisely, the weakly non-degenerate function f constructed in the example is such that the restriction $f|_J$ of f on any compact interval $J \subset I$ in NOT C^2.

Theorem 4 below gathers the various main consequences of Theorems 2 and 3 for the simultaneous Diophantine approximation on weakly non-degenerate planar curves. Before stating the result we introduce some further notation. Given an arithmetic function $\psi : \mathbb{N} \to (0, +\infty)$, let

$$\lambda_\psi := \liminf_{h \to \infty} \frac{-\log \psi(h)}{\log h}$$

denote *the lower order of* $1/\psi$ *at infinity*. Also define the following two sets of ψ-approximable points:

$$\mathcal{S}_f(\psi) := \left\{ x \in I \colon \max\{\|qx\|, \|qf(x)\|\} < \psi(q) \text{ for i.m. } q \in \mathbb{N} \right\},$$

$$\mathcal{S}_f^*(\psi) := \left\{ x \in I \colon \|qx\| \cdot \|qf(x)\| < \psi(q) \text{ for i.m. } q \in \mathbb{N} \right\},$$

where $\|y\| = \min\{|y - p| \colon p \in \mathbb{Z}\}$ and 'i.m.' stands for 'infinitely many'.

Theorem 4. *Let* $\psi : \mathbb{N} \to (0, +\infty)$ *be monotonic and* $f : I \to \mathbb{R}$ *be a weakly non-degenerate function. Then*

(A) $\mathcal{S}_f(\psi)$ *has full Lebesgue measure in* I *whenever* $\sum_{h=1}^\infty \psi(h)^2 = \infty$;
(B) $\mathcal{H}^s(\mathcal{S}_f(\psi)) = \infty$ *whenever* $\sum_{h=1}^\infty h^{1-s}\psi(h)^{s+1} = \infty$ *and* $s \in (\frac{1}{2}, 1)$;
(C) $\dim \mathcal{S}_f(\psi) = s_0 := \frac{2 - \lambda_\psi}{1 + \lambda_\psi}$ *whenever* $\lambda_\psi \in [1/2, 1)$ *and* f *is weakly non-degenerate everywhere apart from a set of Hausdorff dimension* $\leqslant s_0$;
(D) $\dim \mathcal{S}_f^*(\psi) = s_0^* := \frac{2}{1 + \lambda_\psi}$ *whenever* $\lambda_\psi > 1$ *and* f *is weakly non-degenerate everywhere apart from a set of Hausdorff dimension* $\leqslant s_0^*$. *In particular,* $\mathcal{C}_f$ *is strongly extremal.*

The proofs of parts (A), (B) and (C) of Theorem 4 are essentially the same as those of Theorems 1, 3 and 4 in [10] with the only differences being that we use our Theorem 2 instead of [10, Theorem 7] and we use Theorem 3 instead of Huxley's original result (2). Note also that the proofs make use of continuous differentiability of f – a property that will be shown in the next section (Theorem 6). The proof of part (D) of Theorem 4 follows the line of argument of Theorems 6 and 6* from [12]. For the modifications are obvious we leave further details out. Using our Theorem 2 in combination with the ideas of [11] it is also straightforward to state and prove an inhomogeneous version of Theorem 4.

Weakly non-degenerate curves are characterised by the property that locally they can be perturbed into an arbitrarily close 'properly' non-degenerate curve with 'rigid' bounds on their curvature. By these we mean that the constants c_1 and c_2 appearing in (1) are not varying as we perturb the curve. Considering how Diophantine properties of manifolds are affected by small perturbations is not absolutely new. For example, Rynne [23] obtained a negative result by showing that certain Diophantine properties of non-degenerate manifolds are NOT preserved under small perturbations even in the C^k topology. It is likely that establishing positive results

will require imposing some kind of rigidity on the geometry of perturbed manifolds, likewise conditions (1) hold uniformly within $\mathcal{F}(I; c_1, c_2)$. This gives rise to the following

General problem. Find a 'reasonable' generalisation of weak non-degeneracy for manifolds in higher dimensions and prove that such manifolds are (strongly) extremal and/or satisfy the analogues of the Khintchine–Groshev theorem (see [4,8,10,15,16,20] for appropriate terminology and related results).

2. The nature of functions in $\overline{\mathcal{F}}(I; c_1, c_2)$

In this section we make an attempt to understand the main object we study in this paper – functions in $\overline{\mathcal{F}}(I; c_1, c_2)$. In particular, it is crucial to understand whether this class is any bigger than $\mathcal{F}(I; c_1, c_2)$.

Since the functions $f \in \overline{\mathcal{F}}(I; c_1, c_2)$ are obtained as limits of continuous and even twice differentiable functions in the uniform convergence topology, they are continuous. As is well known, differentiability is not preserved by the limit functions in the C^0 topology; for example, any continuous function on a compact interval can be uniformly approximated by a polynomial (Weierstrass' theorem). However, we shall see that functions in $\overline{\mathcal{F}}(I; c_1, c_2)$ are indeed continuously differentiable. Furthermore, they happen to have the second derivatives almost everywhere. On the other hand, we shall see that the second derivative may be non-existent on an everywhere dense set and so may be discontinuous everywhere. The latter fact in particular, shows that the class $\overline{\mathcal{F}}(I; c_1, c_2)$ is genuinely bigger than $\mathcal{F}(I; c_1, c_2)$ and thus the notion of weak non-degeneracy is not vacuous (see also Remark 1 in Section 1). Throughout this section, $I = [x_1, x_2]$ is a compact interval and $c_2 \geqslant c_1 > 0$.

We begin by investigating the convexity properties of functions f in $\mathcal{F}(I; c_1, c_2)$. The function $f : I \to \mathbb{R}$ will be called (c_1, c_2)-*convex* if for every $x \in I$ and every $\delta > 0$ such that $x \pm \delta \in I$

$$c_1 \leqslant \frac{f(x + \delta) - 2f(x) + f(x - \delta)}{\delta^2} \leqslant c_2. \tag{11}$$

In what follows $\mathcal{C}(I; c_1, c_2)$ will be the set of (c_1, c_2)-convex functions.

2.1. Historical note

In the case $c_2 = +\infty$ the r.h.s. of (11) imposes no restriction on f. Consequently, f is called c_1-*convex* or simply *strongly convex*. The strongly convex functions (also known as *uniformly convex functions*) were introduced by Levitin and Poljak [21], and have been widely used over the past 50 years mostly in optimisation and mathematical finance. Geometrically, the function f is c_1-convex if for every x in the interior of I the radius R of the supporting circle of the graph $\mathcal{C}_f$ is bounded above by c_1^{-1} (see also [28,29] for other properties of strongly convex functions). In the case $c_2 < +\infty$ the r.h.s. inequality of (11) implies that R is also bounded below by c_2^{-1}. In the case $f \in C^2(I)$ we have that $f \in \mathcal{C}(I; c_1, c_2)$ if and only if $c_1 \leqslant f''(x) \leqslant c_2$ for all $x \in I$. In fact, the following theorem shows that $\mathcal{C}(I; c_1, c_2)$ coincides with the topological closure of the set of functions satisfying the latter condition.

Theorem 5. $\overline{\mathcal{F}}(I; c_1, c_2) = \mathcal{C}_{\pm}(I; c_1, c_2) := \{\pm f : f \in \mathcal{C}(I; c_1, c_2)\}.$

We will use well-known properties of convolution. Given $\phi, \psi : \mathbb{R} \to \mathbb{R}$, the *convolution* of ϕ and ψ is the function $(\phi \star \psi) : \mathbb{R} \to \mathbb{R}$ defined by

$$(\phi \star \psi)(x) := \int\limits_{-\infty}^{+\infty} \phi(x-t)\psi(t)\, dt.$$

There are various assumption ensuring its existence. We will use the following well-known

Lemma 1. *If $\phi \in C^\infty(\mathbb{R})$, $\int_{-\infty}^{+\infty} |\phi(x)|\, dx < \infty$ and $\psi : \mathbb{R} \to \mathbb{R}$ is bounded and integrable, then $\phi \star \psi \in C^\infty(\mathbb{R})$.*

Proof of Theorem 5. As noted above $\mathcal{F}(I; c_1, c_2) \subset \mathcal{C}_\pm(I; c_1, c_2)$. In fact, the latter is an easy consequence of Taylor's formula. Clearly, taking the limit $f \to f_0$, where $f \in \mathcal{F}(I; c_1, c_2)$ and $f_0 \in \overline{\mathcal{F}}(I; c_1, c_2)$, preserves (11), thus showing the inclusion $\overline{\mathcal{F}}(I; c_1, c_2) \subset \mathcal{C}_\pm(I; c_1, c_2)$. The main substance of the proof is therefore to establish that $\mathcal{C}_\pm(I; c_1, c_2) \subset \overline{\mathcal{F}}(I; c_1, c_2)$. Let $f \in \mathcal{C}(I; c_1, c_2)$. Define $\hat{f} : \mathbb{R} \to \mathbb{R}$ by setting

$$\hat{f}(x) := \begin{cases} f(x) & \text{if } x \in I = [x_1, x_2], \\ f(x_1) & \text{if } x < x_1, \\ f(x_2) & \text{if } x > x_2. \end{cases}$$

Clearly, $\hat{f}$ is uniformly bounded and continuous on $\mathbb{R}$ and identically equal to f on I. Further, since I is compact and $\hat{f}$ is constant outside I, it is easily seen that $\hat{f}$ is uniformly continuous on $\mathbb{R}$. Define $B : \mathbb{R} \to \mathbb{R}$ by setting

$$B(x) := \begin{cases} \exp(-\frac{1}{(x-1)^2} - \frac{1}{(x+1)^2}) & \text{if } |x| < 1, \\ 0 & \text{otherwise.} \end{cases} \tag{12}$$

It is easily verified that $B \in C^\infty(\mathbb{R})$ and is supported on $[-1, 1]$. Then

$$w := \int\limits_{-\infty}^{+\infty} B(x)\, dx = \int\limits_{-1}^{1} B(x)\, dx. \tag{13}$$

Given an $\varepsilon > 0$, define $f_\varepsilon : \mathbb{R} \to \mathbb{R}$ by

$$f_\varepsilon(x) := \frac{1}{w\varepsilon} \int\limits_{-\infty}^{+\infty} B\left(\frac{x-y}{\varepsilon}\right) \hat{f}(y)\, dy. \tag{14}$$

The function $f_\varepsilon(x)$ is $1/(w\varepsilon)$ times the convolution of $\hat{f}(x)$ and $B(\frac{x}{\varepsilon})$. By Lemma 1, $f_\varepsilon(x) \in C^\infty(\mathbb{R})$. Making the change of variables $z = \frac{x-y}{\varepsilon}$ transforms (14) into

$$f_\varepsilon(x) = \frac{1}{w} \int\limits_{-\infty}^{+\infty} B(z) \hat{f}(x - \varepsilon z)\, dz = \frac{1}{w} \int\limits_{-1}^{1} B(z) \hat{f}(x - \varepsilon z)\, dz. \tag{15}$$

By the uniform continuity of $\hat{f}$, for any $\eta > 0$ there is an $\varepsilon > 0$ such that

$$\sup_{|x'-x|\leqslant\varepsilon} |\hat{f}(x') - \hat{f}(x)| < \eta. \tag{16}$$

Then

$$\sup_{x\in\mathbb{R}} |f_\varepsilon(x) - \hat{f}(x)| \overset{(13),\,(15)}{=} \sup_{x\in\mathbb{R}} \left| \frac{1}{w} \int\limits_{-1}^{1} B(z)\big(\hat{f}(x - \varepsilon z) - \hat{f}(x)\big)\,dz \right|$$

$$\leqslant \frac{1}{w} \int\limits_{-1}^{1} B(z)\,dz \sup_{|x'-x|\leqslant\varepsilon} |\hat{f}(x') - \hat{f}(x)|$$

$$\overset{(13),\,(16)}{<} \eta.$$

This means that f_ε converges to $\hat{f}$ uniformly on $\mathbb{R}$ as $\varepsilon \to 0$.

Since $f_\varepsilon \in C^\infty(\mathbb{R})$, using Taylor's formula we verify that for all $x \in \mathbb{R}$

$$\lim_{\delta\to0} \frac{f_\varepsilon(x+\delta) - 2f_\varepsilon(x) + f_\varepsilon(x-\delta)}{\delta^2} = f_\varepsilon''(x). \tag{17}$$

By (15),

$$\frac{f_\varepsilon(x+\delta) - 2f_\varepsilon(x) + f_\varepsilon(x+\delta)}{\delta^2}$$

$$= \frac{1}{w} \int\limits_{-1}^{1} B(z) \frac{\hat{f}(x - \delta z + \delta) - 2\hat{f}(x - \delta z) + \hat{f}(x - \delta z - \delta)}{\delta^2}\,dz. \tag{18}$$

When $x \in [x_1 + 2\delta, x_2 - 2\delta]$, where $[x_1, x_2] = I$, we have that $x - \delta z \pm \delta \in I$ for any $z \in [-1, 1]$. Then, since $\hat{f} \in \mathcal{C}(I; c_1, c_2)$, the fraction within the r.h.s. of (18) is bounded between c_1 and c_2. Consequently, by (13), the l.h.s. of (18) is bounded between c_1 and c_2 for all $x \in [x_1 + 2\delta, x_2 - 2\delta]$. By (17),

$$c_1 \leqslant f_\varepsilon''(x) \leqslant c_2 \tag{19}$$

for all $x \in [x_1 + 2\delta, x_2 - 2\delta]$. Since δ can be made arbitrarily small and $f_\varepsilon''(x)$ is continuous on I, (19) must hold on I. This means that $f_\varepsilon \in \mathcal{F}(I; c_1, c_2)$ and consequently f belongs to $\overline{\mathcal{F}}(I; c_1, c_2)$ as a uniform limit of f_ε. In the case $-f \in \mathcal{C}(I; c_1, c_2)$ taking $-f_\varepsilon$ does the job and completes the proof. $\square$

We now utilise the characterisation of functions in $\overline{\mathcal{F}}(I; c_1, c_2)$ given by Theorem 5 to show that these functions are continuously differentiable.

Theorem 6. $\mathcal{C}(I; c_1, c_2) \subset C^1(I)$.

Proof. Let $f \in \mathcal{C}(I; c_1, c_2)$. As a convex function f has left and right derivatives $f'_+(x)$ and $f'_-(x)$ at each point x of the interior of I (to be denoted as $\operatorname{int} I$) – see e.g. [22, Theorem 1.3.3]. Furthermore, for all $x, y \in \operatorname{int} I$ such that $x < y$

$$f'_-(x) \leqslant f'_+(x) \leqslant f'_-(y) \leqslant f'_+(y). \tag{20}$$

By definition, $f'_-(x) = f'_+(x)$ if and only if f is differentiable at x. Assume for the moment that f is not differentiable at some point $x_0 \in \operatorname{int} I$, that is, by (20),

$$f'_-(x_0) < f'_+(x_0). \tag{21}$$

Define the auxiliary function $t : I \to \mathbb{R}$ by setting

$$t(x) := \begin{cases} f'_-(x) & \text{if } x < x_0, \\ f'_+(x) & \text{if } x \geqslant x_0. \end{cases} \tag{22}$$

This function is known as a *subdifferential* for f (see [22, §5] for its definition and basic properties). By Theorem 1.6.1 of [22], for any $a, b \in I$ we have

$$f(b) - f(a) = \int_a^b t(x)\, dx. \tag{23}$$

The latter equality implies that

$$
\begin{aligned}
\frac{f(x_0 + \delta) + f(x_0 - \delta) - 2f(x_0)}{\delta^2} &= \frac{\int_{x_0}^{x_0+\delta} f'_+(x)\, dx - \int_{x_0-\delta}^{x_0} f'_-(x)\, dx}{\delta^2} \\
&\overset{(20)}{\leqslant} \frac{f'_+(x_0)\delta - f'_-(x_0)\delta}{\delta^2} \\
&= \frac{f'_+(x_0) - f'_-(x_0)}{\delta}.
\end{aligned}
$$

By (21), the latter fraction tends to infinity as $\delta \to 0$. This contradicts to the fact that $f \in \mathcal{C}(I; c_1, c_2)$. The contradiction shows that f is differentiable.

To complete the proof we have to verify that f' is continuous. Property (20) implies that the function f' is monotonically increasing on I. So, by the Monotone Convergence Theorem, for any $x_0 \in \operatorname{int} I$ there exist left and right limits of f' and furthermore we have that

$$\lim_{x \to x_0^-} f'(x) \leqslant \lim_{x \to x_0^+} f'(x). \tag{24}$$

By definition,

$$f'_-(x_0) = \lim_{\delta \to 0+} \frac{f(x_0) - f(x_0 - \delta)}{\delta}. \tag{25}$$

By Theorem 1.6.1 of [22], for $\delta > 0$ we have

$$f(x_0) - f(x_0 - \delta) = \int\limits_{x_0 - \delta}^{x_0} f'(x)\,dx \tag{26}$$

and, by the monotonicity of f',

$$\int\limits_{x_0 - \delta}^{x_0} f'(x)\,dx \leqslant \delta \lim_{x \to x_0^-} f'(x). \tag{27}$$

Combining (25), (26) and (27) shows that

$$f'_-(x_0) \leqslant \lim_{x \to x_0^-} f'(x). \tag{28}$$

Similarly, we establish that

$$\lim_{x \to x_0^+} f'(x) \leqslant f'_+(x_0). \tag{29}$$

By the differentiability of f, $f'_-(x_0) = f'_+(x_0)$. Thus (28), (29) together with (24) imply that $\lim_{x \to x_0^-} f'(x) = \lim_{x \to x_0^+} f'(x)$ and complete the proof. $\square$

Since functions in $\overline{\mathcal{F}}(I; c_1, c_2)$ are continuously differentiable, we are able to simplify property (11) and thus give an alternative description of the class $\overline{\mathcal{F}}(I; c_1, c_2)$. With this goal in mind we now introduce further notation. Let $\mathcal{C}^1(I; c_1, c_2)$ be the set of $C^1(I)$ functions such that

$$c_1 \leqslant \frac{f'(x + \delta) - f'(x)}{\delta} \leqslant c_2 \tag{30}$$

for any $x \in I$ and $\delta > 0$ with $x + \delta \in I$.

Theorem 7. $\overline{\mathcal{F}}(I; c_1, c_2) = \mathcal{C}^1_\pm(I; c_1, c_2) := \{\pm f\colon\ f \in \mathcal{C}^1(I; c_1, c_2)\}.$

Proof. Using the Mean Value Theorem, it is easily seen that $\mathcal{F}(I; c_1, c_2) \subset \mathcal{C}^1_\pm(I; c_1, c_2)$. By Theorem 6, we are able to take the limit (in the uniform convergence topology) within (30) as $f \to f_0$, where $f \in \mathcal{F}(I; c_1, c_2)$ and $f_0 \in \overline{\mathcal{F}}(I; c_1, c_2)$. Thus we trivially have that $\overline{\mathcal{F}}(I; c_1, c_2) \subset \mathcal{C}^1_\pm(I; c_1, c_2)$. To show the opposite inclusion we use functions f_ε similarly to the proof of Theorem 6. The details are easy and are left as an exercise. $\square$

Remark 2. The obvious consequence of Theorem 7, or indeed (30), is that the functions in $\overline{\mathcal{F}}(I; c_1, c_2)$ have bi-Lipschitz derivatives. However, the following example shows that their second derivative may be non-existent on an everywhere dense set.

Example 1. Let $A \subset (x_1, x_2)$ be a countable set (e.g., $A = (x_1, x_2) \cap \mathbb{Q}$), where $[x_1, x_2] = I$. We will construct a function f which fails to have the second derivative exactly on A. Since A is countable, we can fix a bijection $\phi : A \to \mathbb{N}$. Let (c_n) be a sequence of positive numbers such that $\sum_{n=1}^{\infty} c_n < \infty$, e.g., $c_n = n^{-2}$. Let $T := \sum_{n=1}^{\infty} c_n$ and

$$t(x) := 1 + \sum_{a \in A,\, a < x} c_{\phi(a)}. \tag{31}$$

The function $t(x)$ is well defined because the sum $\sum c_n$ is absolutely convergent. It is easy to verify that $t(x)$ is strictly increasing and positive on I. Moreover, $t(x)$ is continuous at any point of $I \setminus A$ and discontinuous at any point of A – see e.g. [17, p. 18]. Since $T = \sum c_n$, $t(x)$ is bounded above by $T + 1$ for any $x \in I$. Now define $v : I \to \mathbb{R}$ by setting

$$v(x) := \int_{x_1}^{x} t(z)\, dz \quad \text{for } x \in I.$$

The function v is well defined as t is continuous almost everywhere. Also v is strictly increasing because $t(x) \geqslant 1$ for all $x \in I$. Also, as t is bounded on I, v is continuous at every point of I, and as t is continuous at every point $A \setminus I$, v is differentiable at every point of this set. On the contrary, if $a \in A$, by the definition of t, one readily computes that $v(a + \delta) - v(a - \delta) \geqslant c_{\phi(a)}$ for any $\delta > 0$ and therefore v is not differentiable at a. Finally, let

$$f(x) := \int_{x_1}^{x} v(z)\, dz \quad \text{for } x \in I.$$

As v is continuous, f is continuously differentiable and $f' = v$. However, by what we have seen above, f fails to have the second derivative on A. Our final goal is to verify that f satisfies (30). Given the definition of f and v, (30) transforms into

$$c_1 \leqslant \frac{1}{\delta} \int_{x}^{x+\delta} t(z) \leqslant c_2$$

when $x \in I$, $\delta > 0$ and $x + \delta \in I$. The latter inequalities are satisfied with $c_1 = 1$ and $c_2 = T + 1$ because $1 \leqslant t(x) \leqslant 1 + T$ for all $x \in I$. Thus, $f \in \mathcal{C}^1_{\pm}(I; c_1, c_2) = \overline{\mathcal{F}}(I; c_1, c_2)$.

Remark 3. By Alexandrov's theorem [22, Theorem 3.11.2], any convex function has the second derivative almost everywhere. Thus, the functions in $\overline{\mathcal{F}}(I; c_1, c_2)$ are twice differentiable almost everywhere. It is easy to deduce from the fact that $\mathcal{C}(I; c_1, c_2) = \overline{\mathcal{F}}(I; c_1, c_2)$ that if the second derivative of $f \in \overline{\mathcal{F}}(I; c_1, c_2)$ exists at some point x_0, it necessarily satisfies the inequalities $c_1 \leqslant |f''(x_0)| \leqslant c_2$. Although, f'' exists at every point except a set A of Lebesgue measure 0, this exceptional set A can be everywhere dense and so f'' may be discontinues everywhere on I. Note also that the above example can be modified to show that the set of points where the second derivative does not exist is an uncountable set of Hausdorff dimension 1.

3. Reduction to C^2 functions

The goal of this section is to show that it is sufficient to prove Theorem 2 for $\mathcal{F}(I; c_1, c_2)$ only. This follows from the following

Theorem 8. *Let $\mathcal{F}$ be a set of continuous functions on an interval I and $\overline{\mathcal{F}}$ be the closure of $\mathcal{F}$ in the uniform convergence topology. Let positive numbers c, Q, δ, ρ and a subinterval $J \subset I$ be fixed. Then*

$$\inf_{\bar{f} \in \overline{\mathcal{F}}} \left| \Delta_{\bar{f}}^c(Q, \delta, J, \rho) \cap J \right| = \inf_{f \in \mathcal{F}} \left| \Delta_f^c(Q, \delta, J, \rho) \cap J \right|. \tag{32}$$

Proof. Since $\mathcal{F} \subset \overline{\mathcal{F}}$, the l.h.s. of (32) is less than or equal to the r.h.s. of (32). By the definition of $\Delta_f^\delta(Q, \delta, J, \rho)$, to complete the proof it suffices to verify that for any $\bar{f} \in \overline{\mathcal{F}}$ there exists $f \in \mathcal{F}$ such that $\mathcal{R}_f^c(Q, \delta, J) \subset \mathcal{R}_{\bar{f}}^c(Q, \delta, J)$.

Let $\mathcal{R}^*$ denote the set of coprime integer triples (q, p_1, p_2) such that $cQ \leqslant q \leqslant Q$, $p_1/q \in J$ and $|\bar{f}(p_1/q) - p_2/q| \leqslant 1 + \delta/Q$. It is easy to see that $\mathcal{R}^*$ is finite and strictly larger than $\mathcal{R}_{\bar{f}}^c(Q, \delta, J)$. Therefore

$$\varepsilon^* := \min\left\{ \left| \bar{f}(p_1/q) - p_2/q \right| - \delta/Q \colon (q, p_1, p_2) \in \mathcal{R}^* \setminus \mathcal{R}_{\bar{f}}^c(Q, \delta, J) \right\}$$

is positive and well defined. Let $\varepsilon = \min\{1, \varepsilon^*\}$. Take any function $f \in \mathcal{F}$ such that $\sup_{x \in I} |f(x) - \bar{f}(x)| < \varepsilon$. Since $\overline{\mathcal{F}}$ is the closure of $\mathcal{F}$ in the $C^0(I)$ topology, such a function f exists. Using the definitions of ε and f one readily verifies that $\mathcal{R}_f^c(Q, \delta, J) \subset \mathcal{R}^*$. Assume for the moment that there is a $(q, p_1, p_2) \in \mathcal{R}_f^c(Q, \delta, J) \setminus \mathcal{R}_{\bar{f}}^c(Q, \delta, J)$. Then, by the definition of ε^*, we have that

$$\left| \bar{f}(p_1/q) - p_2/q \right| \geqslant \delta/Q + \varepsilon^*. \tag{33}$$

On the other hand, since $(q, p_1, p_2) \in \mathcal{R}_f^c(Q, \delta, J)$, we have that

$$\left| \bar{f}(p_1/q) - p_2/q \right| \leqslant \left| \bar{f}(p_1/q) - f(p_1/q) \right| + \left| f(p_1/q) - p_2/q \right|$$
$$< \varepsilon + \delta/Q \leqslant \varepsilon^* + \delta/Q.$$

This contradicts to (33), thus showing the inclusion $\mathcal{R}_f^c(Q, \delta, J) \subset \mathcal{R}_{\bar{f}}^c(Q, \delta, J)$ and completing the proof. $\square$

Similar arguments are applied to Theorem 3. The details are easy and left to the reader.

4. An explicit version of Theorem BKM

Theorem BKM mentioned in the above heading is Theorem 1.4 from [16] due to Bernik, Kleinbock and Margulis. We will be interested in the case $n = 2$. The goals of this section are (i) to generalise it to weakly non-degenerate maps; and (ii) to make it effective and indeed fully explicit. Our approach to achieving these goals develops the ideas of [5] and [16].

4.1. Statement of results

We will be interested in maps $\mathbf{g} = (g_1, g_2) : I \to \mathbb{R}^2$ given by

$$g_1(x) = xf'(x) - f(x), \qquad g_2(x) = -f'(x) \tag{34}$$

for some function $f \in \overline{\mathcal{F}}(I; c_1, c_2)$. This definition coincides with the one of [10, §4]. Geometrically, the vector $(g_1(x), g_2(x), 1) \in \mathbb{R}^3$ is defined to be the cross-product of $(1, x, f(x))$ and $(0, 1, f'(x))$ and thus is orthogonal to the latter two vectors.

Given positive numbers δ, K, T and a subinterval $J \subset I$, let $B_{\mathbf{g}}(J, \delta, K, T)$ denote the set of $x \in J$ for which there exists $(q, p_1, p_2) \subset \mathbb{Z}^3 \setminus \{0\}$ such that

$$\begin{cases} |qg_1(x) + p_1g_2(x) + p_2| \leqslant \delta, \\ |qg_1'(x) + p_1g_2'(x)| \leqslant K, \\ |q| \leqslant T. \end{cases} \tag{35}$$

Theorem 9. *Let c_1 and c_2 be positive constants, $I \subset \mathbb{R}$ be a compact interval and*

$$L := \max_{x \in I} |x|. \tag{36}$$

Then for any $f \in \overline{\mathcal{F}}(I; c_1, c_2)$, any interval $J \subseteq I$ and any choice of δ, K, T satisfying

$$0 < \delta \leqslant 1, \quad K > 0, \quad T > 1, \quad \delta K T \leqslant 1 \tag{37}$$

we have that

$$\left| B_{\mathbf{g}}(J, \delta, K, T) \right| \leqslant E(\delta K T)^{\frac{1}{6}} |J|, \tag{38}$$

where $\mathbf{g}$ is given by (34),

$$E := \frac{648C}{\sqrt{\rho}}, \tag{39}$$

$$C = \max\{C_0\sqrt{32}, 24\sqrt{6C_0 M}\}, \qquad C_0 = \frac{4c_2}{c_1}, \qquad M = \sqrt{1 + 4L^2}, \tag{40}$$

$$\rho = \min\left\{1, c_1, \frac{c_1|J|\theta}{32M} \max\left\{\frac{16}{\delta}, \frac{|J|}{K}\right\}, \frac{c_1^2|J|^2 T}{32\theta}\right\}, \qquad \theta = (\delta K T)^{\frac{1}{3}}. \tag{41}$$

Remark 4. For T sufficiently large the constant E appearing in (38) is determined by c_1, c_2 and L only and thus is independent from J, δ, K, T. In order to see this, use the inequality $\max(x, y) \geqslant (xy)^{1/2}$ valid for all positive x, y to get

$$\frac{c_1|J|\theta}{32M} \max\left\{\frac{16}{\delta}, \frac{|J|}{K}\right\} \geqslant \frac{c_1|J|\theta}{32M} \left(\frac{16|J|}{\delta K}\right)^{1/2} \stackrel{(41)}{=} \frac{c_1|J|\theta}{32M} \left(\frac{16|J|T}{\theta^3}\right)^{1/2}.$$

Then, since $\theta < 1$, it becomes clear that $\rho = \min\{1, c_1\}$ and so

$$E = \frac{648\, C}{\min(1, \sqrt{c_1})} \quad \text{when } T \geqslant \max\{64M^2|J|^{-3},\, 32c_1^{-1}|J|^{-2}\}. \tag{42}$$

Furthermore, in the case $\delta \leqslant K$ we have a better estimate for T in terms of $|J|$. To see this, note that $\delta^2 \leqslant \delta K = \theta^3 T^{-1} \leqslant \theta^2 T^{-1}$ and so $\delta \leqslant \theta T^{-1/2}$. Then

$$\frac{c_1|J|\theta}{32M} \max\left\{\frac{16}{\delta}, \frac{|J|}{K}\right\} \geqslant \frac{16c_1|J|\theta T^{1/2}}{32M\theta} = \frac{c_1|J|T^{1/2}}{2M}$$

and one readily computes that

$$E = \frac{648\, C}{\min(1, \sqrt{c_1})} \quad \text{when } T \geqslant \frac{\max\{4M^2,\, 32c_1^{-2}\}}{|J|^2} \quad \text{and} \quad \delta \leqslant K. \tag{43}$$

Remark 5. Theorem 9 is the main stepping stone to the proof of Theorem 2. Furthermore, its value is not limited to this application. For example, Theorem 9 can be used to extended the main result of [16] (due to Bernik, Kleinbock and Margulis) to the set of weakly non-degenerate planar curves. Yet another application lies within the results of [7] (due to Beresnevich, Bernik and Götze) on the distribution of close conjugate algebraic numbers which can now be improved towards full effectiveness in the case of quadratic and integer cubic algebraic numbers.

Remark 6. Despite the fact that the functions $f \in \mathcal{F}(I; c_1, c_2)$ (and consequently $\mathbf{g}$) are initially defined on the interval I only, we can always treat them as C^2 functions defined on the whole real line preserving condition (1). Indeed, let $T_2(z, x) := \sum_{i=0}^{2} \frac{1}{i!} f^{(i)}(z)(x - z)$ denote the Taylor polynomial of degree 2 and consider the auxiliary function

$$\tilde{f}(x) = \begin{cases} T_2(x_1, x) & \text{if } x < x_1, \\ f(x) & \text{if } x_1 \leqslant x \leqslant x_2, \\ T_2(x_2, x) & \text{if } x_2 < x, \end{cases}$$

where $[x_1, x_2] = I$. It is then easily verified that $\tilde{f}$ is $C^2(\mathbb{R})$, satisfies (1) for all $x \in \mathbb{R}$ and coincides with f on I. Hence the above claim follows.

4.2. (C, α)-good functions

The property of being (C, α)-good introduced in [20] by Kleinbock and Margulis lies at the heart of the proof of Theorem 9. In this subsection we recall the key definition and various auxiliary statements from [16] and [20]. We also establish a new lemma that provides sufficient conditions for a function to be (C, α)-good – Lemma 6 below.

Let C and α be positive numbers and V be a subset of $\mathbb{R}^d$. The function $f : V \to \mathbb{R}$ is said to be (C, α)-*good on* V if for any open ball $B \subset V$ and any $\varepsilon > 0$ one has

$$\left|\left\{x \in B \colon \left|f(x)\right| < \varepsilon \cdot \sup_{x \in B}\left|f(x)\right|\right\}\right| \leqslant C\varepsilon^\alpha |B|. \tag{44}$$

Here, as before, $|A|$ denotes the Lebesgue measure of $A \subset \mathbb{R}^d$. Within this paper we shall only use the above definition in the case $d = 1$. Several elementary properties of (C, α)-good functions are now recalled.

Lemma 2. *(See Lemma 3.1 in [16].)*

(a) *If f is (C, α)-good on V, then so is λf for any $\lambda \in \mathbb{R}$;*
(b) *If $f_1, \ldots, f_k$ are (C, α)-good on V, then so is $\max_{1 \leqslant i \leqslant k} |f_i|$;*
(c) *If f is (C, α)-good on V and $c_1 \leqslant \frac{|f(x)|}{|g(x)|} \leqslant c_2$ for all $x \in V$, then g is $(C(c_2/c_1)^\alpha, \alpha)$-good on V;*
(d) *If f is (C, α)-good on V, then f is (C', α')-good on V' for every $C' \geqslant C$, $\alpha' \leqslant \alpha$ and $V' \subset V$.*

Lemma 3. *(See Lemma 3.2 in [20].) For any $k \in \mathbb{N}$, any polynomial $f \in \mathbb{R}[x]$ of degree not greater than k is $(2k(k+1)^{1/k}, 1/k)$-good on $\mathbb{R}$.*

Before presenting the final lemma of this subsection, the following two technical statements are established.

Lemma 4. *Let J be an interval, $\lambda > 0$ and $\theta : J \to \mathbb{R}$ be a C^1 function such that $\inf_{x \in J} |\theta'(x)| \geqslant \lambda$. Then $\sup_{x \in J} |\theta(x)| \geqslant \frac{1}{2}\lambda|J|$.*

Proof. Let y_1 and y_2 be the endpoints of J. Then, by the Mean Value Theorem, for any sufficiently small $\varepsilon > 0$ we have that $|\theta(y_1 + \varepsilon) - \theta(y_2 - \varepsilon)| \geqslant \lambda(y_2 - y_1 - 2\varepsilon) = \lambda(|J| - 2\varepsilon)$. Therefore, $|\theta(y_1 + \varepsilon)| + |\theta(y_2 - \varepsilon)| \geqslant \lambda(y_2 - y_1 - 2\varepsilon) = \lambda(|J| - 2\varepsilon)$. Hence

$$2 \sup_{x \in J} |\theta(x)| \geqslant \lambda(|J| - 2\varepsilon). \tag{45}$$

For $\varepsilon > 0$ is arbitrarily, (45) implies $\sup_{x \in J} |\theta(x)| \geqslant \frac{1}{2}\lambda|J|$. $\square$

Lemma 5. *Let f be a C^1 function on an interval B such that*

$$\sup_{x \in B} |f(x)| \geqslant 2 \inf_{x \in B} |f(x)|. \tag{46}$$

Then

$$\sup_{x \in B} |f(x)| \leqslant 2 \int_B |f'(x)| \, dx. \tag{47}$$

Proof. By the Fundamental Theorem of Calculus, for any $y_1, y_2 \in B$ we have that $f(y_2) = f(y_1) + \int_{y_1}^{y_2} f'(x) \, dx$. Then, using triangle inequality gives $|f(y_2)| \leqslant |f(y_1)| + \int_B |f'(x)| \, dx$. Further, taking supremum over $y_2 \in B$ and infimum over $y_1 \in B$ gives the estimate

$$\sup_{x \in B} |f(x)| \leqslant \inf_{x \in B} |f(x)| + \int_B |f'(x)| \, dx \overset{(46)}{\leqslant} \frac{1}{2} \sup_{x \in B} |f(x)| + \int_B |f'(x)| \, dx$$

which readily implies (47). $\square$

Lemma 6. *Let κ_0, κ_1 and κ_2 be some positive constants and f be a C^1 function on an interval I such that for any subinterval $B \subset I$*

$$\sup_{x \in B} |f(x)| \geqslant \kappa_0 |B|^2 \tag{48}$$

and

$$\sup_{x \in B} |f'(x)| \leqslant \kappa_1 \inf_{x \in B} |f'(x)| + \kappa_2 |B|. \tag{49}$$

Assume that f' has at most r roots in I. Then f is $(C_\kappa, \frac{1}{2})$-good on I with

$$C_\kappa := \max\left\{4, 4(\kappa_1 \kappa + \kappa_2)\kappa^{-1}, (r+1)\sqrt{2(\kappa_1 \kappa + \kappa_2)\kappa_0^{-1}}\right\}, \quad \kappa > 0.$$

Proof. Fix any $\kappa > 0$. By definition, in order to prove that f is $(C_\kappa, \frac{1}{2})$-good on I we have to verify that for any interval $B \subset I$ the set

$$B_\varepsilon := \left\{x \in B: |f(x)| < \varepsilon \cdot \sup_{x \in B} |f(x)|\right\} \tag{50}$$

satisfies

$$|B_\varepsilon| \leqslant C_\kappa \varepsilon^{\frac{1}{2}} |B|. \tag{51}$$

Since $C_\kappa \geqslant 4$, for any $\varepsilon \geqslant \frac{1}{2}$ the r.h.s. of (51) does not fall below $|B|$ and (51) is a priori true. Henceforth, we may assume that $\varepsilon < \frac{1}{2}$. Consequently, if $\sup_{x \in B} |f(x)| \leqslant 2 \inf_{x \in B} |f(x)|$ then the set in the l.h.s. of (51) is empty and (51) is trivially satisfied. Otherwise, by Lemma 5, we have the inequality

$$\sup_{x \in B} |F(x)| \leqslant 2 \int_B |F'(x)| \, dx \tag{52}$$

assumed for the rest of the proof, which will depend upon the magnitude of $\lambda := \inf_{x \in B} |f'(x)|$. By (49), $\sup_{x \in B} |f'(x)| \leqslant \kappa_1 \lambda + \kappa_2 |B|$. Combining the latter inequality with (52) gives

$$\sup_{x \in B} |f(x)| < 2(\kappa_1 \lambda + \kappa_2 |B|)|B|. \tag{53}$$

Case (i). Assume that $\lambda \geqslant \kappa |B|$. Then f' does not change sign on B and hence f is monotonic. Therefore, for any $\tau > 0$, the set $J = \{x \in B: |f(x)| < \tau\}$ is an interval (possibly empty). Then, by Lemma 4, $\tau \geqslant \frac{1}{2}\lambda |J|$, that is $|\{x \in B: |f(x)| < \tau\}| \leqslant \frac{2\tau}{\lambda}$. Taking $\tau = \varepsilon \cdot \sup_{x \in B} |f(x)|$ ensures that $J = B_\varepsilon$ and gives

$$|B_\varepsilon| \leqslant \frac{2\varepsilon \cdot \sup_{x \in B} |f(x)|}{\lambda} \overset{(53)}{<} \frac{4\varepsilon(\kappa_1 \lambda + \kappa_2 |B|)|B|}{\lambda}.$$

By the hypothesis that $\lambda \geqslant \kappa |B|$, we have that $\frac{(\kappa_1 \lambda + \kappa_2 |B|)}{\lambda} \leqslant \frac{(\kappa_1 \kappa + \kappa_2)}{\kappa}$ and further obtain the required estimate

$$|B_\varepsilon| \leqslant \frac{4(\kappa_1 \kappa + \kappa_2)}{\kappa} \varepsilon |B| \overset{(\varepsilon < \frac{1}{2})}{<} \frac{4(\kappa_1 \kappa + \kappa_2)}{\kappa} \varepsilon^{1/2} |B| \leqslant C_\kappa \varepsilon^{1/2} |B|.$$

Case (ii). Assume that $\lambda < \kappa |B|$. Since $f'(x)$ has at most r roots in B, the interval B can be split into at most $(r+1)$ subintervals such that f is monotonic on each of them. Consequently, for every $0 < \varepsilon < \frac{1}{2}$ the set B_ε is the union of at most $r+1$ intervals. Let B' denote the biggest interval in B_ε. Then

$$|B_\varepsilon| \leqslant (r+1)|B'|. \tag{54}$$

By (48),

$$\sup_{x \in B'} |f(x)| \geqslant \kappa_0 |B'|^2. \tag{55}$$

On the other hand, by the definition of B', we have that

$$\sup_{x \in B'} |f(x)| \leqslant \varepsilon \cdot \sup_{x \in B} |f(x)| \overset{(53)}{\leqslant} 2\varepsilon (\kappa_1 \lambda + \kappa_2 |B|)|B| \overset{\lambda < \kappa |B|}{\leqslant} 2\varepsilon (\kappa_1 \kappa + \kappa_2)|B|^2. \tag{56}$$

Comparing (55) and (56) gives a bound on $|B'|$, which together with (54) establishes (51) and thus completes the proof. $\quad\square$

4.3. Properties of certain families of functions

In this subsection we investigate certain families of functions for being (C, α)-good and other relevant properties. Unless otherwise stated f, g_1, g_2 and I are the same as at the beginning of Section 4. Although all the statements are established for f, in view of Remark 6 (on p. 3077) they are true for $\tilde{f}$ and with I replaced by any interval $\tilde{I}$, in particular, for $\tilde{I} = 3^3 I$, which is of greatest interest to us.

Lemma 7. *Let $f \in \mathcal{F}(I; c_1, c_2)$ and L be given by* (36)*. Define the constants*

$$M = \sqrt{1 + 4L^2}, \qquad C_0 = 4c_2 c_1^{-1} \quad \text{and} \quad C_1 = 2 \max\{C_0, \sqrt{32 C_0 M}\}. \tag{57}$$

Let $a, b \in \mathbb{R}$ satisfy $a^2 + b^2 \geqslant 1$ and $\eta : I \to \mathbb{R}$ satisfy

$$\eta'(x) = (ax + b) f''(x), \tag{58}$$

that is η is an antiderivative of $(ax + b) f''(x)$. Then

(a) *η' is $(C_0, 1)$-good on I;*

(b) *for any interval $B \subset I$* $\displaystyle \sup_{x \in B} |\eta'(x)| \geqslant \frac{c_1 |B|}{2M};$ (59)

(c) η is $(C_1, \frac{1}{2})$-good on I;

(d) for any interval $B \subset I$ $\sup_{x \in B} |\eta(x)| \geqslant \dfrac{c_1 |B|^2}{32M}$. $\qquad$ (60)

Proof. Part (a) is readily implied by Lemmas 3 and 2(c). Further, to prove part (b) we distinguish two cases. If $|a| \leqslant \frac{|b|}{2L}$ then $1 \leqslant a^2 + b^2 \leqslant b^2(\frac{1}{4L^2} + 1)$. Consequently $|b| \geqslant \frac{2L}{M}$ and therefore

$$\sup_{x \in B} |ax + b| \geqslant \inf_{x \in B} |ax + b| \geqslant |b| - |a| \sup_{x \in B} |x| \geqslant |b| - |a|L \geqslant \frac{|b|}{2} \geqslant \frac{L}{M}. \qquad (61)$$

By (36), $|B| \leqslant 2L$. Therefore, (61) together with (1) implies (59) in the case $|a| \leqslant \frac{|b|}{2L}$. Otherwise, $|a| > \frac{|b|}{2L}$ and then $1 \leqslant a^2 + b^2 < a^2(1 + 4L^2) = (aM)^2$. It follows that $|a| > \frac{1}{M}$. Therefore,

$$\sup_{x \in B} |ax + b| = |a| \sup_{x \in B} |x - (-b/a)| \geqslant |a| \frac{|B|}{2} \geqslant \frac{1}{2M} |B|. \qquad (62)$$

The latter together with (1) implies (59) and completes the proof of part (b).

We now prove part (d). As with the proof of part (b) we distinguish the following two cases: $|a| \leqslant \frac{|b|}{2L}$ and $|a| > \frac{|b|}{2L}$. In the first case, by (61) and the inequality $|B| \leqslant 2L$ implied by (36), we have that $|\eta'(x)| \geqslant \frac{c_1 |B|}{2M}$ for all $x \in B$. By Lemma 4, we then obtain (60). In the second case, split B into three subintervals: B_l, B_m and B_r, where B_m is the middle half of B; B_l is the left quarter of B; and B_r is the right quarter of B. Then applying (62) to B_m gives

$$\sup_{x \in B_m} |ax + b| \geqslant \frac{|B_m|}{2M} = \frac{|B|}{4M}. \qquad (63)$$

In the case under consideration $a \neq 0$ and therefore the function $ax + b$ is strictly monotonic. It follows that the supremum in (63) is attained at the endpoints of B_m. Hence, either $\inf_{x \in B_l} |ax + b|$ or $\inf_{x \in B_r} |ax + b|$ is bounded away from 0 by the right-hand side of (63). Then, by (1) and (58), we have

$$\max\left\{ \inf_{x \in B_l} |\eta'(x)|, \; \inf_{x \in B_r} |\eta'(x)| \right\} \geqslant \frac{c_1 |B|}{4M}. \qquad (64)$$

Applying Lemma 4 with J being equal to either B_l or B_r, $\theta = \eta$ and $\lambda =$ r.h.s. of (64) we again obtain (60).

Finally, in order to prove part (c) we will appeal to Lemma 6 with $f = \eta$. In view of Lemma 2(a), without loss of generality we can assume that $a^2 + b^2 = 1$ and so $|a| \leqslant 1$. By (58), for any $y_1, y_2 \in B$, we have that $\eta'(y_2) = \eta'(y_1)\frac{f''(y_2)}{f''(y_1)} + a(y_2 - y_1)f''(y_2)$. Then, using the triangle inequality, inequalities (1), $|y_2 - y_1| \leqslant |B|$ and $|a| \leqslant 1$ we get $|\eta'(y_2)| \leqslant \frac{c_2}{c_1}|\eta'(y_1)| + c_2|B|$. Taking supremum over $y_2 \in B$ and infimum over $y_1 \in B$ gives (49) with $\kappa_1 = c_2/c_1$ and $\kappa_2 = c_2$. Also (60) verifies (48) with $\kappa_0 = c_1/(32M)$. It is also easily seen that $r = 1$. Then, by Lemma 6 with $\kappa = c_1$ we get the conclusion of part (c). $\quad \square$

If g_1 and g_2 are given by (34), then $\eta(x) = c + ag_1(x) + bg_2(x)$, where $a, b, c \in \mathbb{R}$, satisfies $\eta'(x) = ag_1'(x) + bg_2'(x) = (ax - b)g''(x)$. Therefore, by Lemma 7, we get the following

Corollary 1. *Let g_1 and g_2 be given by* (34), $a, b, c \in \mathbb{R}$ *and* $a^2 + b^2 \geqslant 1$. *Let C_0, C_1 and M be the same as in Lemma 7. Then*

(a) $ag_1' + bg_2'$ *is* $(C_0, 1)$*-good on I;*
(b) $\sup_{x \in B} |ag_1'(x) + bg_2'(x)| \geqslant \frac{c_1 |B|}{2M}$ *for any subinterval $B \subset I$;*
(c) $ag_1 + bg_2 + c$ *is* $(C_1, \frac{1}{2})$*-good on I;*
(d) $\sup_{x \in B} |ag_1(x) + bg_2(x) + c| \geqslant \frac{c_1 |B|^2}{32M}$ *for any subinterval $B \subset I$.*

Lemma 8. *Let $a, b \in \mathbb{R}$ and $\tilde{\eta}(x) := (f(x) + ax + b) f''(x)$ for $x \in I$. Let $C_2 = C_0 \sqrt{32}$. Then*

(a) $\tilde{\eta}$ *is* $(C_2, \frac{1}{2})$*-good on I;*
(b) *for any interval $B \subset I$* $\quad \sup_{x \in B} |\tilde{\eta}(x)| \geqslant \dfrac{c_1^2}{32} |B|^2.$ $\qquad\qquad\qquad\qquad$ (65)

Proof. Let $\theta(x) = f(x) + ax + b$. Clearly $\theta''(x) = f''(x)$. Split B into three subintervals: B_l, B_m and B_r, where B_m is the middle half of B; B_l is the left quarter of B; and B_r is the right quarter of B. By (1) and Lemma 4, $\sup_{x \in B_m} |\theta'(x)| \geqslant \frac{c_1}{2} |B_m| = \frac{c_1}{4} |B|$. In view of (1), θ' is monotonic. Therefore, $\inf_{x \in B'} |\theta'(x)| \geqslant \frac{c_1}{4} |B|$ for at least one choice of B' from B_r and B_l. Applying Lemma 4 to B' further gives the estimate

$$\sup_{x \in B'} |\theta(x)| \geqslant \frac{1}{2} |B'| \frac{c_1}{4} |B| = \frac{c_1}{32} |B|^2. \qquad\qquad\qquad (66)$$

Using the identity $\tilde{\eta}(x) = \theta(x) f''(x)$ and (1) gives (65).

Further, for any $y_1, y_2 \in B$ we have $\theta'(y_2) = \theta'(y_1) + (f'(y_2) - f'(y_1))$. By the Mean Value Theorem and (1), we have $|f'(y_2) - f'(y_1)| \leqslant c_2 |B|$. Further, with reference to the former equality, taking supremum over $y_2 \in B$ and infimum over $y_1 \in B$ gives (49) with $\kappa_1 = 1$ and $\kappa_2 = c_2$. Also, (66) ensures (48) with $\kappa_0 = c_1/32$. Applying Lemma 6 with $\kappa = c_2$ gives that θ is $(8\sqrt{2C_0}, \frac{1}{2})$-good on I, where C_0 is defined by (57). Finally, since $\tilde{\eta}(x) = \theta(x) f''(x)$, using Lemma 2(c) and (1) establishes the statement of part (a). $\quad\square$

Our final statement of this subsection is concerned with the skew-gradient of pairs of functions

$$\mathbf{u} \cdot \hat{\mathbf{g}}(x) \quad \text{with } \mathbf{u} \in \mathbb{Z}^3 \setminus \{0\}, \qquad\qquad\qquad (67)$$

where $\hat{\mathbf{g}}(x) := (g_1(x), g_2(x), 1)$ and $\mathbf{u} \cdot \hat{\mathbf{g}}(x)$ is the scalar product of $\mathbf{u}$ and $\hat{\mathbf{g}}(x)$. The skew-gradient of a pair of functions (γ_1, γ_2) as defined in [16, §4] is given by

$$\tilde{\nabla}(\gamma_1, \gamma_2)(x) := \gamma_1(x) \gamma_2'(x) - \gamma_1'(x) \gamma_2(x).$$

Lemma 9. *Let $C_2 = C_0 \sqrt{32}$ be the same as in Lemma 8. Then for any $\mathbf{u}, \mathbf{w} \in \mathbb{Z}^3 \setminus \{0\}$ with $\mathbf{u} \wedge \mathbf{w} \neq 0$ and any subinterval $B \subset I$*

(a) $\tilde{\nabla}(\mathbf{u} \cdot \hat{\mathbf{g}}, \mathbf{w} \cdot \hat{\mathbf{g}})$ *is* $(C_2, \frac{1}{2})$*-good on I.*
(b) $\sup_{x \in B} |\tilde{\nabla}(\mathbf{u} \cdot \hat{\mathbf{g}}, \mathbf{w} \cdot \hat{\mathbf{g}})(x)| \geqslant \min \left\{ \dfrac{c_1^2 |B|^2}{32}, \dfrac{c_1 |B|}{2M} \right\}.$ $\qquad\qquad\qquad$ (68)

Proof. Given any $\mathbf{u}, \mathbf{w} \in \mathbb{Z}^3 \setminus \{0\}$ such that $\mathbf{u} \wedge \mathbf{w} \neq 0$, by the Laplace identity (see, e.g., [25, Lemma 6D, p. 105]),

$$\tilde{\nabla}(\mathbf{u} \cdot \hat{\mathbf{g}}, \mathbf{w} \cdot \hat{\mathbf{g}})(x) := \begin{vmatrix} \mathbf{u} \cdot \hat{\mathbf{g}}(x) & \mathbf{w} \cdot \hat{\mathbf{g}}(x) \\ \mathbf{u} \cdot \hat{\mathbf{g}}'(x) & \mathbf{w} \cdot \hat{\mathbf{g}}'(x) \end{vmatrix} = \big(\hat{\mathbf{g}}(x) \wedge \hat{\mathbf{g}}'(x)\big) \cdot (\mathbf{u} \wedge \mathbf{w}). \tag{69}$$

By (34) and the definition of exterior product, one easily verifies that $\hat{\mathbf{g}}(x) \wedge \hat{\mathbf{g}}'(x) = f''(x)(f(x), -x, 1)$. Also $\mathbf{u} \wedge \mathbf{w} = (p, q, r) \in \mathbb{Z}^3 \setminus \{0\}$. Thus, by (69),

$$\tilde{\nabla}(\mathbf{u} \cdot \hat{\mathbf{g}}, \mathbf{w} \cdot \hat{\mathbf{g}})(x) = f''(x)\big(pf(x) - qx + r\big). \tag{70}$$

If $p = 0$ then Lemma 7(a) implies Lemma 9(a) via Lemma 2(d); Lemma 7(b) implies Lemma 9(b). If $p \neq 0$ then $|p| \geqslant 1$. By this fact and Lemma 2, without loss of generality we can assume that $p = 1$ as otherwise we would divide (70) through by p. In this case Lemma 8 with $\tilde{\eta} = \tilde{\nabla}(\mathbf{u} \cdot \hat{\mathbf{g}}, \mathbf{w} \cdot \hat{\mathbf{g}})(x)$ completes the proof. $\square$

4.4. Proof of Theorem 9

Theorem 9 will be derived from a general result due to Kleinbock and Margulis appearing as Theorem 5.2 in [20]. In order to state it we recall some notation from [20]. In what follows $\mathcal{C}(\mathbb{Z}^k)$ will denote the set of all non-zero complete sublattices of $\mathbb{Z}^k$. An integer lattice $\Lambda \subset \mathbb{Z}^k$ is called *complete* if it contains all integer points lying in the linear space generated by Λ. Given a lattice $\Lambda \subset \mathbb{R}^k$ and a basis $\mathbf{w}_1, \ldots, \mathbf{w}_r$ of Λ, the multivector $\mathbf{w}_1 \wedge \cdots \wedge \mathbf{w}_r$ is uniquely defined up to sign since any two basis of Λ are related by a unimodular transformation. Therefore, the following function on the set of non-zero lattices is well defined:

$$\|\Lambda\| := |\mathbf{w}_1 \wedge \cdots \wedge \mathbf{w}_r|_\infty, \tag{71}$$

where $|\cdot|_\infty$ denotes the supremum norm on $\bigwedge(\mathbb{R}^k)$.

Theorem KM. *(See Theorem 5.2 in [20].) Let $d, k \in \mathbb{N}$, $C, \alpha > 0$ and $0 < \rho \leqslant 1$ be given. Let B be a ball in $\mathbb{R}^d$ and $h : 3^k B \to \mathrm{GL}_k(\mathbb{R})$ be given. Assume that for any $\Lambda \in \mathcal{C}(\mathbb{Z}^k)$*

(i) *the function $\mathbf{x} \mapsto \|h(\mathbf{x})\Lambda\|$ is (C, α)-good on $3^k B$, and*
(ii) *$\sup_{\mathbf{x} \in B} \|h(\mathbf{x})\Lambda\| \geqslant \rho$.*

Then there is a constant N_d depending on d only such that for any $\varepsilon > 0$ one has

$$\left| \left\{ \mathbf{x} \in B : \min_{\mathbf{a} \in \mathbb{Z}^k \setminus \{0\}} \big| h(\mathbf{x})\mathbf{a} \big|_\infty \leqslant \varepsilon \right\} \right| \leqslant kC \big(3^d N_d\big)^k \left(\frac{\varepsilon}{\rho}\right)^\alpha |B|. \tag{72}$$

We will use this general result in the case $k = 3$ and $d = 1$. Note that the Besicovitch constant N_d appearing in (72) equals 2 in the case $d = 1$.

Proof of Theorem 9. Since the right-hand side of (38) is independent of f and $\mathcal{F}(I; c_1, c_2)$ is dense in $\overline{\mathcal{F}}(I; c_1, c_2)$ (in the uniform convergence topology), it suffices to prove Theorem 9 for $f \in \mathcal{F}(I; c_1, c_2)$. Let $J \subset I$ be an interval and $\theta = (\delta KT)^{1/3}$. Define

$$t_1 = \frac{\theta}{\delta}, \qquad t_2 = \frac{\theta}{K}, \qquad t_3 = \frac{\theta}{T},$$

$$g_{\mathbf{t}} = \mathrm{diag}(t_1, t_2, t_3) \quad \text{and} \quad G_x = \begin{pmatrix} g_1(x) & g_2(x) & 1 \\ g_1'(x) & g_2'(x) & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{73}$$

It easily follows from the above definitions that

$$B(J, \delta, K, T) = \left\{ x \in J: \min_{\mathbf{a} \in \mathbb{Z}^3 \setminus \{0\}} |h(x)\mathbf{a}|_\infty \leqslant \theta \right\}, \quad \text{where } h(x) = g_{\mathbf{t}} G_x. \tag{74}$$

It is also readily seen that $\det g_{\mathbf{t}} = 1$ and

$$\det h(x) = \det G_x = -g_2'(x) \overset{(1)}{=} -f''(x) \neq 0. \tag{75}$$

Therefore, $h(x) \in \mathrm{GL}_3(\mathbb{R})$. In view of Remark 6 we will regard h as a map defined on $3^3 J$.

Our next goal is to verify conditions (i) and (ii) of Theorem KM for the specific choice of h made by (74). Fix a $\Gamma \in \mathcal{C}(\mathbb{Z}^3)$. Let $r = \dim \Gamma > 0$. We will consider the three cases $r = 1, 2, 3$ separately. It is easily seen that C defined by (40) satisfies $C = \max\{C_0, C_1, \tilde{C}_2\}$, where C_0 and C_1 are defined by (57) and $\tilde{C}_2$ is defined in the same way as C_2 within Lemma 8 but with M replaced by $27M$.

Case (1). Let $r = 1$. Then the basis of Γ consists of just one integer vector, say $\mathbf{w} = {}^t(w_1, w_2, w_3) \neq 0$ (here and elsewhere ^{t} denotes transposition). Consequently, $h(x)\mathbf{w}$ is a basis of $h(x)\Gamma$ and $\|h(x)\Gamma\| = |h(x)\mathbf{w}|_\infty$. Using (73) we get

$$h(x)\mathbf{w} = \begin{pmatrix} t_1 \mathbf{w} \cdot \hat{\mathbf{g}}(x) \\ t_2 \mathbf{w} \cdot \hat{\mathbf{g}}'(x) \\ t_3 w_1 \end{pmatrix}, \tag{76}$$

where $\mathbf{w} \cdot \hat{\mathbf{g}}(x) = w_1 g_1(x) + w_2 g_2(x) + w_3$ is the scalar product of $\mathbf{w}$ and $\hat{\mathbf{g}}(x) = {}^t(g_1(x), g_2(x), 1)$.

Subcase (1a). Assume that $w_1 = w_2 = 0$. Then $h(x)\mathbf{w} = {}^t(1, 0, 0)$ and it is easily verified using (44) and Lemma 2(b) that $\|h(x)\Gamma\|$ is $(C, 1/2)$-good on $3^3 J$. It is also clear that

$$\|h(x)\Gamma\| = 1 \quad \text{if } w_1^2 + w_2^2 = 0. \tag{77}$$

Subcase (1b). Assume that $(w_1, w_2) \neq \mathbf{0}$. Since $w_1, w_2 \in \mathbb{Z}$, we have $w_1^2 + w_2^2 \geqslant 1$. Then using Corollary 1(a)+(c) and Lemma 3 we verify that every coordinate function in (76) is $(C, \frac{1}{2})$-good on $3^3 J$. Consequently, by Lemma 2(b), $\|h(x)\Gamma\|$ is $(C, \frac{1}{2})$-good on $3^3 J$. Further, applying Corollary 1(b)+(d) to the first and second coordinate functions in (76) gives

$$\sup_{x \in J} \|h(x)\Gamma\| \geqslant \max\left\{ \frac{t_1 c_1 |J|}{2M}, \frac{t_2 c_1 |J|^2}{32M} \right\} \quad \text{if } w_1^2 + w_2^2 \neq 0. \tag{78}$$

Case (2). Let $r = 2$. Then the basis of Γ consists of two integer vectors, say $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{w} = (w_1, w_2, w_3)$ with $\mathbf{u} \wedge \mathbf{w} \neq 0$. Consequently, $h(x)\mathbf{u}$ and $h(x)\mathbf{w}$ is a basis of $h(x)\Gamma$ and

$\|h(x)\Gamma\| = |h(x)\mathbf{u} \wedge h(x)\mathbf{w}|_\infty$. Using (76) and a similar expression for $h(x)\mathbf{u}$ one readily verifies that

$$
h(x)\mathbf{u} \wedge h(x)\mathbf{w} = \begin{pmatrix} t_1 t_2 \tilde{\nabla}(\mathbf{u} \cdot \hat{\mathbf{g}}, \mathbf{w} \cdot \hat{\mathbf{g}})(x) \\ t_1 t_3 (w_1 \mathbf{u} - u_1 \mathbf{w}) \cdot \hat{\mathbf{g}}(x) \\ t_2 t_3 (w_1 \mathbf{u} - u_1 \mathbf{w}) \cdot \hat{\mathbf{g}}'(x) \end{pmatrix}. \tag{79}
$$

Using Corollary 1(a)+(c) and Lemma 9(a) we immediately verify $\|h(x)\Gamma\|$ is $(C, \frac{1}{2})$-good on $3^3 J$. Further, by Lemma 9(b),

$$
\sup_{x \in J} \|h(x)\Gamma\| \geqslant \sup_{x \in J} |t_1 t_2 \tilde{\nabla}(\mathbf{u} \cdot \hat{\mathbf{g}}, \mathbf{w} \cdot \hat{\mathbf{g}})(x)| \geqslant t_1 t_2 \frac{c_1^2}{32} |J|^2. \tag{80}
$$

Case (3). Let $r = 3$. Then, $\Gamma = \mathbb{Z}^3$. Consequently,

$$
\|h(x)\Gamma\| = |\det h(x)| = |\det G_x| = |g''(x)| \geqslant c_1. \tag{81}
$$

Completion of the proof. The upshot of (77), (78), (80) and (81) is that $\sup_{x \in J} \|h(x)\Gamma\| \geqslant \rho$, where ρ is given by (41). Thus Theorem KM is applicable with this value of ρ, $\alpha = \frac{1}{2}$ and C given by (40). Then, by (72) with $k = 3$ and $d = 1$ and (74), we get (38). $\quad\square$

5. Proof of Theorem 2

We follow the proof of Theorem 7 in [10] replacing the use of Lemma 6 of [10] with our Theorem 9. Recall that, by Theorem 8, it suffices to consider functions in $\mathcal{F}(I; c_1, c_2)$ only. Thus we fix any $f \in \mathcal{F}(I; c_1, c_2)$ and fix any non-empty interval $J \subseteq I$ of length $|J| \leqslant \frac{1}{2}$. Since the set of rational points and their denominators is invariant under translations by integer points, without loss of generality we can assume that $J \subset [-\frac{1}{2}; \frac{1}{2}]$. Consequently, $M \leqslant \sqrt{2}$ and $C \leqslant 36 c_2 / c_1$, where M and C are defined by (40). Then

$$
\frac{648C}{\min(1, \sqrt{c_1})} \leqslant \hat{E}
$$

where $\hat{E}$ is the constant defined by (6). Furthermore, (9) ensures that either (42) or (43) is applicable. Henceforth,

$$
E \leqslant \hat{E}. \tag{82}
$$

Let δ and Q satisfy (8) and (9). Let $\mathbf{g}$ be given by (34) and let $B_{\mathbf{g}}(\ldots)$ be defined in the same way as in Theorem 9, that is the set of $x \in J$ such that there exists a non-zero integer solution (q, p_1, p_2) to (35). By Theorem 9 and inequality (82),

$$
\left| B_{\mathbf{g}}\big(J, c_0 \delta, c_2 (c_0 Q \delta)^{-1}, 2 c_0 Q\big) \right| \leqslant \hat{E} |J| (2 c_0 c_2)^{\frac{1}{6}} \overset{(6)}{=} \frac{1}{4} |J|. \tag{83}
$$

Therefore the set $G := \frac{3}{4} J \setminus B_{\mathbf{g}}(J, c_0 \delta, c_2 (c_0 Q \delta)^{-1}, 2 c_0 Q)$ satisfies

$$
|G| \geqslant \frac{1}{2} |J|, \tag{84}
$$

where $\frac{3}{4}J$ is the interval J scaled by $\frac{3}{4}$. Take $x \in G$. By Minkowski's linear forms theorem, there is a coprime triple $(q, p_1, p_2) \in \mathbb{Z}^3 \setminus \{0\}$ satisfying the system of inequalities

$$\begin{cases} |qg_1(x) + p_1 g_2(x) + p_2| \leqslant c_0 \delta, \\ |qg_1'(x) + p_1 g_2'(x)| \leqslant c_2 (c_0 Q \delta)^{-1}, \\ \quad 0 \leqslant q \leqslant Q. \end{cases} \tag{85}$$

By the definition of G,

$$q > 2c_0 Q. \tag{86}$$

By (34) and the second inequality of (85), we have that $|qxf''(x) - p_1 f''(x)| < c_2 (c_0 Q \delta)^{-1}$. This together with (86) and (1) implies that

$$\left| x - \frac{p_1}{q} \right| \leqslant \frac{c_2}{2c_1 c_0^2 Q^2 \delta} = \frac{C_1}{Q^2 \delta} \overset{(8)}{\leqslant} \frac{1}{8} |J|. \tag{87}$$

Since $x \in \frac{3}{4}J$, $\frac{p_1}{q} \in J$. By Taylor's formula,

$$f\left(\frac{p_1}{q} \right) = f(x) + f'(x)\left(\frac{p_1}{q} - x \right) + \frac{1}{2} f''(\tilde{x})\left(\frac{p_1}{q} - x \right)^2 \tag{88}$$

for some $\tilde{x}$ between x and p_1/q. Thus $\tilde{x} \in J$. Using (34) and (88) we transform the first inequality of (85) into

$$\left| p_2 - qf\left(\frac{p_1}{q} \right) + \frac{q}{2} f''(\tilde{x})\left(x - \frac{p_1}{q} \right)^2 \right| \leqslant c_0 \delta \tag{89}$$

(see [10, p. 391] for details). Thus

$$\left| qf\left(\frac{p_1}{q} \right) - p_2 \right| \leqslant \left| p_2 - qf\left(\frac{p_1}{q} \right) + \frac{q}{2} f''(\tilde{x})\left(x - \frac{p_1}{q} \right)^2 \right| + \left| \frac{q}{2} f''(\tilde{x})\left(x - \frac{p_1}{q} \right)^2 \right|.$$

In view of (1), (87) and (89), $|qf(\frac{p_1}{q}) - p_2| \leqslant c_0 \delta + \frac{Q}{2} c_2 (\frac{C_1}{Q^2 \delta})^2$. By (8), this further transforms into $|qf(\frac{p_1}{q}) - p_2| \leqslant 2c_0 \delta$. This inequality and (86) imply $|f(\frac{p_1}{q}) - \frac{p_2}{q}| \leqslant \frac{\delta}{Q}$. Thus, we have shown that $(q, p_1, p_2) \in \mathcal{R}_f^{c_0}(Q, \delta, J)$. Therefore, in view of (87) we have that

$$G \subset \Delta_f^{c_0}(Q, \delta, J, \rho) = \bigcup_{(q, p_1, p_2) \in \mathcal{R}_f^{c_0}(Q, \delta, J)} \{x \colon |x - p_1/q| \leqslant \rho\} \quad \text{when } \rho = \frac{C_1}{\delta Q^2}.$$

By (84), this shows (10) and completes the proof.

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